Finite-size effects of β-deformed SYM at strong coupling

Changrim Ahn
Institute for the Early Universe
Ewha Womans University

with

D. Bombardelli (U. Porto)
M. Kim (Sogang U.)

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Motivation

- All-loop integrability in the spectrum of

  N=4 SYM

- Is this also true for "one-parameter extension"

  β-deformed SYM ?

(cf) XXX → XXZ → XYZ spin chains

Outline

- Review: S-matrix approach to N=4 SYM
  - Weak coupling
  - Strong coupling
- N=1 "beta"-deformed SYM
  - Twisted S-matrix with twisted BC
  - Weak coupling
  - Controversy over dual classical string solutions
  - Strong coupling Luscher correction
- Conclusion

$\mathcal{N}$=4 SYM

$$S = \frac{\mathrm{Tr}}{g_{\mathrm{YM}}^2} \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu}^2 + (D_\mu \Phi^a)^2 + V(X,Y,Z) + \bar{\chi} \not{D} \chi - i \bar{\chi} \Gamma_a [\Phi^a, \chi] \right\}$$

$$X = \Phi_1 + i\Phi_2, \quad Y = \Phi_3 + i\Phi_4, \quad Z = \Phi_5 + i\Phi_6$$

$$V_{N=4} = |ZX - XZ|^2 + |XY - YX|^2 + |YZ - ZY|^2$$

- CFT
- Exact results based on INTEGRABILITY

Integrability → Factorized scattering theory

YBE, Symmetry

↓

S-matrix

↓

Asymptotic Bethe ansatz, TBA

↙ ↘

Strong coupling limit:
Classical string theory

Weak coupling limit:
Perturbative gauge theory

S-matrix

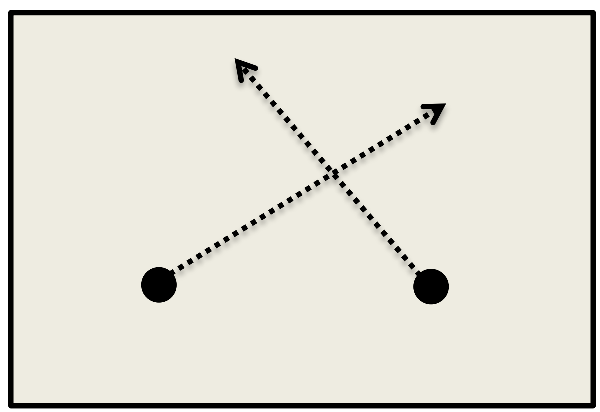
- SYM side : scattering of fields on the spin chain

$$\overrightarrow{p_1} \qquad\qquad\qquad\qquad \overleftarrow{p_2}$$

$$\mathrm{Tr}\left[\cdots ZZZZZZZZZ\chi_1 ZZZZZZ\cdots ZZZZZZ\chi_2 ZZZZZZZZZZ\cdots\right]$$

$$\left\{X, \overline{X}, Y, \overline{Y}, \chi^\alpha, D_\mu\right\}$$

- String side : scattering on the world sheet



- Symmetry of the excitations: su(2|2) x su(2|2)

$$\left(\begin{array}{c|c} \mathbb{L}_a{}^b & \mathbb{Q}_\alpha{}^b \\ \hline \mathbb{Q}^{\dagger\beta}_a & \mathbb{R}_\alpha{}^\beta \end{array}\right), \quad \left(\begin{array}{c|c} \mathbb{L}_{\dot a}{}^{\dot b} & \mathbb{Q}_{\dot\alpha}{}^{\dot b} \\ \hline \mathbb{Q}^{\dagger\dot\beta}_{\dot a} & \mathbb{R}_{\dot\alpha}{}^{\dot\beta} \end{array}\right)$$

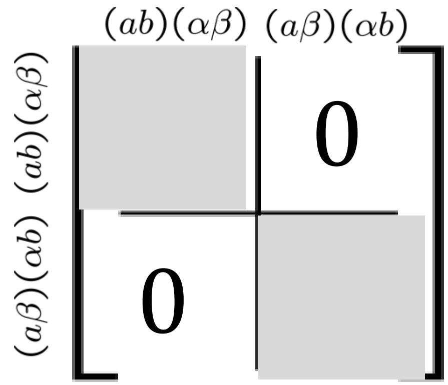
- From symmetry to S-matrix

[Beisert 2008]

$$\left[\mathbf{S}(p_1,p_2), \left(\begin{array}{c|c} \mathbb{L}_a{}^b & \mathbb{Q}_\alpha{}^b \\ \hline \mathbb{Q}^{\dagger\beta}_a & \mathbb{R}_\alpha{}^\beta \end{array}\right)\right] = 0$$

