Solving AdS/CFT Y- and T-system.

Sébastien Leurent
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[arXiv:1110.0562] N. Gromov, V. Kazakov, SL & D. Volin

[arXiv:1007.1770] V. Kazakov & SL

[arXiv:1010.2720] N. Gromov, V.Kazakov, SL & Z.Tsuboi

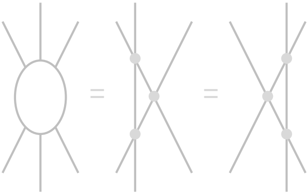
Nordita, 7 February 2012

FiNLIE for AdS/CFT Y- and T-system.

[arXiv:1110.0562] N. Gromov, V. Kazakov, SL & D. Volin

Outline

Bethe Ansatz



Several spin chains and 2-D QFT are integrable,
i.e. there exists a Bethe Ansatz :

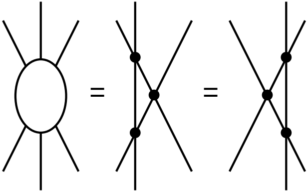
- Expression of the wave function (planar waves + phase shifts)
- Quantization condition ($e^{iLp_i} = \prod_{j\neq i} S_{j,i}$)

⇝ Energy $E = \sum_i E_i$

⇒ Yang Baxter equation (factorization into 2-points interactions)

More details

Bethe Ansatz



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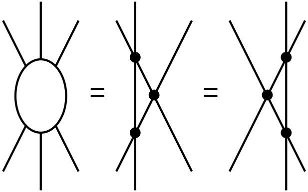

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AdS/CFT

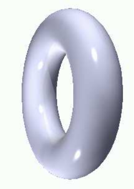
For AdS/CFT, integrability comes from a mapping
Long single-trace operators ↔ Spin-chain states

[Beisert Eden Staudacher 07]

Outline

"Thermodynamic Bethe Ansatz"

Finite-size vacuum energy from a "Double Wick rotation"



Periodic space (size L), infinite time-period $R \to \infty$: Path integral

$$Z \simeq e^{-RE_0(L)} \qquad (R \to \infty)$$

Periodic space of size $R \gg 1$ and time period L:

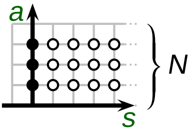
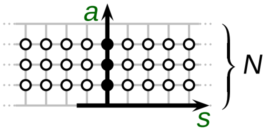
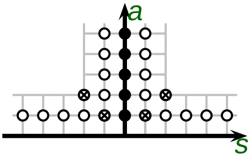
$$\Rightarrow \text{free energy} \quad f(L) = E_0(L)$$

To compute the free energy at finite temperature, introduce density of each type of particles as a function of rapidity.

Classification of Bound states
Y-functions

up to a change of variables

$Y_{(a,s)}(u)$= density of particles of type (a,s) and rapidity $u \in \mathbb{C}$.

$SU(N)$ Gross-Neveu	$SU(N)\times SU(N)$ Principal Chiral Model	AdS_5/CFT_4
		

- $Y_{a,s}{}^{+}\, Y_{a,s}{}^{-} = \frac{1+Y_{a,s+1}}{1+(Y_{a+1,s})^{-1}}\,\frac{1+Y_{a,s-1}}{1+(Y_{a-1,s})^{-1}}$ "Y-system equation"

$$Y_{a,s}{}^{\pm} \equiv Y_{a,s}(u \pm i/2)$$

[Gromov Kazakov Vieira 09] [Bombardelli Fioravanti Tateo 09][Arutyunov Frolov 09]

$$E = -\sum_{a,s} \int E_{a,s}(u)\ \log\left(1 + Y_{a,s}(u)\right)\ du$$

- + analyticity condition

▸ Integral form : TBA equation

Y-systems
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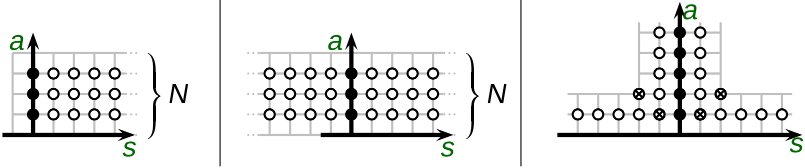

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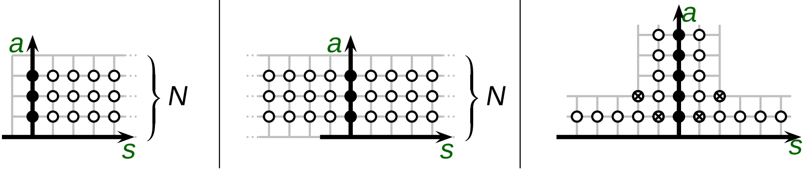

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Y-system ↔ Hirota equation

(also called T-system)

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Hirota

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  [Gromov Kazakov Tsuboi 10]
  [Benichou 11] ▸ More details
- Finite parameterization
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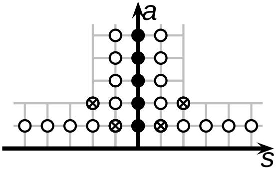

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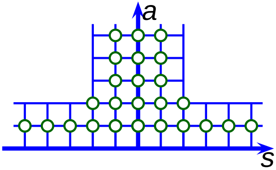

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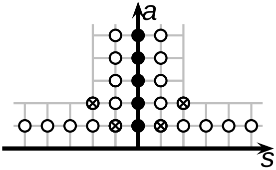

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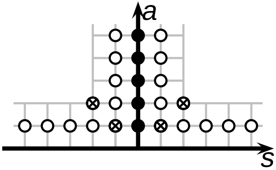

