q-Deformation of the $AdS_5 \times S^5$ Superstring S-matrix

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Common Trends in Gauge fields, Strings and Integrable models

Exact Results in Gauge/String Dualities

NORDITA

10th February 2012

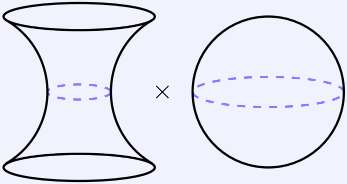
Based on

arXiv:0912.2958, arXiv:1104:2423 with Arkady Tseytlin

arXiv:1107.0628, arXiv:1112.4485 with Tim Hollowood and J. Luis Miramontes

Introduction

Planar AdS/CFT



$\longleftrightarrow \quad F_{\mu\nu}, \ \Psi_a, \ \bar{\Psi}_{\dot{a}}, \ \Phi_I$

String coupling $g_s \to 0$ | Rank of YM gauge group $N \to \infty$

Free string theory | Planar gauge theory

Integrability!

One coupling remaining

string tension – $\sqrt{\lambda}$ | λ – 't Hooft coupling

Aim

Solve free string theory on $AdS_5 \times S^5$

- Direct approach – *Light-cone gauge-fixed superstring.*

  Lack of 2-d Lorentz invariance.

  Many successes including S-matrix, phase, TBA, . . .

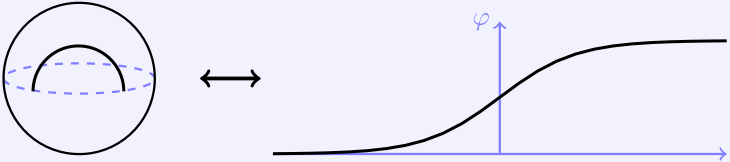
- Alternative approach? – *Pohlmeyer-reduced superstring.*

  Classically equivalent 2-d Lorentz-invariant action

  – describing same physical degrees-of-freedom

  – and with equivalent integrable structure.

Pohlmeyer reduction



Key features

- Solves all constraints $\longrightarrow$ describes physical degrees of freedom.
- Preserves classical integrability – equivalent integrable structure.
- Preserves 2-d Lorentz invariance and UV-finiteness.
- Standard kinetic terms for fermions – hidden 2-d SUSY.
- Special UV-finite massive integrable models

  — deserves study regardless of quantum equivalence.

Some history

The Pohlmeyer reduction relates the $O(n)$ sigma models (classical relativistic field theories) to integrable Hamiltonian systems.

- $O(3)$ sigma model classically related to sine-Gordon.
- $O(4)$ sigma model classically related to complex sine-Gordon.
- Discovery of the integrability of the classical $O(n)$ sigma model.
- Various developments, including use of Backlünd transformation to generate solutions and higher conserved charges.

Pohlmeyer '76
Luscher, Pohlmeyer '78
Pohlmeyer, Rehren '79
Eichenherr, Pohlmeyer '79

Reduction assumes conformal invariance, which for these models is broken at the quantum level.

- Quantum $O(3)$ sigma model and quantum sine-Gordon are different.
- Integrability extends to quantum level in both theories.

Polyakov '77
Zamolodchikov, Zamolodchikov '79

Pohlmeyer reduction – in string theory

- $O(n+1)$ sigma model can be interpreted as describing strings moving on S^n.
- Construction of classical string solutions in constant curvature backgrounds – de Sitter and anti de Sitter

Barbashov, Nesterenko '81
de Vega, Sanchez '83

- Construction of classical string solutions representing semiclassical closed string states in AdS/CFT context.

Hofman, Maldacena '06
Dorey et al '06
Jevicki, Spradlin, Volovich et al '07
BH, Iwashita, Tseytlin '09
Hollowood, Miramontes '09 ...

- Construction of Euclidean open-string world-surfaces related to $\mathcal{N}=4$ super Yang-Mills scattering amplitudes at strong coupling.

Alday, Maldacena '09
Alday, Gaiotto, Maldacena '09
Dorn et. al '09
Jevicki, Jin '09 ...

Pohlmeyer reduction – group theoretic

Classical strings on S^2 (S^3) equivalent to (complex) sine-Gordon.

Pohlmeyer '76

$S^2 = \frac{SO(3)}{SO(2)}$ ↓ $S^3 = \frac{SO(4)}{SO(3)}$

Classical strings on S^n, equivalently the symmetric space $\frac{SO(n+1)}{SO(n)}$, ...

related to the $\frac{SO(n)}{SO(n-1)}$ gWZW theory plus integrable potential.

Bakas, Park, Shin '95; Grigoriev, Tseytlin '07

Green-Schwarz action ↓ $\frac{PSU(2,2|4)}{USp(2,2) \times USp(4)}$

Superstrings on $AdS_5 \times S^5$ ...

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Superstrings on $AdS_5 \times S^5 \ldots$

$\mathbf{AdS_5 \times S^5}$ superstring

$AdS_5 \times S^5$ superstring worldsheet sigma model

Metsaev, Tseytlin '98

$$\frac{\hat{F}}{G} = \frac{PSU(2,2|4)}{USp(2,2)\times USp(4)}$$

Bosonic part of the coset is $\frac{SU(2,2)}{USp(2,2)} \times \frac{SU(4)}{USp(4)} \cong AdS_5 \times S^5$.