- Yang-Baxter equation is satisfied

- S : 16 x 16 matrix

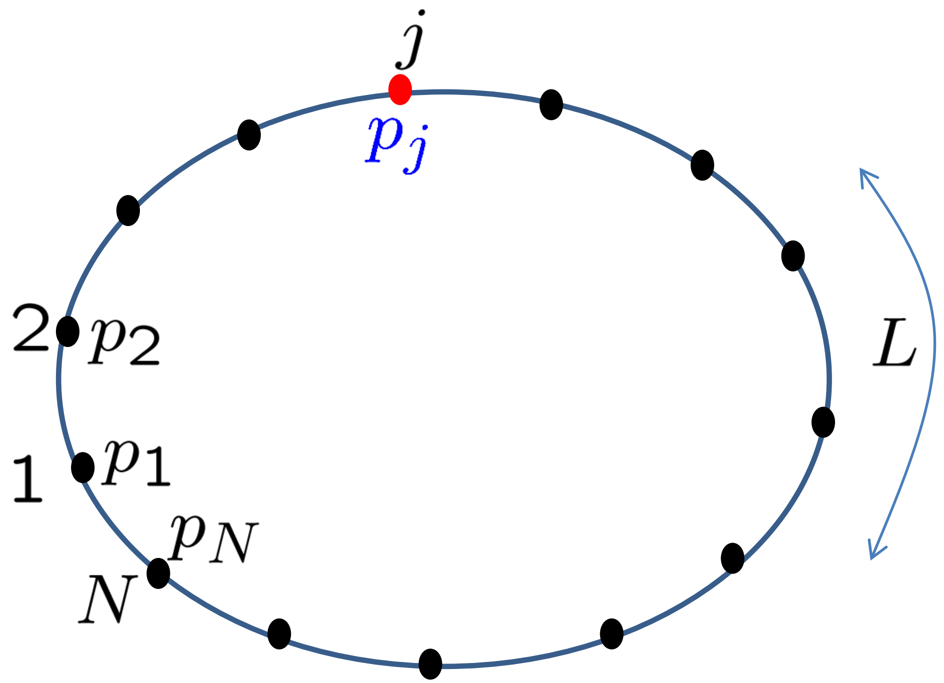


$$
\begin{aligned}
S^{a\,a}_{a\,a} &= A\,, \quad S^{\alpha\,\alpha}_{\alpha\,\alpha} = D\,,\\
S^{a\,b}_{a\,b} &= \frac{1}{2}(A-B)\,, \quad S^{b\,a}_{a\,b} = \frac{1}{2}(A+B)\,,\\
S^{\alpha\,\beta}_{\alpha\,\beta} &= \frac{1}{2}(D-E)\,, \quad S^{\beta\,\alpha}_{\alpha\,\beta} = \frac{1}{2}(D+E)\,,\\
S^{\alpha\,\beta}_{a\,b} &= -\frac{1}{2}\epsilon_{ab}\epsilon^{\alpha\beta}C\,, \quad S^{a\,b}_{\alpha\,\beta} = -\frac{1}{2}\epsilon^{ab}\epsilon_{\alpha\beta}F\,,\\
S^{a\,\alpha}_{a\,\alpha} &= G\,, \quad S^{\alpha\,a}_{a\,\alpha} = H\,, \quad S^{a\,\alpha}_{\alpha\,a} = K\,, \quad S^{\alpha\,a}_{\alpha\,a} = L
\end{aligned}
$$