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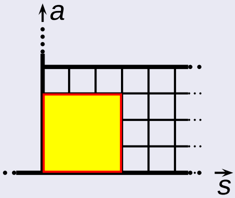
Lattice's meaning

Each point in the (a, s)-lattice is associated to a representation of the symmetry group
(to the right : SU(N) example)



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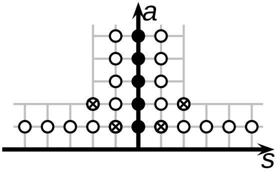

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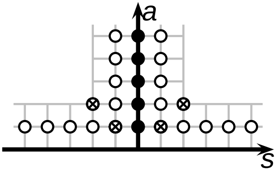

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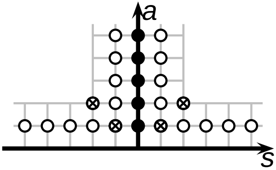


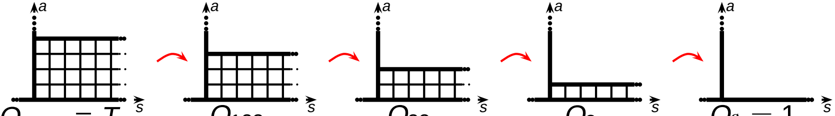
Outline

Q-functions and Hasse diagram

example of the SU(4) (a,s)-lattice

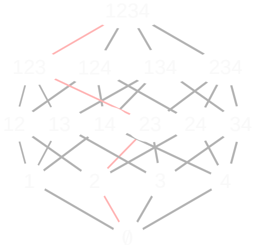
- “Undressing” procedure (Bäcklund Transformation)

su(4) ⊃ su(3) ⊃ su(2) ⊃ su(1)



$Q_{1234} \equiv T_{0,0}$ Q_{123} Q_{23} Q_{2} $Q_{\emptyset} = 1$

- N! different “nesting paths” define 2^N Q-functions



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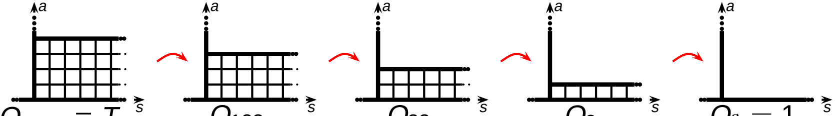

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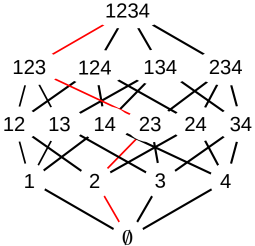

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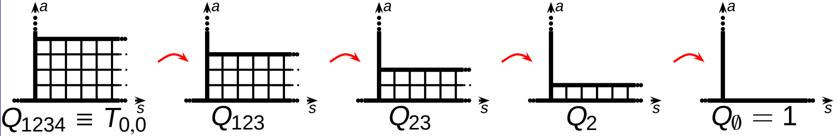

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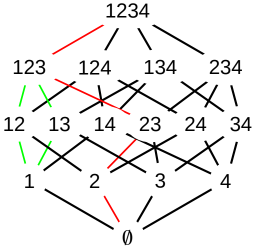

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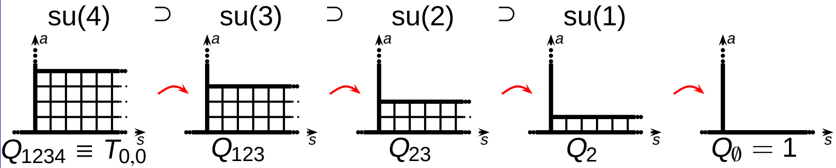

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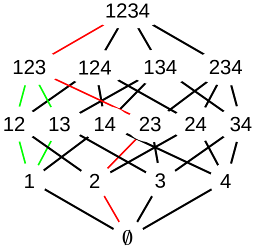

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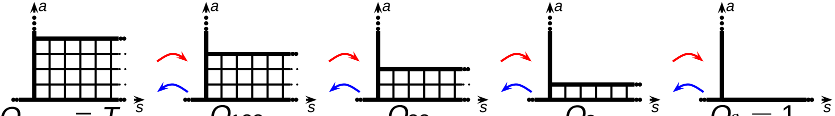

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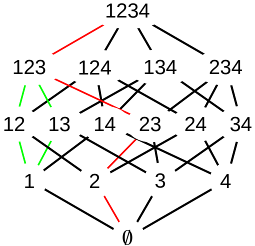

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General solution of Hirota equation

example of the $SU(3)$ (a,s)-lattice

- The general solution of Hirota equation for $SU(3)$ reads (up to a gauge transformation)

$$T_{3,s} = Q_{123}^{[+s]} Q_{\emptyset}^{[-s]} = \begin{vmatrix} Q_1^{[+s+2]} & Q_2^{[+s+2]} & Q_3^{[+s+2]} \\ Q_1^{[+s]} & Q_2^{[+s]} & Q_3^{[+s]} \\ Q_1^{[+s-2]} & Q_2^{[+s-2]} & Q_3^{[+s-2]} \end{vmatrix}$$

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a

s

[Krichever Lipan Wiegmann Zabrodin 96]
[Bazhanov Frassek Lukowski Meneghelli Staudacher 10]
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- Notation :

$$Q_{\bullet}^{[+s]} \wedge Q_{\bullet\bullet}^{[-s]} \equiv Q_1^{[+s]} Q_{23}^{[-s]} - Q_2^{[+s]} Q_{13}^{[-s]} + Q_3^{[+s]} Q_{12}^{[-s]}$$