$\mathbb{Z}_4$ decomposition of algebra:

$$\hat{\mathfrak{f}} = \mathfrak{psu}(2,2|4) = \bigoplus_{i=1}^{4} \hat{\mathfrak{f}}_i \,, \qquad [\hat{\mathfrak{f}}_i, \hat{\mathfrak{f}}_j] \subset \hat{\mathfrak{f}}_{i+j \mod 4}$$

$$\hat{\mathfrak{f}}_0 = \mathfrak{g} = \mathfrak{usp}(2,2) \oplus \mathfrak{usp}(4)$$

$\hat{\mathfrak{f}}_{1,3}$ fermionic

$\hat{\mathfrak{f}}_2$ bosonic part of coset

Take group-valued field and consider Maurer-Cartan one-form

$$f \in PSU(2,2|4) \qquad \mathcal{J} = f^{-1} df \in \hat{\mathfrak{f}}$$

$\mathbb{Z}_4$ decomposition:

$$\mathcal{J} = \mathcal{A} + \mathcal{Q}_1 + \mathcal{P} + \mathcal{Q}_3$$

- For correct number of bosonic degrees of freedom require a G-gauge symmetry $\quad f \to f\, g(x^\pm)$

$\mathcal{A}$ transforms as a connection, $\mathcal{P}$ & $\mathcal{Q}_{1,3}$ transform covariantly.

Action is constructed from supertraces over products of $\mathcal{P}$ and $\mathcal{Q}_{1,3}$

$$\mathcal{S} = \frac{\sqrt{\lambda}}{2\pi} \int d^2x \, \sqrt{|h|} h^{ab} \, \mathrm{STr}\left(\mathcal{P}_a \mathcal{P}_b\right) + \epsilon^{ab} \, \mathrm{STr}\left(\mathcal{Q}_{1\,a} \mathcal{Q}_{3\,b}\right)$$

- For correct number of fermionic degrees of freedom require κ-symmetry — fixes coefficient of fermionic term.

$AdS_5 \times S^5$ superstring – Pohlmeyer reduction

Fix conformal gauge . . . $\sqrt{|h|}h^{ab} = \eta^{ab}$

and write equations of motion in first order form;
supplement by flatness condition for $\mathcal{J}$ $d\mathcal{J} + \mathcal{J} \wedge \mathcal{J} = 0$

Virasoro constraints

$$\mathrm{STr}\left(\mathcal{P}_\pm \mathcal{P}_\pm\right) = 0$$

Fermionic equations of motion

$$[\mathcal{P}_+,\ \mathcal{Q}_{1\,-}] = 0 \qquad [\mathcal{P}_-,\ \mathcal{Q}_{3\,+}] = 0$$

Bosonic eom & flatness equation projected on $\hat{\mathfrak{f}}_2$

$$\partial_+\mathcal{P}_- + [\mathcal{A}_+,\ \mathcal{P}_-] + [\mathcal{Q}_{3\,+},\ \mathcal{Q}_{3\,-}] = 0$$

$$\partial_-\mathcal{P}_+ + [\mathcal{A}_-,\ \mathcal{P}_+] + [\mathcal{Q}_{1\,+},\ \mathcal{Q}_{1\,-}] = 0$$

Flatness equation projected on $\hat{\mathfrak{f}}_0$, $\hat{\mathfrak{f}}_1$, $\hat{\mathfrak{f}}_3$

Virasoro constraints

Introduce a constant matrix $T \in \hat{\mathfrak{f}}_2$ such that

$$\mathrm{STr}\,(T^2) = 0 \ .$$

Solve Virasoro by using the G-gauge symmetry and residual conformal diffeomorphisms

$$\mathcal{P}_+ = \mu T \ , \qquad \mathcal{P}_- = \mu g^{-1} T g \ ,$$

$$g \in G = USp(2,2) \times USp(4) \ .$$

μ is an arbitrary scale parameter

– remnant of fixing residual conformal diffeomorphisms .

Algebra

- The matrix T defines an additional $\mathbb{Z}_2$ decomposition

$$\hat{\mathfrak{f}} = \hat{\mathfrak{f}}^{\parallel} \oplus \hat{\mathfrak{f}}^{\perp}$$

$$\left\{\hat{\mathfrak{f}}^{\parallel}, T\right\} = 0 \ , \qquad [\hat{\mathfrak{f}}^{\perp}, \ T] = 0 \ .$$

$$[\hat{\mathfrak{f}}^{\parallel}, \hat{\mathfrak{f}}^{\parallel}] \subset \hat{\mathfrak{f}}^{\perp} \qquad [\hat{\mathfrak{f}}^{\parallel}, \hat{\mathfrak{f}}^{\perp}] \subset \hat{\mathfrak{f}}^{\parallel} \qquad [\hat{\mathfrak{f}}^{\perp}, \hat{\mathfrak{f}}^{\perp}] \subset \hat{\mathfrak{f}}^{\perp}$$

- $\hat{\mathfrak{f}}_0^{\perp} = [\mathfrak{su}(2)]^4$ is an algebra

  – denote $\mathfrak{h}$ and the corresponding group H

Fermionic equations of motion

Solved by fixing partially κ-symmetry gauge

$$\mathcal{Q}_{1\,-} = \mathcal{Q}_{3\,+} = 0 \ ,$$

With residual κ-symmetry and solving flatness equations projected onto $\hat{\mathfrak{f}}^{\perp}_{1,3}$ can fix

$$\mathcal{Q}_{1\,+} \equiv \sqrt{\mu}\,\Psi_{_R} \in \hat{\mathfrak{f}}^{\parallel}_{1} \ , \qquad g\,\mathcal{Q}_{3\,-}g^{-1} \equiv \sqrt{\mu}\,\Psi_{_L} \in \hat{\mathfrak{f}}^{\parallel}_{3} \ .$$