$$
\begin{aligned}
A &= S_0\frac{x_2^- - x_1^+}{x_2^+ - x_1^-}\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\,,\\
B &= -S_0\left[\frac{x_2^- - x_1^+}{x_2^+ - x_1^-} + 2\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^-x_2^- - x_1^+x_2^+)}\right]\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\,,\\
C &= S_0\frac{2ix_1^-x_2^-(x_1^+ - x_2^+)\eta_1\eta_2}{x_1^+x_2^+(x_1^- - x_2^+)(1 - x_1^-x_2^-)}\,, \qquad D = -S_0\,,\\
E &= S_0\left[1 - 2\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- + x_2^+)}{(x_1^- - x_2^+)(x_1^-x_2^- - x_1^+x_2^+)}\right]\,,\\
F &= S_0\frac{2i(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- - x_2^+)(1 - x_1^-x_2^-)\tilde\eta_1\tilde\eta_2}\,,\\
G &= S_0\frac{(x_2^- - x_1^-)}{(x_2^+ - x_1^-)}\frac{\eta_1}{\tilde\eta_1}\,, \qquad H = S_0\frac{(x_2^+ - x_2^-)}{(x_1^- - x_2^+)}\frac{\eta_1}{\tilde\eta_2}\,,\\
K &= S_0\frac{(x_1^+ - x_1^-)}{(x_1^- - x_2^+)}\frac{\eta_2}{\tilde\eta_1}\,, \qquad L = S_0\frac{(x_1^+ - x_2^+)}{(x_1^- - x_2^+)}\frac{\eta_2}{\tilde\eta_2}
\end{aligned}
$$

$$\eta_1 = \eta(p_1)e^{ip_2/2}\,, \quad \eta_2 = \eta(p_2)\,, \quad \tilde\eta_1 = \eta(p_1)\,, \quad \tilde\eta_2 = \eta(p_2)e^{ip_1/2}$$

- Asymptotic Bethe ansatz
  - PBC

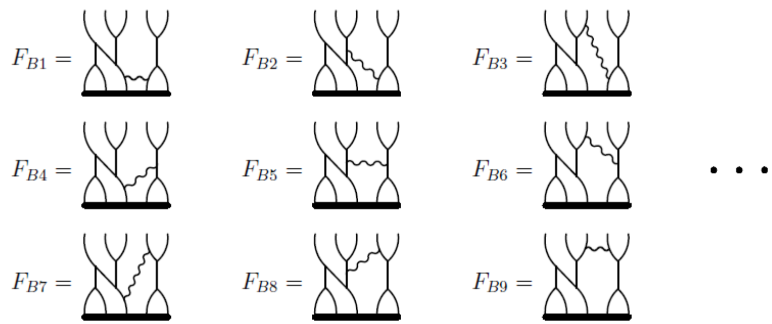


  - At each crossing, S-matrix $$e^{ip_jL}\prod_{k\neq j,1}^{N} S(p_j,p_k)=1$$
  - Diagonalize "transfer" matrix → Beisert-Staudacher BAE

Weak coupling side

- (ex) su(2) Konishi $\mathrm{Tr}\,[ZZXX]\,,\quad \mathrm{Tr}\,[ZXZX]$
- Perturbative gauge theory computation

[Fiamberti, Santambrogio, Sieg, Zanon (2008)]



Asymptotic BAE

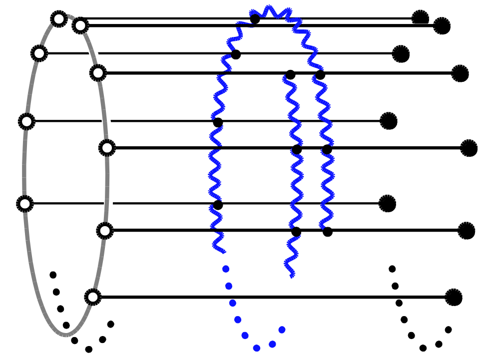
$$\Delta = \underbrace{4 + 12g^2 - 48g^4 + 336g^6 - (2820 + 288\zeta(3))g^8}_{\text{Asymptotic BAE}}$$

$$+ \underbrace{(324 + 864\zeta(3) - 1440\zeta(5))}_{\text{Finite-size effect}}g^8 + \ldots$$

Finite-size effect

Finite-size Effects

- Wrapping effects



- Asymptotic Bethe ansatz is valid for infinite length

- “finite-size” methods based on S-matrix
  - Luscher correction
  - Thermodynamic Bethe ansatz / Y-system / NLIE