General solution of Hirota equation

example of the $SU(3)$ (a,s)-lattice

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$$T_{a,0} = Q_{1234}^{[+a]} Q_{\emptyset}^{[-a]} \qquad a \geq 0$$

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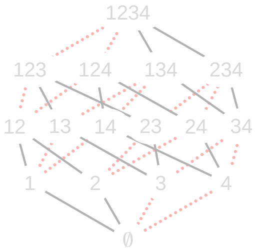
$$T_{2,s} = Q_{12}^{[+s]} Q_{34}^{[-s]} \qquad s \geq 2$$

$$T_{1,s} = Q_{\bullet}^{[+s]} \wedge Q_{\bullet 34}^{[-s]} \qquad s \geq 1$$

$$T_{0,s} = Q_{\emptyset}^{[+s]} Q_{1234}^{[-s]} \qquad s \geq 0$$



[Kazakov Sorin Zabrodin 08] [Kazakov S.L. Tsuboi 10]
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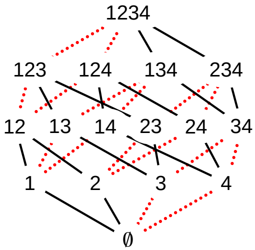

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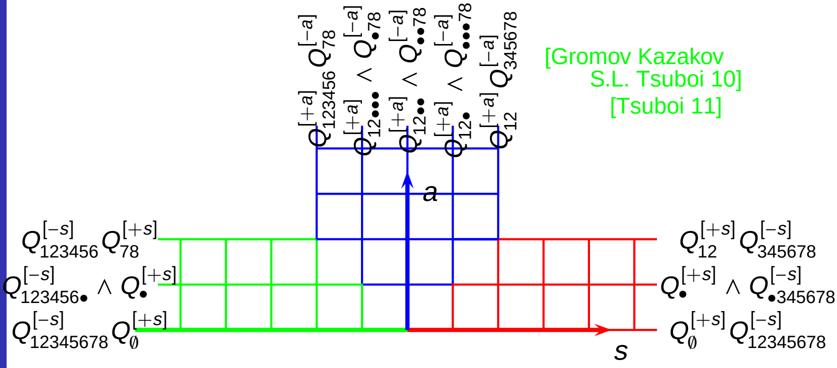
General solution of Hirota equation for AdS/CFT.



[Gromov Kazakov S.L. Tsuboi 10]
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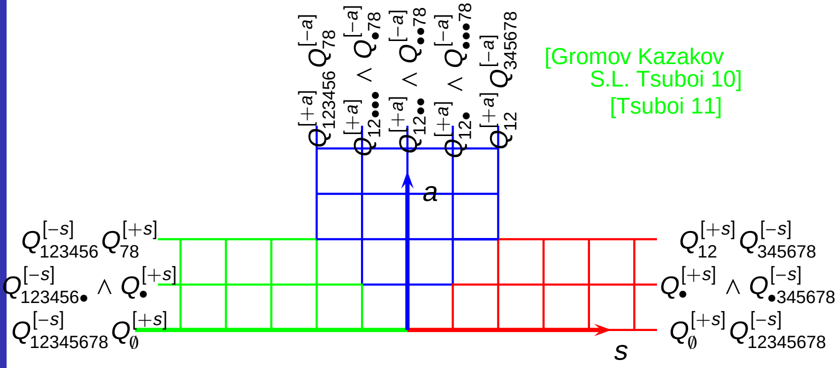
⇒ For several Y-systems, all Y- and T-functions are expressed in terms of a finite number of functions of the spectral parameter $u \in \mathbb{C}$.

- How should Q-functions themselves be expressed ?

General solution of Hirota equation $\rightsquigarrow$ FiNLIE.



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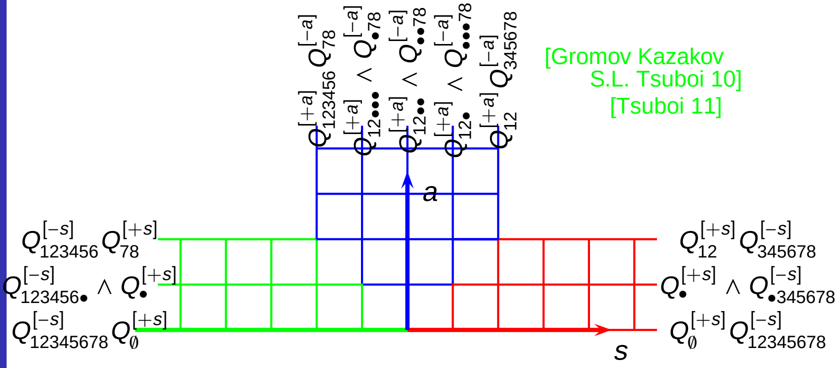

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Outline

Analyticity constraints on Q-functions

An Example in AdS/CFT Case

- Y-functions are meromorphic inside some strips : In the case of AdS/CFT, one can show that

for instance, $Y_{1,s}$ is meromorphic if $|\mathrm{Im}(u)| < \frac{s-1}{2}$

$\rightsquigarrow T_{1,s}$ is holomorphic if $|\mathrm{Im}(u)| < \frac{s}{2}$

when $s \geq 1$,

$$T_{1,s} = \begin{vmatrix} Q_1^{[+s]} & Q_2^{[+s]} \\ Q_{1345678}^{[-s]} & Q_{2345678}^{[-s]} \end{vmatrix} = \begin{vmatrix} 1 & Q_2^{[+s]} \\ 1 & Q_{2345678}^{[-s]} \end{vmatrix}$$