Bosonic equations of motion

Solved by

$$g\mathcal{A}_{+}g^{-1} - \partial_{+}gg^{-1} \equiv A_{+} \in \mathfrak{h} \ , \qquad \mathcal{A}_{-} \equiv A_{-} \in \mathfrak{h} \ .$$

$AdS_5 \times S^5$ superstring Pohlmeyer reduction

Virasoro constraints ✓

Fermionic equations of motion ✓

Flatness equation projected on $\hat{\mathfrak{f}}^{\perp}_{1,3}$ ✓

Bosonic eom & flatness equation projected on $\hat{\mathfrak{f}}_2$ ✓

Flatness equation projected on $\hat{\mathfrak{f}}_0$, $\hat{\mathfrak{f}}^{\|}_1$, $\hat{\mathfrak{f}}^{\|}_3$

PR $AdS_5 \times S^5$ superstring – Action

- Reduced theory equations of motion have $H \times H$-gauge symmetry
- Gauge-fix to leave single factor, then the resulting equations come from the following action

$$\begin{aligned}\mathcal{S} = -\frac{k}{4\pi}\mathrm{STr}\Big[&\frac{1}{2}\int d^2x\, g^{-1}\partial_+ g g^{-1}\partial_- g - \frac{1}{3}\int d^3x\, \epsilon^{mnl} g^{-1}\partial_m g g^{-1}\partial_n g g^{-1}\partial_l g \\ &+ \int d^2x\, (A_+\partial_- g g^{-1} - A_- g^{-1}\partial_+ g - g^{-1}A_+ g A_- + A_+ A_-) \\ &+ \int d^2x\, (\Psi_{_L} T D_+ \Psi_{_L} + \Psi_{_R} T D_- \Psi_{_R} + \mu\, g^{-1}\Psi_{_L} g \Psi_{_R} + \mu^2\, g^{-1} T g T)\Big]\end{aligned}$$

- $g \in G = USp(2,2) \times USp(4)$
- $A_\pm \in \mathfrak{h} = [\mathfrak{su}(2)]^4$
- $\Psi_{_R} \in \hat{\mathfrak{f}}_1^{||}$
- $\Psi_{_L} \in \hat{\mathfrak{f}}_3^{||}$

Grigoriev, Tseytlin '07, '08

PR $AdS_5 \times S^5$ superstring – Comments

- 2-d Lorentz invariant, classically integrable theory

  – existence of a Lax connection

  Grigoriev, Tseytlin '07
  Hollowood, Miramontes '10, '11; Schmidtt '10, '11

- H-gauge symmetry

  – theory describes $\dim G - \dim H = 8$ bosonic degrees of freedom

- 16 2-d Majorana-Weyl fermionic variables

  – first order equations of motion

  – theory describes 8 fermionic degrees of freedom

- Linearised equations of motion

  – $8 + 8$ degrees of freedom with mass μ

  Grigoriev, Tseytlin '07

- Apparently no supersymmetry – target-space or spacetime
- Hidden supersymmetry – Gauge-fixed/modified action has non-local supersymmetry

  Grigoriev, Tseytlin '07; Schmidtt '10
  Goykhman, Ivanov '11; Hollowood, Miramontes '11

PR $AdS_5 \times S^5$ superstring – Going Quantum

$\frac{USp(2,2)}{SU(2)^2} \times \frac{USp(4)}{SU(2)^2}$ gauged WZW model

plus integrable potential with fermionic extension.

Grigoriev, Tseytlin '07; Mikhailov, Schafer-Nameki '07

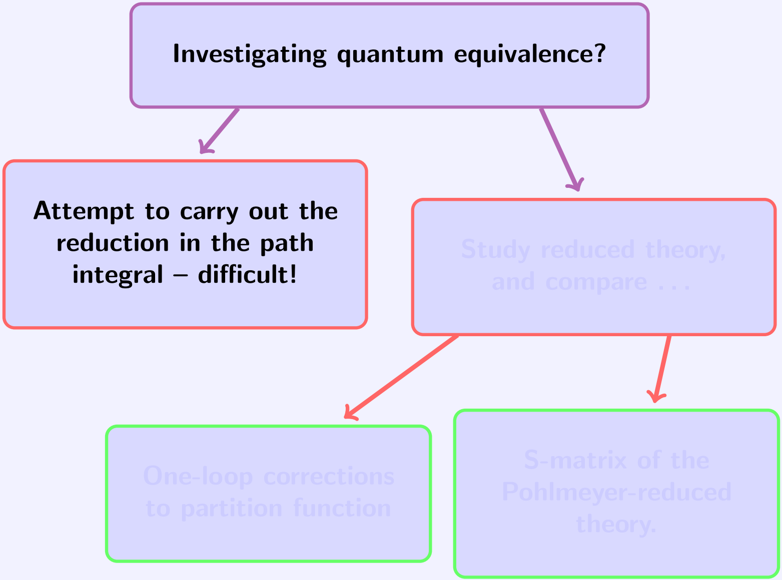
The reduction is at the level of the equations of motion — that is it doesn't know about the string tension, $\propto \sqrt{\lambda}$.
When we integrate up to an action, we introduce a new coupling, k, *a priori* unrelated to λ.