Results from Luscher formula

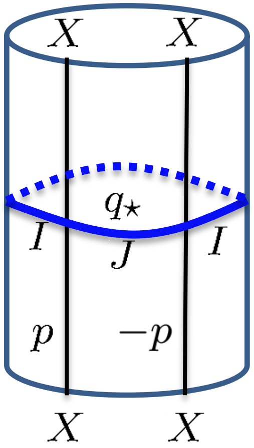
For the su(2) Konishi [Bajnok, Janik (2008)]

$$\delta\Delta = -\sum_{Q=1}^{\infty}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\,\frac{4^L g^{2L}}{(Q^2+q^2)^L}\,\sum_{I,J}(-1)^{F_I}\left[\mathcal{S}^{(Q1)}(q_\star(q),p)_{IX}^{JX}\cdot\mathcal{S}^{(Q1)}(q_\star(q),-p)_{JX}^{IX}\right]$$

L=4

$q_\star(q) = -i\tilde{\epsilon}(q)$

$$\sum_Q \sum_{I,J}$$

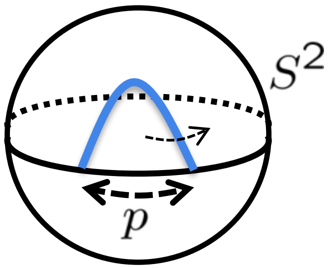


$$\delta\Delta = (324 + 864\zeta(3) - 1440\zeta(5))g^8$$

Strong coupling side

- Classical solutions of the string theory

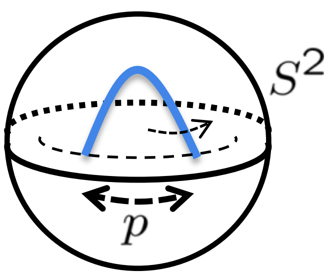
(ex) Giant magnon [Hofman, Maldacena (2006)]



$$E, J \to \infty \qquad E - J = 2g \sin\frac{p}{2}$$

- Giant magnon with finite-size J

[Arutyunov, Frolov, Zamaklar (2007); Klose, McLoughlin (2008)]



$$E - J \approx 2g \sin\frac{p}{2} \underbrace{- 8g \sin^3\frac{p}{2} \exp\left[-\left(\frac{J}{g \sin\frac{p}{2}} + 2\right)\right]}_{\text{Finite-size effect}} + \dots$$

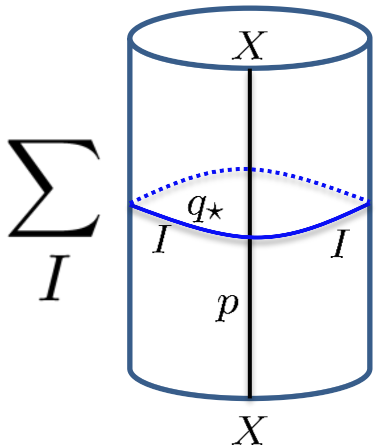
$$\infty > J \gg g \gg 1$$

Luscher formula

[Ambjorn, Janik, Kristjansen (2006); Janik, Lukowski (2007)]

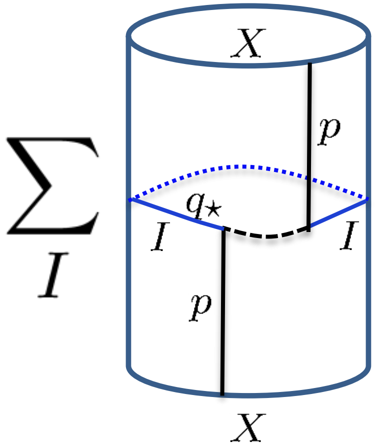
- Only fundamental mirror particles contribute

$$\delta\Delta = -\int_{-\infty}^{\infty}\frac{dq}{2\pi}\left(1-\frac{\epsilon'(p)}{\epsilon'(q_\star)}\right)e^{-iq_\star(q)J}\ \sum_I(-1)^{F_I}\left[\mathcal{S}^{(11)}(q_\star(q),p)^{IX}_{IX}\right]$$



- A leading classical contribution is obtained by a pole of S-matrix

$$\begin{aligned}\delta\Delta &= -i\left(1-\frac{\epsilon'(p)}{\epsilon'(q_\star(\bar q))}\right)e^{-iq_\star(\bar q)J}\ \sum_I(-1)^{F_I}\operatorname*{Res}_{q=\bar q}\left[\mathcal{S}^{(11)}(q_\star(q),p)^{IX}_{IX}\right]\\ &= -8g\sin^3\frac{p}{2}\ \exp\left[-\left(\frac{J}{g\sin\frac{p}{2}}+2\right)\right]\end{aligned}$$