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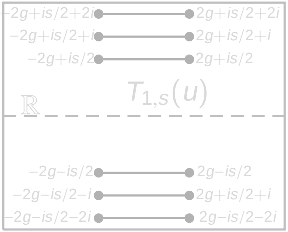

- up to a gauge transformation

& under reality assumption

- Q-functions are analytic on whole half planes ($\mathbb{R} + i\,\mathbb{R}^+$)

Analyticity constraints on Q-functions

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$$\frac{1}{-1+\frac{T_{1,s}^{+}T_{1,s}^{-}}{T_{1,s+1}T_{1,s-1}}} = Y_{1,s} \text{ is meromorphic if } |\mathrm{Im}(u)| < \frac{s-1}{2}$$

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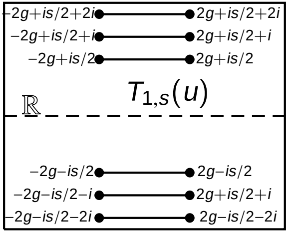

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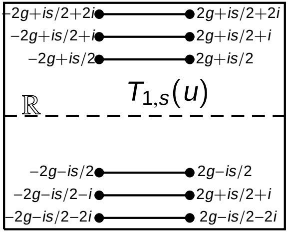

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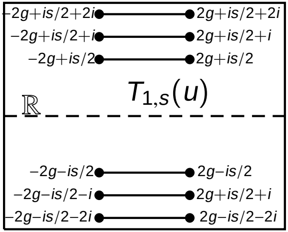

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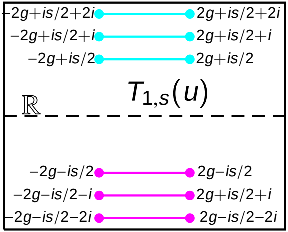

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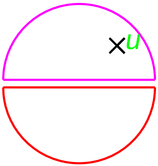
Riemann-Hilbert Problem

General statement

If $F(u)$ and $G(u)$ are analytic when $\mathrm{Im}(u) \geq 0$ (resp $\mathrm{Im}(u) \leq 0$) and $F(u), G(u) \xrightarrow[|u|\to\infty]{} 0$ at least as a power law,

then $$\frac{1}{2i\pi}\int_{-\infty}^{\infty}\frac{F(v)-G(v)}{v-u}\mathrm{d}v = \begin{cases} F(u) & \text{if } \mathrm{Im}(u) > 0 \\ G(u) & \text{if } \mathrm{Im}(u) < 0 \end{cases}$$



Example : if $Q(u)$ is analytic on the upper-half-plane, and $Q(u) = \underbrace{au^2 + bu + c}_{P(u)} + O(1/u)$ at $u \to \infty$,

then
$Q(u) = P(u) + \frac{1}{2i\pi}\int_{\mathbb{R}}\frac{\rho(v)}{v-u}\mathrm{d}v$
where $\rho = 2\mathrm{Re}(Q - P)$

$(F(u) = Q(u) - P(u),\ G = -\overline{Q(\bar{u})} + \overline{P(\bar{u})})$

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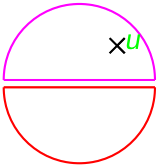

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Riemann-Hilbert Problem

$\rightsquigarrow$FiNLIE

[Gromov Kazakov Vieira 08] [Kazakov S.L. 10][Gromov Kazakov S.L. Volin 11]

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FiNLIE-equations

Appropriate choices of F and G allow to derive non-trivial integral equations from analyticity constraints.

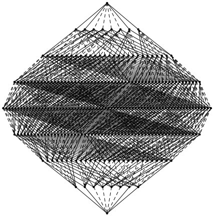
For AdS/CFT, these equations can be shown to be equivalent to the TBA-equations.

For $O(4)$ Principal Chiral Model, they are also equivalent to DdV-equations.

Outline

≪Splitting≫ the (a,s)-lattice

and the Hasse diagram



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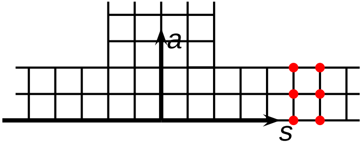
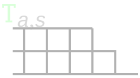
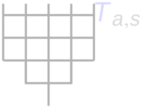
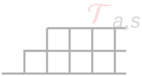
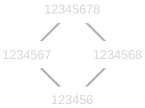


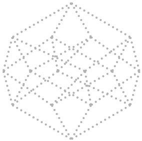
Split

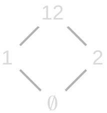












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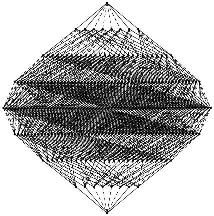

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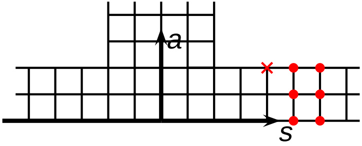

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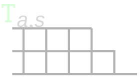
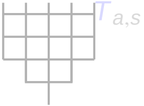
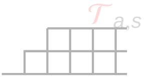
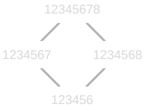
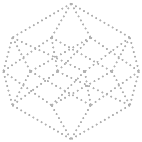
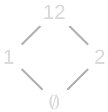












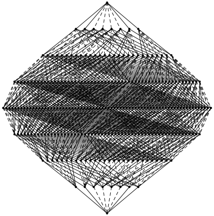
FiNLIE for AdS/CFT spectrum.