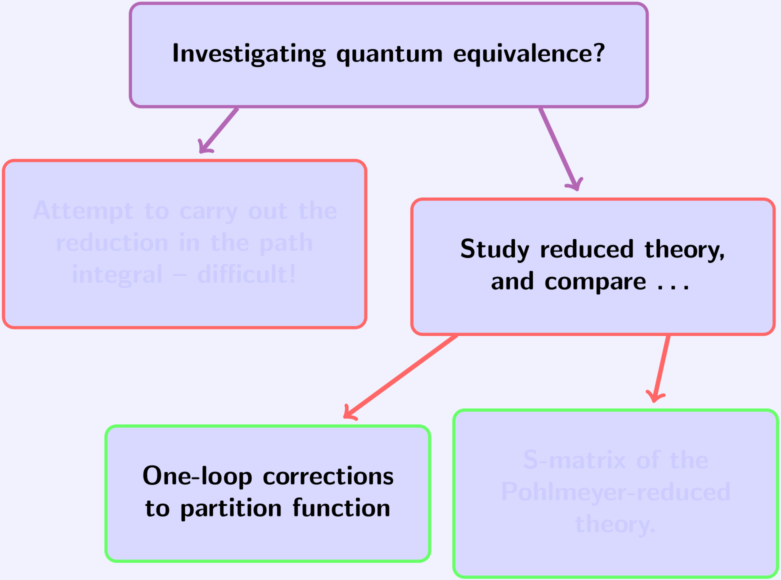
- Classical reduction assumes conformal invariance – for the $O(n)$ sigma model this is broken at quantum level.
- $AdS_5 \times S^5$ superstring has no conformal anomoly so Pohlmeyer reduced theory may be related at quantum level.

Grigoriev, Tseytlin '07
Mikhailov, Schafer-Nameki '07

First test – both superstring and PR theory are UV-finite. ☺

Roiban, Tseytlin '09


Investigating quantum equivalence?
Attempt to carry out the reduction in the path integral – difficult!
Study reduced theory, and compare . . .
One-loop corrections to partition function
S-matrix of the Pohlmeyer-reduced theory.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One-loop corrections to partition function

S-matrix of the Pohlmeyer-reduced theory.

Perturbative S-matrix

Work with a particular choice for T

$$T = \frac{i}{2}\mathrm{diag}(1,1,-1,-1,1,1,-1,-1)$$

Parametrise the group-element g in terms of an algebra element

$$g = \exp(X+\xi) \qquad X \in \hat{\mathfrak{f}}_0^{\parallel}\,, \quad \xi \in \hat{\mathfrak{f}}_0^{\perp}$$

Fix H-gauge symmetry $A_+ = 0$

Integrate out unphysical degrees of freedom, ξ, perturbatively using constraint equation $\left(g^{-1}\partial_+ g + [\Psi_R T,\, \Psi_R]\right)|_{\mathfrak{h}} = 0$

A_- decouples. After integration by parts and field redefinitions, left with local (quartic) action for the fields

$$X \qquad \Psi_L \qquad \Psi_R$$

BH, Tseytlin '09

Quartic local Lagrangian

$$
\begin{aligned}
\mathcal{L} = \mathrm{STr}\Big(& \frac{1}{2}\partial_+ X \partial_- X - \frac{\mu^2}{2} X^2 + \Psi_L T \partial_+ \Psi_L + \Psi_R T \partial_- \Psi_R + \mu \Psi_L \Psi_R \\
& + \frac{1}{12}[X, \partial_+ X][X, \partial_- X] + \frac{\mu^2}{24}[X, [X, T]]^2 \\
& - \frac{1}{4}[\Psi_L T, \Psi_L][X, \partial_+ X] - \frac{1}{4}[\Psi_R, T\Psi_R][X, \partial_- X] \\
& - \frac{\mu}{2}[X, \Psi_R][X, \Psi_L] + \frac{1}{2}[\Psi_L T, \Psi_L][\Psi_R, T\Psi_R]\Big) + \ldots
\end{aligned}
$$

Residual $[SU(2)]^4$ global symmetry
$\longrightarrow$ can write fields in representations.

Quartic local Lagrangian

$$\begin{aligned}\mathcal{L} = \mathrm{STr}\Big(& \frac{1}{2}\partial_+ X \partial_- X - \frac{\mu^2}{2} X^2 + \Psi_L T \partial_+ \Psi_L + \Psi_R T \partial_- \Psi_R + \mu \Psi_L \Psi_R \\ & + \frac{1}{12}[X, \partial_+ X][X, \partial_- X] + \frac{\mu^2}{24}[X, [X, T]]^2 \\ & - \frac{1}{4}[\Psi_L T, \Psi_L][X, \partial_+ X] - \frac{1}{4}[\Psi_R, T\Psi_R][X, \partial_- X] \\ & - \frac{\mu}{2}[X, \Psi_R][X, \Psi_L] + \frac{1}{2}[\Psi_L T, \Psi_L][\Psi_R, T\Psi_R]\Big) + \dots\end{aligned}$$

Residual $[SU(2)]^4$ global symmetry

$\longrightarrow$ can write fields in representations.