β deformed SYM

[Leigh, Strassler (1995)]

$$S = \frac{\mathrm{Tr}}{g_{\mathrm{YM}}^2} \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu}^2 + (D_\mu \Phi^a)^2 + V_\beta(X,Y,Z) + \bar{\chi} \not{D} \chi - i \bar{\chi} \Gamma_a [\Phi^a, \chi] \right\}$$

$$V_\beta = \left| e^{i\pi\beta} ZX - e^{-i\pi\beta} XZ \right|^2 + \left| e^{i\pi\beta} XY - e^{-i\pi\beta} YX \right|^2 + \left| e^{i\pi\beta} YZ - e^{-i\pi\beta} ZY \right|^2$$

- *N*=1 super-CFT

- Dual to string theory on Lunin-Maldacena background

$$AdS_5 \times S^5_\beta$$

- Is this all-loop INTEGRABLE ?

Conjecture: Integrability

[CA, Bajnok, Bombardelli, Nepomechie (2010a)]

- Drinfeld-Reshetikhin twisted S-matrix from YBE

$$\tilde{S}(p_1, p_2) = F \cdot \mathcal{S}(p_1, p_2) \cdot F$$

$$\mathcal{S} = S_{su(2|2)} \otimes S_{su(2|2)}, \qquad F = e^{2\pi i \beta (h \otimes \mathbb{I} \dot{\otimes} \mathbb{I} \otimes h - \mathbb{I} \otimes h \dot{\otimes} h \otimes \mathbb{I})}, \qquad h = \mathrm{diag}(\frac{1}{2}, -\frac{1}{2}, 0, 0)$$

- Twisted boundary condition

$$M = \mathbb{I} \otimes e^{4 i \pi \beta J h}$$

- Reproduce (3-parameter) deformed Beisert-Roiban BAE

Weak coupling side

- (ex) su(2) Konishi
- Perturbative gauge theory computation

[Fiamberti, Santambrogio, Sieg, Zanon (2008)]

$$\Delta = \overbrace{4 + D_1 g^2 + D_2 g^4 + D_3 g^6 + D_4 g^8}^{\text{Asymptotic BAE}} + \delta\Delta$$

$$\begin{aligned}
D_1 &= 6(1+\delta), \quad D_2 = -\frac{3}{\delta} - 15 - 21\delta - 9\delta^2 \\
D_3 &= -\frac{3}{4\delta^3} + \frac{153}{4\delta} + 114 + \frac{495}{4}\delta + 54\delta^2 + \frac{27}{4}\delta^3 \\
D_4 &= \frac{3\,(1+\delta)^4}{8\delta^5(1+3\delta)^2}(-1 - 2\delta + 49\delta^2 + 84\delta^3 \\
&- 1359\delta^4 - 5562\delta^5 - 2673\delta^6 + 1944\delta^7) \\
&+ \left(-\frac{9}{\delta} + 27 + 54\delta - 90\delta^2 - 189\delta^3 - 81\delta^4\right)\zeta(3)
\end{aligned}$$

$$\delta \equiv \frac{\sqrt{5 + 4\cos(4\pi\beta)}}{3}$$

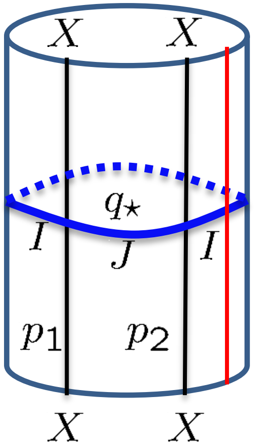
$$\delta\Delta = g^8 \underbrace{\left[-54(1+\delta)^3(-5+3\delta)\zeta(3) - 360(1+\delta)^2\zeta(5) + \frac{81\,(1-3\,\delta)^2\,(1+\delta)^4}{(1+3\,\delta)^2} \right]}_{\text{Finite-size effect}}$$