S. Leurent

TBA approach
- Bethe Ansatz
- TBA
- Y vs T

FiNLIE
- Q-functions
- Wronskian solution of Hirota equation
- Analyticity of Q-functions

Symmetries
- ≪Splitting≫ the (a,s)-lattice
- New symmetries
- FiNLIE



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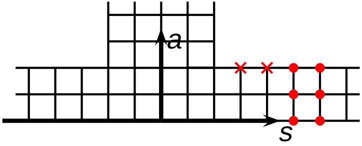

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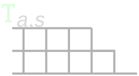

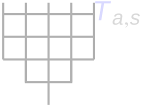

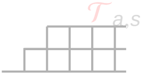

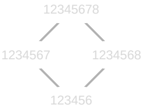

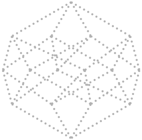

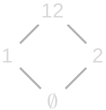

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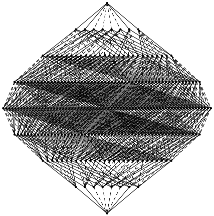

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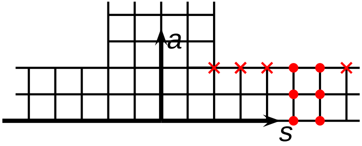

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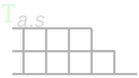

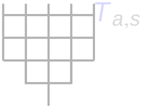

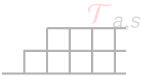

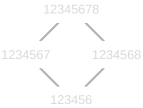

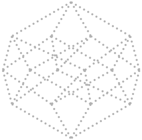

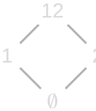

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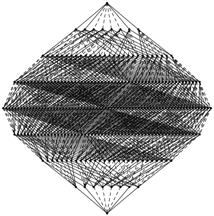

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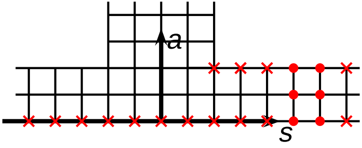

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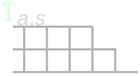

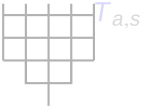

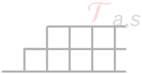

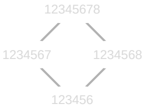

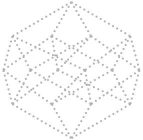

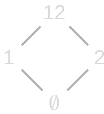

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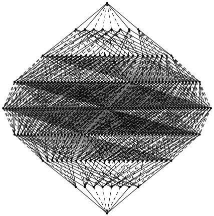

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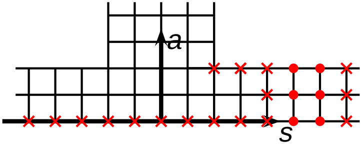

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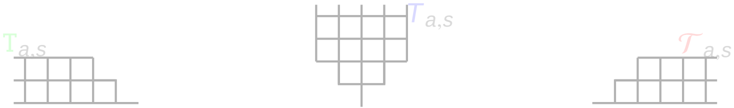

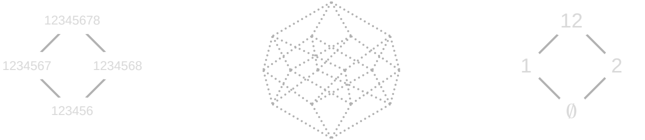

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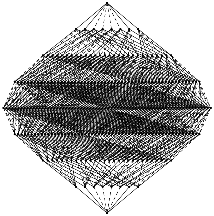

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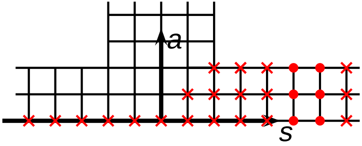

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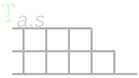

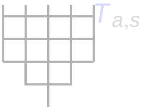

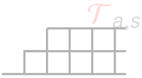

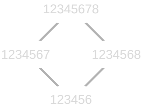

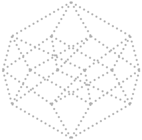

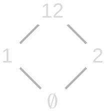

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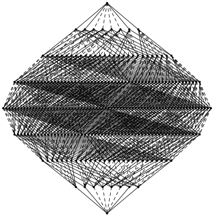

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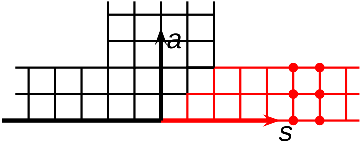

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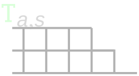

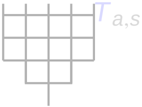

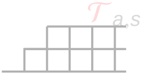

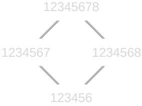

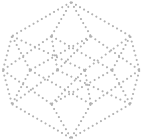

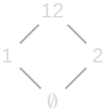

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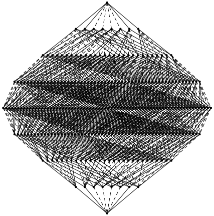

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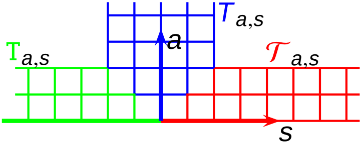

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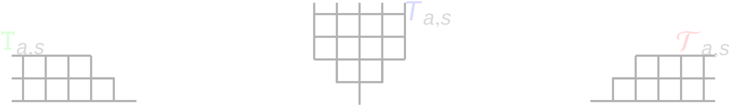