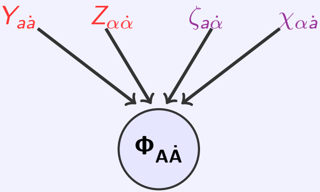
Symmetries and packaging

Similar to light-cone gauge-fixed string...

Klose, McLoughlin, Roiban, Zarembo '06
Arutyunov, Frolov, Zamaklar '06

$$X = Y + Z \qquad \Psi = \zeta + \chi$$

$$\begin{pmatrix} SU(2)_a & Y & 0 & \zeta \\ Y & SU(2)_{\dot a} & \chi & 0 \\ 0 & \chi & SU(2)_\alpha & Z \\ \zeta & 0 & Z & SU(2)_{\dot\alpha} \end{pmatrix}$$



$$A = (a, \alpha) \qquad \dot A = (\dot a, \dot\alpha)$$

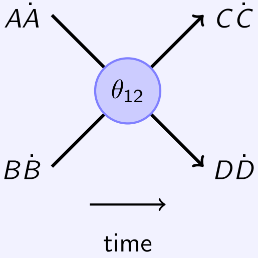
Perturbative scattering

The excitations we want to scatter are

$$\Phi_{A\dot{A}}(\vartheta) \qquad A, \dot{A} = 1,\ 2,\ 3,\ 4$$

We want to scatter two of these particles

$$S_{A\dot{A},B\dot{B}}^{C\dot{C},D\dot{D}}(\theta_{12}) \quad =$$



$\approx$ Integrability/ hidden SUSY?

cf. light-cone gauge-fixed string theory
Klose, McLoughlin, Roiban, Zarembo '06
Arutyunov, Frolov, Zamaklar '06

Conventions ...

- $p = \sinh\vartheta$, where p is the spatial momenta and ϑ is the rapidity.
- $\theta_{ij} = \vartheta_i - \vartheta_j$.

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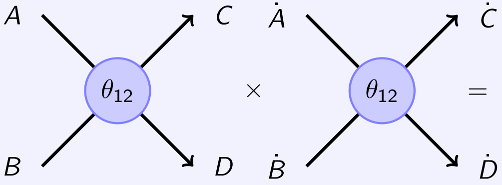

$$= \quad S_{AB}^{CD}(\theta_{12})\ S_{\dot{A}\dot{B}}^{\dot{C}\dot{D}}(\theta_{12})$$

Conventions . . .

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Perturbative S-matrix

$$S_{AB}^{CD}(\theta,k) = \begin{cases} K_1(\theta,k)\delta_a^c\delta_b^d \; + \; K_2(\theta,k)\delta_a^d\delta_b^c \\ K_3(\theta,k)\delta_\alpha^\gamma\delta_\beta^\delta \; + \; K_4(\theta,k)\delta_\alpha^\delta\delta_\beta^\gamma \\ K_5(\theta,k)\epsilon_{ab}\epsilon^{\gamma\delta} \quad K_6(\theta,k)\epsilon_{\alpha\beta}\epsilon^{cd} \\ K_7(\theta,k)\delta_a^d\delta_\beta^\gamma \quad K_8(\theta,k)\delta_\alpha^\delta\delta_b^c \\ K_9(\theta,k)\delta_a^c\delta_\beta^\delta \quad K_{10}(\theta,k)\delta_\alpha^\gamma\delta_b^d \end{cases}$$

$$K_i = \; P(\theta,k) \; \hat{K}_i$$

$$P(\theta,k) = 1 + \; \frac{\pi \,\mathrm{cosech}\,\theta}{4k^2}\Big(i\big[2+(i\pi-2\theta)\coth\theta\big] - \pi\,\mathrm{cosech}\,\theta\Big)$$

Perturbative S-matrix

$$\hat{K}_1(\theta,k) = \hat{K}_3(\theta,-k) = 1 + \frac{i\pi}{2k}\tanh\frac{\theta}{2} - \frac{5\pi^2}{8k^2} - \frac{i\pi\theta}{2k^2} + \mathcal{O}(k^{-3})$$

$$\hat{K}_2(\theta,k) = \hat{K}_4(\theta,-k) = -\frac{i\pi}{k}\coth\theta + \frac{\pi^2}{2k^2} + \frac{i\pi\theta}{k^2} + \mathcal{O}(k^{-3})$$

$$\hat{K}_5(\theta,k) = -\hat{K}_6(\theta,-k) = -\frac{i\pi}{2k}\,\mathrm{sech}\,\frac{\theta}{2} + \mathcal{O}(k^{-3})$$

$$\hat{K}_7(\theta,k) = -\hat{K}_8(\theta,-k) = -\frac{i\pi}{2k}\,\mathrm{cosech}\,\frac{\theta}{2} + \mathcal{O}(k^{-3})$$

$$\hat{K}_9(\theta,k) = \hat{K}_{10}(\theta,-k) = 1 + \mathcal{O}(k^{-3})$$

Perturbative S-matrix – Comments

- Unitary and crossing-symmetric – by construction.
- Satisfies group factorisation, but **not** Yang-Baxter equation!
- YBE "anomaly": clash between relativistic invariance, trigonometric structure and non-abelian $[SU(2)]^4$ symmetry.
- Functions K closely related to a particular limit of q-deformed $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ R-matrix of quantum-deformed Hubbard model

  Beisert, Koroteev '08

  Beisert '10
- PR theory should be integrable…

  q-deformation suggests a possible resolution.
- **Further evidence:**
  - lower dimensional models;

    PR $AdS_2 \times S^2$ and PR $AdS_3 \times S^3$, where H is trivial and abelian respectively, have undeformed and q-deformed supersymmetry respectively.