Results from Luscher formula

[CA, Bajnok, Bombardelli, Nepomechie (2010b)]

$$\delta\Delta = -\sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{4^L g^{2L}}{(Q^2+q^2)^L} \sum_{I,J} (-1)^{F_I} \left[M_I^I \; \tilde{\mathcal{S}}^{(Q1)}(q_\star(q), p_1)_{IX}^{JX} \cdot \tilde{\mathcal{S}}^{(Q1)}(q_\star(q), p_2)_{JX}^{IX} \right]$$

$$\sum_Q \sum_{I,J}$$



$$\delta\Delta = g^8 \left[-54(1+\delta)^3(-5+3\delta)\zeta(3) - 360(1+\delta)^2\zeta(5) + \frac{81\,(1-3\,\delta)^2\,(1+\delta)^4}{(1+3\,\delta)^2} \right]$$

Dual string theory

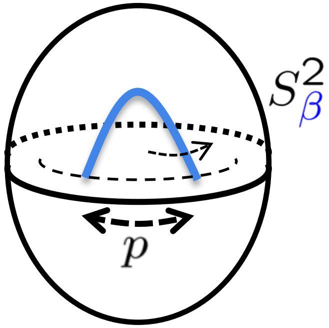
- Classical string theory on $AdS_5 \times S^5_\beta$
  - TsT transformation [Lunin, Maldacena (2005)]
  - Classical integrability [Frolov (2005)]
  - Antisymmetric tensor field $B_{\mu\nu}$
  - Classical string solutions can be constructed

Controversy over classical string computation

- Giant magnon with finite-size J in the Lunin-Maldacena

[Bykov, Frolov (2008)]

$$E - J \approx 2g \sin\frac{p}{2} \underbrace{- 8g \sin^3\frac{p}{2} \cdot \cos\Phi \cdot \exp\left[-\left(\frac{J}{g\sin\frac{p}{2}} + 2\right)\right]}_{\text{Finite-size effect}} + \dots$$



$$\Phi = \frac{2\pi(n_2 - \beta J)}{2^{3/2}\cos^3\frac{p}{4}}$$

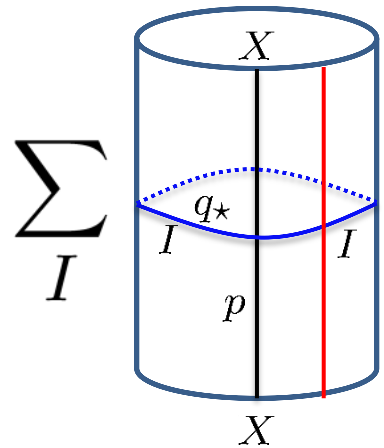
- Reanalysis of the same computation [CA, Bozhilov (2010)]

$$\Phi = 2\pi(n_2 - \beta J)$$

Luscher formula

- F-term with twisted S-matrix and BC

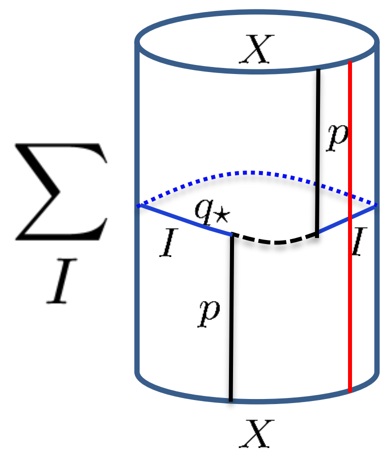
$$\delta\Delta = -\int_{-\infty}^{\infty}\frac{dq}{2\pi}\left(1-\frac{\epsilon'(p)}{\epsilon'(q_\star)}\right)e^{-iq_\star(q)J}\sum_I(-1)^{F_I}\left[M_I^I\ \tilde{\mathcal{S}}^{(11)}(q_\star(q),p)_{IX}^{IX}\right]$$



Checked with algebraic curve computation

- A leading classical contribution is obtained by a pole of S-matrix

$$\begin{aligned}\delta\Delta &= -i\left(1-\frac{\epsilon'(p)}{\epsilon'(q_\star(\bar{q}))}\right)e^{-iq_\star(\bar{q})J}\sum_I(-1)^{F_I}\,\underset{q=\bar{q}}{\mathrm{Res}}\left[M_I^I\ \tilde{\mathcal{S}}^{(11)}(q_\star(q),p)_{IX}^{IX}\right]\\ &= -8g\sin^3\frac{p}{2}\ \cos(2\pi\beta J)\ \exp\left[-\left(\frac{J}{g\sin\frac{p}{2}}+2\right)\right]\end{aligned}$$



Conclusion

- Resolved a controversy over classical string solution

- Beta-deformed SYM is all-loop integrable

- Various open problems

THANK YOU!