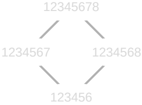

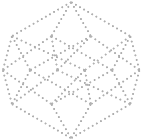

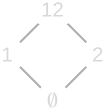

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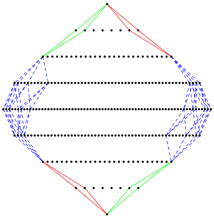

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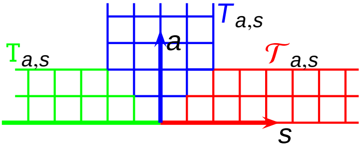

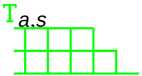

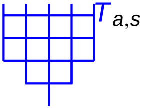

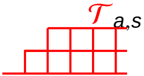

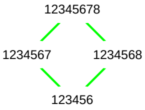

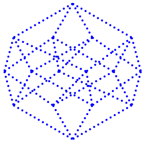

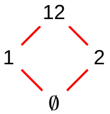

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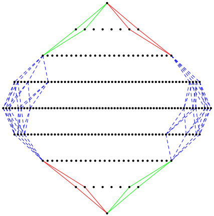

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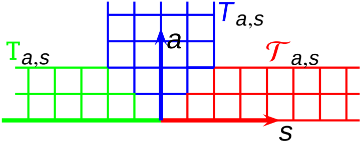

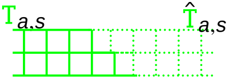

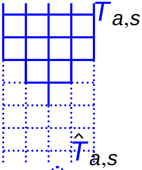

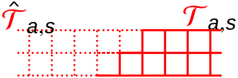

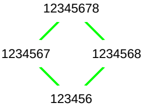

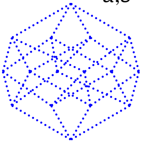

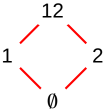

Outline

Symmetries ⇝ Classical limit

In the classical limit, $T_{a,s}(u) = \chi_{a,s}\left(\Omega(u)\right)$ where $\Omega \in U(2,2|4)$.
characters in rectangular irreps [Gromov Kazakov Tsuboi 10]

- Actually, the $PSU(2,2|4)$ symmetry imposes more constraints :
  - $\mathrm{sdet} = 1$
  - invariance under a $\mathbb{Z}_4$ transformation

  That gives extra symmetries of the characters (generalizing to symmetries of T-functions at finite size).

$\mathbb{Z}_4$ symmetry of the classical limit

$$\Omega = \hat{C}^{-1}(\Omega^{-1})^T\hat{C} \qquad \text{(or } \{\lambda_i\} = \{1/\lambda_i\} \text{ for } \Omega\text{'s eigenvalues)}$$

[Bena Polchinksi Roiban]

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$$\mathcal{T}_{1,s} = -\hat{\mathcal{T}}_{1,-s} \qquad \mathrm{T}_{1,s} = -\hat{\mathrm{T}}_{1,-s}$$

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characters in rectangular irreps [Gromov Kazakov Tsuboi 10]

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  - $\mathrm{sdet} = 1$
  - invariance under a $\mathbb{Z}_4$ transformation

  That gives extra symmetries of the characters (generalizing to symmetries of T-functions at finite size).

$\mathbb{Z}_4$ symmetry of the classical limit

$$\Omega = \hat{C}^{-1}(\Omega^{-1})^T \hat{C} \qquad \text{(or } \{\lambda_i\} = \{1/\lambda_i\} \text{ for } \Omega\text{'s eigenvalues)}$$

[Bena Polchinksi Roiban]

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$$\mathcal{T}_{1,s} = -\hat{\mathcal{T}}_{1,-s} \qquad \mathbb{T}_{1,s} = -\hat{\mathbb{T}}_{1,-s}$$

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$$\hat{\mathcal{T}}_{1,0} = 0 \Rightarrow \quad q = -\bar{q}$$

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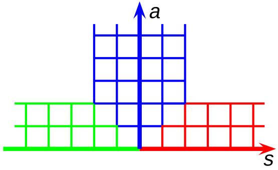
$$q(u) = -iu + \frac{1}{2i\pi}\int_{-2g}^{2g}\frac{\rho(v)}{v-u}$$



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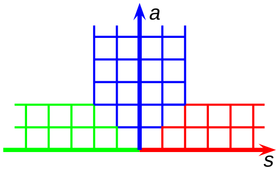

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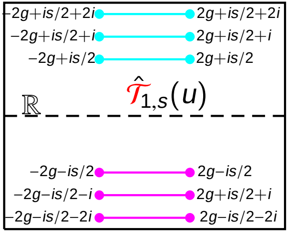

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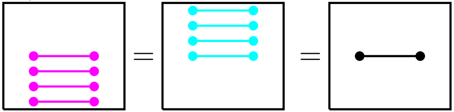

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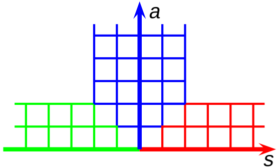

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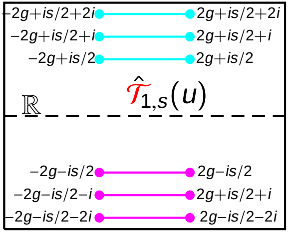

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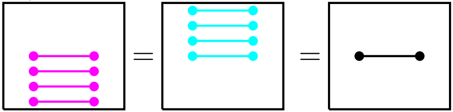

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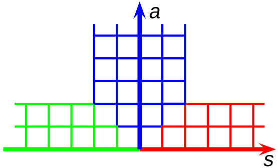