    BH, Tseytlin '11
  - semi-classical analysis of soliton spectrum.

    Hollowood, Miramontes '11

Q-deformed S-matrix of the PR theory

R-matrix of the q-deformed Hubbard model depends on two parameters g and q

$g \propto \sqrt{\lambda}$

Beisert, Koroteev '08

Q-deformed R-matrix of PR theory given by strong-coupling limit

$$g \to \infty$$

and identifying q with the coupling k

$$q = \exp(\frac{i\pi}{k})$$

Beisert '10
BH, Tseytlin '11

Q-deformed S-matrix of PR theory given by solving unitarity and crossing-symmetry for the phase ✓

BH, Tseytlin '11

Use bootstrap to construct scattering of boundstates ✓

BH, Hollowood, Miramontes '11

$$\Phi_A = (\phi_a, \psi_\alpha) , \qquad x = \exp(\theta) , \qquad q = \exp(\frac{i\pi}{k}) .$$

$$S(x) \left|\phi^a \phi^a\right\rangle = A \left|\phi^a \phi^a\right\rangle , \qquad S(x) \left|\psi^\alpha \psi^\alpha\right\rangle = D \left|\psi^\alpha \psi^\alpha\right\rangle ,$$

$$S(x) \left|\phi^1 \phi^2\right\rangle = \frac{q(A-B)}{q^2+1} \left|\phi^2 \phi^1\right\rangle + \frac{q^2 A + B}{q^2+1} \left|\phi^1 \phi^2\right\rangle + \frac{C}{1+q^2} \left|\psi^3 \psi^4\right\rangle - \frac{qC}{1+q^2} \left|\psi^4 \psi^3\right\rangle ,$$

$$S(x) \left|\phi^4 \phi^3\right\rangle = \frac{q(A-B)}{q^2+1} \left|\phi^1 \phi^2\right\rangle + \frac{q^2 B + A}{q^2+1} \left|\phi^2 \phi^1\right\rangle - \frac{qC}{1+q^2} \left|\psi^3 \psi^4\right\rangle + \frac{q^2 C}{1+q^2} \left|\psi^4 \psi^3\right\rangle ,$$

$$S(x) \left|\psi^3 \psi^4\right\rangle = \frac{q(D-E)}{q^2+1} \left|\psi^4 \psi^3\right\rangle + \frac{q^2 D + E}{q^2+1} \left|\psi^3 \psi^4\right\rangle + \frac{F}{1+q^2} \left|\phi^1 \phi^2\right\rangle - \frac{qF}{1+q^2} \left|\phi^2 \phi^1\right\rangle ,$$

$$S(x) \left|\psi^4 \psi^3\right\rangle = \frac{q(D-E)}{q^2+1} \left|\psi^3 \psi^4\right\rangle + \frac{q^2 E + D}{q^2+1} \left|\psi^4 \psi^3\right\rangle - \frac{qF}{1+q^2} \left|\phi^1 \phi^2\right\rangle + \frac{q^2 F}{1+q^2} \left|\phi^2 \phi^1\right\rangle ,$$

$$S(x) \left|\phi^a \psi^\alpha\right\rangle = G \left|\psi^\alpha \phi^a\right\rangle + H \left|\phi^a \psi^\alpha\right\rangle , \qquad S(x) \left|\psi^\alpha \phi^a\right\rangle = K \left|\psi^\alpha \phi^a\right\rangle + L \left|\phi^a \psi^\alpha\right\rangle$$

$$A = P(\theta, k) \frac{(qx-1)(x+1)}{q^{1/2} x} \qquad D = P(\theta, k) \frac{(q-x)(x+1)}{q^{1/2} x}$$

$$B = P(\theta, k) \frac{q^3 - (q^3 - 2q^2 + 2q - 1)x - x^2}{q^{3/2} x} \qquad E = P(\theta, k) \frac{q^3 x^2 - (q^3 - 2q^2 + 2q - 1)x - 1}{q^{3/2} x}$$

$$C = F = P(\theta, k) \frac{i(q-1)(q^2+1)(x-1)}{q^{3/2} x^{1/2}} \qquad G = L = P(\theta, k)(x - x^{-1})$$

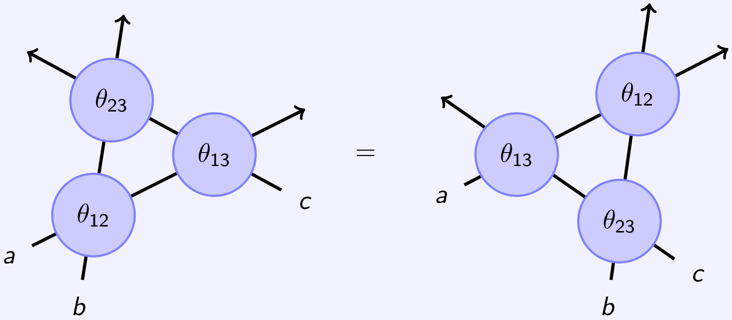
$$H = K = P(\theta, k) \frac{(q-1)(x+1)}{q^{1/2} x^{1/2}}$$