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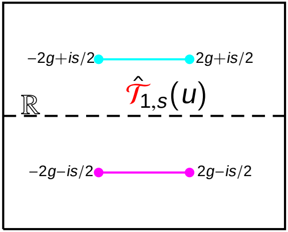
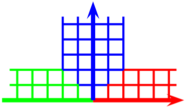


Outline

AdS/CFT FiNLIE

$$\begin{aligned}
T_{a,+1} &= q_1^{[+a]}\bar{q}_2^{[-a]} + q_2^{[+a]}\bar{q}_1^{[-a]} + q_3^{[+a]}\bar{q}_4^{[-a]} + q_4^{[+a]}\bar{q}_3^{[-a]} \,, \\
T_{a,0} &= q_{12}^{[+a]}\bar{q}_{12}^{[-a]} + q_{34}^{[+a]}\bar{q}_{34}^{[-a]} - q_{14}^{[+a]}\bar{q}_{14}^{[-a]} \\
&\quad - q_{23}^{[+a]}\bar{q}_{23}^{[-a]} - q_{13}^{[+a]}\bar{q}_{24}^{[-a]} - q_{24}^{[+a]}\bar{q}_{13}^{[-a]} \,, \\
q_\emptyset q_{ij} &= q_i^+ q_j^- - q_j^+ q_i^- \,, \\
q_{ijk} q_i &= q_{ij}^+ q_{ik}^- - q_{ik}^+ q_{ij}^- \,.
\end{aligned}$$

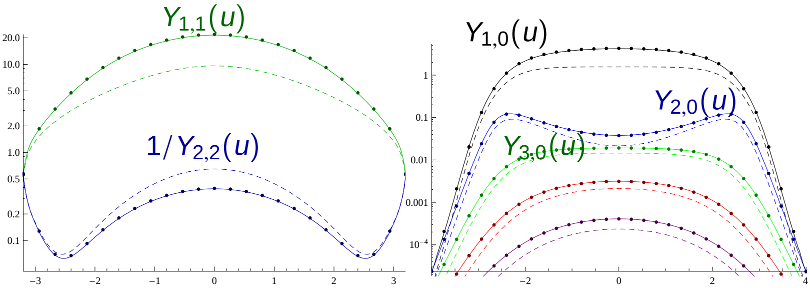


$$Y_{1,1} = -\sqrt{\frac{R^{(+)}}{R^{(-)}}\frac{B^{(-)}}{R^{(+)}}\frac{\mathcal{T}_{1,2}}{T_{2,1}}\left(\frac{T_{1,0}}{Q^+Q^-}\right)^{1+\check{*}\mathcal{Z}}\left(\frac{Q^2}{T_{0,0}}\right)^{*\frac{1}{2}(\mathcal{Z}_1+\mathcal{K}_1)}\left(\frac{T_{1,1}}{\mathcal{T}_{1,1}}\right)^{*\!\!\not\mathcal{K}_1}} \,.$$

$$U^2 = \frac{\Lambda^2 T_{00}^-}{\hat{x}^{L-2} Y_{1,1} Y_{2,2} T_{1,0}}\left(\frac{Y_{1,1}Y_{2,2}-1}{\rho/\mathcal{F}^+}\right)^{\hat{*}2\mathcal{Z}}\left(\frac{T_{2,1}\mathcal{T}_{1,1}^-}{\hat{T}_{1,1}^-\mathcal{T}_{1,2}Y_{2,2}}\right)^{*2\Psi}$$

Numeric implementation of FiNLIE

Numerical Y-functions for Konishi state ($g = 1.6$):
Dots are obtained from FiNLIE and lines from standard Y-system iterations. The asymptotic expression is dashed, .



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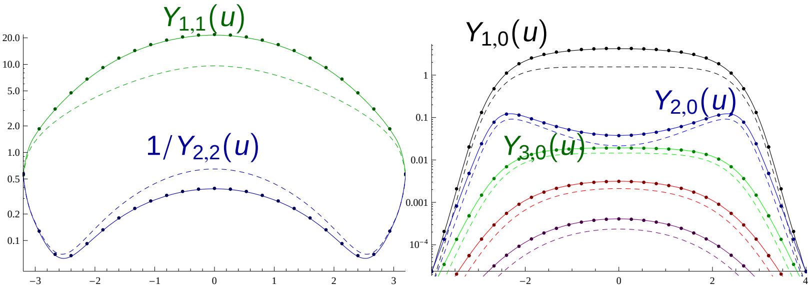

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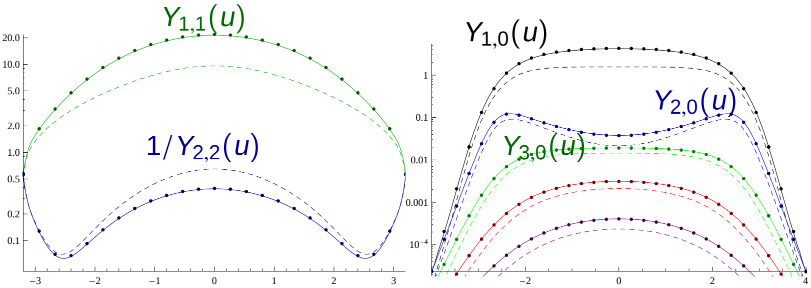

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Conclusion

- A better understanding of Y-system
  - analytic properties
  - new symmetries
  - Finite set of NLIEs
  - $\partial \log T_{0,0} \xrightarrow{u\to\infty} \frac{2E}{u}$
  - Exact Bethe equations arise as absence of poles of T-functions

- to be continued
  - currently restricted to symmetric sl_2 "sector" states
  - $\begin{cases} \text{numeric efficiency} \\ \text{best FiNLIE formulation} \end{cases}$ are to be studied
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Thank you !

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Bethe Ansatz

Quantization condition in a periodic box of size L

Back

- For one particle, the wave function is periodic iff $e^{i\ L\ p} = 1$
- For two particles, the Bethe Ansatz is

$$\psi(x_1, x_2) = \begin{cases} e^{i(p_1 x_1 + p_2 x_2)} + S(p_1, p_2) \times e^{i(p_2 x_1 + p_1 x_2)} & \text{if } x_1 \lessapprox x_2 \\ S(p_1, p_2) \times e^{i(p_1 x_1 + p_2 x_2)} + e^{i(p_1 x_1 + p_2 x_2)} & \text{if } x_1 \gtrapprox x_2 \end{cases}$$

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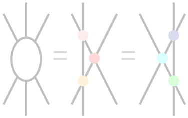

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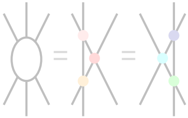

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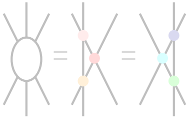

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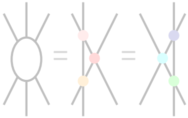

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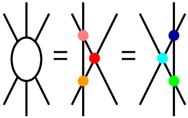

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Main conditions

- many conserved charges
- unidimensional space (eg spin chain)
- $L \gg$ interaction range

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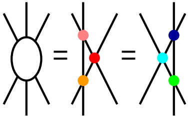

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Examples

This ansatz describes

- several 2-dimensional field-theories such as the Principal Chiral Model
- Spin chains under some condition on the form the Hamiltonian.

Spectrum of an integrable theory

- Bethe equation : $\forall i, e^{iLp_i} = \prod_{j \neq i} S_{j,i}$
- $E = \sum_i E_i$

For relativistic models, $p_i = m_a \sinh \theta_i$, $E_i = m_a \cosh \theta_i$.

- The spectrum is identified by finding the rapidities (θ_i) of a number of particles (solution of Bethe equation), and then deducing energy.

- This works when the periodic “box” is big

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For relativistic models, $p_i = m_a \sinh \theta_i$, $E_i = m_a \cosh \theta_i$.

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Spectrum of an integrable theory

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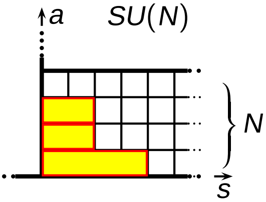
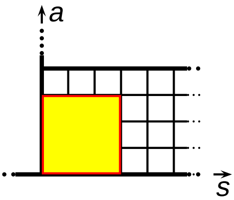
Hirota equation and characters of the symmetry group

Back

- mapping : young-tableau $\leftrightarrow$ representation of the symmetry group



- Lattice node $\leftrightarrow$ "rectangular" representations



Hirota equation and characters of the symmetry group

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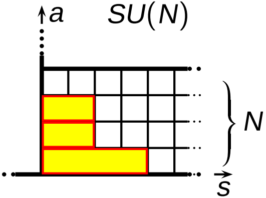
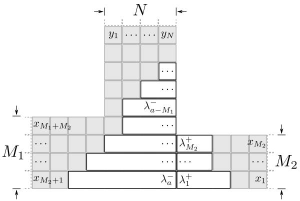


$SU(M_1, M_2|N)$



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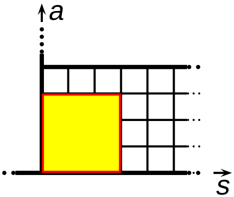

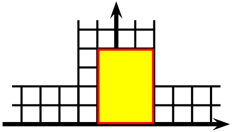

Hirota equation and characters of the symmetry group

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Characters

The characters associated to rectangular representations satisfy $\chi_{a,s}{}^2 = \chi_{a,s+1}\chi_{a,s-1} + \chi_{a+1,s}\chi_{a-1,s}$

The Hirota equation $T_{a,s}{}^+ T_{a,s}{}^- = T_{a,s+1}T_{a,s-1} + T_{a+1,s}T_{a-1,s}$ is generalisation of this relation.

[Benichou 11]

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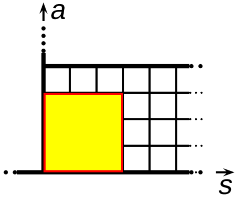

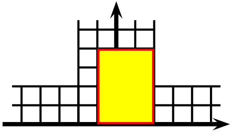

Thermodynamic Bethe Ansatz

$\rightsquigarrow$ Equations of the form

$$Y_{a,s}(u) = -L\ E_{a,s}(u) + \sum_{a',s'} K_{a,s}^{(a',s')} \star \log\left(1 + Y_{a',s'}(u)^{\pm 1}\right)$$
$$+\langle \textit{Source Terms} \rangle$$

- Vacuum energy

$$E_0 = -\sum_{a,s} \int E_{a,s}(u)\ \log\left(1 + Y_{a,s}(u)\right)\ \mathrm{d}u$$

Back to the presentation

- Extra assumption : Excited states obey the same equations.
  Each state corresponds to a different solution of Y-system, characterized by its zeroes and poles

- AdS/CFT case : both $E_{a,s}$ and $K_{a,s}^{(a',s')}$ have several square-root

$\Rightarrow$ TBA-equations contain analyticity information under a form which is hard to decode (infinite sums)

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