$$P(\theta, k) = \frac{1}{2(q^{1/2} - q^{-1/2})} \sqrt{\frac{\sinh(\frac{\theta}{2} + \frac{i\pi}{2k})}{\sinh(\frac{\theta}{2} - \frac{i\pi}{2k})} \cdot \frac{\cosh(\frac{\theta}{2} - \frac{i\pi}{2k})}{\cosh(\frac{\theta}{2} + \frac{i\pi}{2k})}}$$

$$\exp\left[-2\int_0^\infty \frac{dt}{t} \frac{\cosh^2(t(1-\frac{1}{k}))\sinh(t(1-\frac{\theta}{i\pi}))\sinh(\frac{t\theta}{i\pi})}{\sinh t \cosh^2 t}\right]$$

Q-deformed S-matrix of the PR theory.

Satisfies the Yang-Baxter equation by construction ☺



Interpolating R-matrix/S-matrix

Pohlmeyer-reduced R-matrix appears as $\lambda \to \infty$ limit of q-deformed light-cone gauge string R-matrix.

Beisert, Koroteev '08; Beisert '10
BH, Tseytlin '11; BH, Hollowood, Miramontes '11

Suggests the existence of an **interpolating S-matrix**

– matrix structure completely fixed by R-matrix

Beisert, Koroteev '08

– overall phase fixed by physical requirements

BH, Hollowood, Miramontes '11

Interpolating S-matrix – Phase

Phase of light-cone gauge-fixed S-matrix

including . . . Arutyunov, Frolov, Staudacher '04
Beisert, Staudacher '05
Beisert, Tseytlin '05
Janik '06
Hernandez, Lopez '06
Arutyunov, Frolov '06
Freyhult, Kristjansen '06
Beisert, Hernandez, Lopez '06
Beisert '06
Beisert, Eden, Staudacher '06
Arutyunov, Frolov, Zamaklar '06
Kostov, Serban, Volin '07
Dorey, Hofman, Maldacena '07
Gromov, Vieira '07
Arutyunov, Frolov '09
Volin '09
Review article – Vieira, Volin '10

Long story!

Q-deformed S-matrix phase construction follows a similar logic.

Interpolating S-matrix – Single particle states

Single particle states labelled by a pair of variables, $x^{\pm}$, which satisfy the non-trivial constraint equation

$$q^{-1}\left(x^{+}+\frac{1}{x^{+}}\right)-q\left(x^{-}+\frac{1}{x^{-}}\right)=(q-q^{-1})\left(\xi+\frac{1}{\xi}\right)$$

$$\xi=-i\tilde{g}(q-q^{-1})\ ,\qquad \tilde{g}^{2}=\frac{g^{2}}{1-g^{2}(q-q^{-1})^{2}}\ .$$

Beisert, Koroteev '08
Beisert, Galleas, Matsumoto '11
BH, Hollowood, Miramontes '11

Interpolating S-matrix – Single particle states

Solve constraint equation by introducing a function $x(u)$

$$x + \frac{1}{x} + \xi + \frac{1}{\xi} = \frac{1}{\xi}\left(\frac{g}{\tilde{g}}\right)^2 q^{-2iu}$$

Branch points at $q^{-2iu_\pm} = \left(\frac{g}{\tilde{g}}\right)^2 (\xi \mp 1)^2$

$$x^\pm = x(u \pm \frac{i}{2})$$

- When $q = e^{\frac{i\pi}{k}}$ – imaginary u direction is periodic with period k $\rightarrow$ u is complex coordinate on cylinder

  String limit: $q \rightarrow 1$, $k \rightarrow \infty \longrightarrow$ periodic direction decompactifies as expected.

  PR limit: $g \rightarrow \infty \longrightarrow u$ cylinder becomes identified with usual rapidity cylinder.

Interpolating S-matrix – Crossing equation

The crossing equation for the phase is given by

$$\sigma^\gamma(x_1^\pm, x_2^\pm)\sigma(x_1^\pm, x_2^\pm) = \frac{x_2^- + \xi}{x_2^+ + \xi}\frac{x_1^- - x_2^+}{x_1^- - x_2^-}\frac{1 - \frac{1}{x_1^+ x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}}$$

Beisert, Koroteev '08
BH, Hollowood, Miramontes '11

First factor on LHS is analytic continuation of dressing phase along contour γ which

- first passes through branch cut corresponding to $|x_1^+| = 1$
- second passes through branch cut corresponding to $|x_1^-| = 1$

Solution of crossing equation

Following light-cone gauge-fixed string theory ...

$$\sigma(x_1^\pm, x_2^\pm) = \exp i\left[\chi(x_1^+, x_2^+) - \chi(x_1^-, x_2^+) - \chi(x_1^+, x_2^-) + \chi(x_1^-, x_2^-)\right]$$

Rewrite as a Riemann-Hilbert problem that can be solved:

$$\chi(x_1, x_2) = i \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z - x_1} \oint_{|z'|=1} \frac{dz'}{2\pi i} \frac{1}{z' - x_2} \log \frac{\Gamma_{q^2}(1 + iu(z) - iu(z'))}{\Gamma_{q^2}(1 - iu(z) + iu(z'))}$$

BH, Hollowood, Miramontes '11

$$\Gamma_{q^2}(1 + x) = \frac{1 - q^{2x}}{1 - q^2} \Gamma_{q^2}(x)$$

String limit: $q \to 1 \Rightarrow \Gamma_{q^2}(x) \to \Gamma(x)$ and as expected we get the light-cone gauge-fixed string phase ☺

Solution of crossing equation

PR limit: $g \to \infty \to$ possible to show that the phase reduces to the correct relativistic expression ☺

Modulo subtleties . . . ☹

- To get the correct relativistic crossing equation $S(\theta) = S(i\pi - \theta)$ we need to set

$$\vartheta = \frac{\pi}{k} u$$

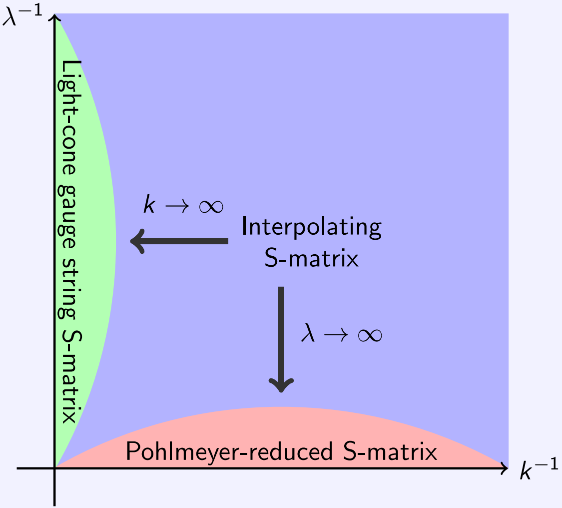
With this choice – if bound-states of light-cone gauge-fixed string theory transform in $\langle n, 0\rangle$ representations then bound states of the relativistic limit transform in $\langle 0, n\rangle$ representations.

Poles moving on and off "physical strip" as we interpolate?

Or take $\vartheta = -\frac{\pi}{k} u$ – no relativistic crossing?

More details discussed in BH, Hollowood, Miramontes '11 including alternative contours to γ etc.

Interpolating S-matrix



Beisert, Koroteev '08; Beisert '10
BH, Tseytlin '09, '10, '11; BH, Hollowood, Miramontes '11

Symmetry algebra of the PR theory

The Pohlmeyer-reduced theory S-matrix is invariant under a larger symmetry algebra.

Hollowood, Miramontes '11
BH, Hollowood, Miramontes '11

$$\mathcal{U}_q(\widehat{\mathfrak{su}}(2|2)^{(\sigma)})$$

σ is $\mathbb{Z}_4$ automorphism of $\mathfrak{su}(2|2)$ that defines twisting.

$$\mathfrak{su}(2|2) = \bigoplus_{m=0}^{3} \mathfrak{su}(2|2)_m \qquad \sigma(\mathfrak{su}(2|2)_m) = i^m \mathfrak{su}(2|2)_m$$

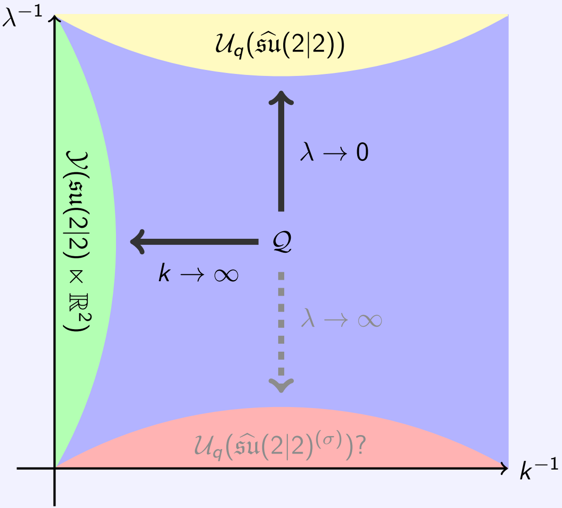
$$\widehat{\mathfrak{su}}(2|2)^{(\sigma)} = \bigoplus_{M=-\infty}^{\infty} z^M \mathfrak{su}(2|2)_{M \text{ mod } 4} \oplus \mathbb{C}\mathfrak{C}$$

As level 2 contains only the central element

the levels $\{-2,\ -1,\ 0,\ 1,\ 2\}$ form the familiar finite subalgebra

$$\mathfrak{psu}(2|2) \ltimes \mathbb{R}^2$$

Interpolating symmetry algebra



Beisert, Galleas, Matsumoto '11
de Leeuw, Matsumoto, Regelskis '11
de Leeuw, Regelskis, Torrielli '11

Conclusions

- S-matrix of the PR superstring on $AdS_5 \times S^5$ has been constructed.
- Underlying infinite dimensional symmetry algebra has been understood.

- Interpolating S-matrix, including the phase, has been derived.
- Interpolating symmetry algebra has been constructed.
- Quantum connection between light-cone gauge-fixed and Pohlmeyer-reduced superstring ☺.

- Open questions
  - PR S-matrix - relation between perturbative and q-deformed S-matrices.
  - Matrix unitarity of q-deformed PR S-matrix.
  - Relation of S-matrix story to semi-classical partition function computations.
  - Other limits of interpolating S-matrix – double scalings of g and k.
  - Mirror interpolating S-matrix and its $g \to \infty$ limit.
  - Regularisation of the string theory — TBA.
  - Physical meaning of interpolating theory?

T h a n k y o u !