Conformal blocks in 2d CFT, the Calogero-Sutherland model and the AGT conjecture

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Conformal blocks of some 2d CFT

FQHE states with non-abelian statistics

[Estienne, Bernevig, Santachiara, 10]

$$g \rightarrow -g$$
$$b \rightarrow ib$$

AGT conjecture

(Nekrasov's partition function ~ Liouville conformal blocks)

[Alba, Fateev, Litvinov, Tarnopolsky 10]

Integrable structure of the CS model

[Alday, Gaiotto, Tachikawa, 09]

AGT conjecture

Link between a supersymetric 4d theory and a 2d CFT
(and a 1d integrable model)

n-point correlation function of Liouville theory ⟷ Nekrasov's instanton partition function of a gauge theory with gauge group $su(2)_1 \otimes \ldots \otimes su(2)_{n-3}$

$$Z^{u(2)} = Z^{u(1)} Z^{su(2)}$$

$$\langle V_{\Delta_{-1}}(\infty)\, V_{\Delta_0}(1)\, V_{\Delta_1}(q_1) \ldots V_{\Delta_{n-3}}(q_1 \ldots q_{n-3})\, V_{\Delta_{n-2}}(0)\rangle = c \prod_i f(\Delta_i) \int \prod_i a_i^2 da_i\, |Z(q|\Delta, \widetilde{\Delta})|^2$$

[Nekrasov, 02]

$$\alpha_i = Q/2 + a_i$$

$$Q = b + \frac{1}{b}$$

$$\Delta_i = m_i(Q - m_i), \quad \widetilde{\Delta}_i = \alpha_i(Q - \alpha_i)$$

AGT dictionary:

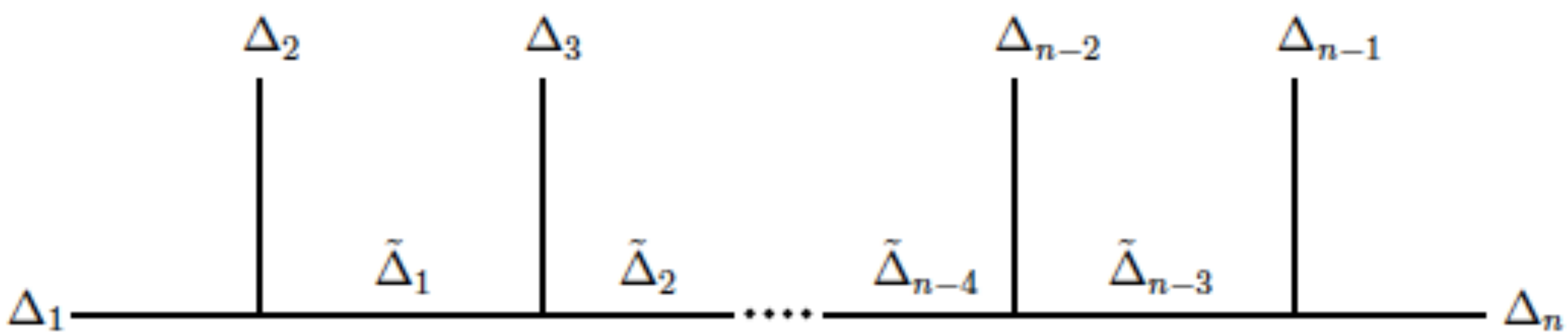
[Alday, Gaiotto, Tachikawa, 09]

Gauge theory	Liouville theory
Deformation parameters ϵ_1, ϵ_2	Liouville parameters $\epsilon_1 : \epsilon_2 = b : 1/b$ $c = 1 + 6Q^2,\ Q = b + 1/b$
four free hypermultiplets	a three-punctured sphere
Mass parameter m associated to an $SU(2)$ flavor	Insertion of a Liouville exponential $e^{2m\phi}$
one $SU(2)$ gauge group with UV coupling τ	a thin neck (or channel) with sewing parameter $q = \exp(2\pi i \tau)$
Vacuum expectation value a of an $SU(2)$ gauge group	Primary $e^{2\alpha\phi}$ for the channel, $\alpha = Q/2 + a$
Instanton part of Z	Conformal blocks
One-loop part of Z	Product of DOZZ factors
Integral of $\lvert Z_{\text{full}}^2 \rvert$	Liouville correlator

$$\langle V_{\Delta_{-1}}(\infty)\ V_{\Delta_0}(1)\ V_{\Delta_1}(q_1)\ldots V_{\Delta_{n-3}}(q_1\ldots q_{n-3})\ V_{\Delta_{n-2}}(0)\rangle = c\prod_i f(\Delta_i)\int\prod_i a_i^2 da_i\ |Z(q|\Delta,\widetilde{\Delta})|^2$$

CFT side: computing the conformal blocks

$$\langle V_{\Delta_{-1}}(\infty)\, V_{\Delta_0}(1)\, V_{\Delta_1}(q_1)\ldots V_{\Delta_{n-3}}(q_1\ldots q_{n-3})\, V_{\Delta_{n-2}}(0)\rangle = c\prod_i f(\Delta_i)\int\prod_i a_i^2 da_i\, |Z(q|\Delta,\widetilde{\Delta})|^2$$



[Belavin, Polyakov, Zamolodchikov, 84]

- insert complete set of states $\sum_{\mu_i} |\mu_i\rangle\langle\mu_i|$ in the intermediate channels

$$|\mu_i\rangle \sim L_{-n_1}\ldots L_{-n_k}|\widetilde{\Delta}_i\rangle$$

- compute the matrix elements elements $\frac{\langle\nu_i|V_{\Delta_{i-1}}(1)|\mu_{i+1}\rangle}{\langle\nu_i|\mu_{i+1}\rangle}$

- compare with the gauge theory result **[Nekrasov, 02]**

⟶ proof of the AGT conjecture **[Alba, Fateev, Litvinov, Tarnoplosky, 10]**

2d CFT and the Fractional Quantum Hall Effect

holomorphic correlators ⟷ wave functions for the FQHE

e.g. u(1) CFT correlators ⟷ Laughlin wave function with filling fraction 1/m

free boson (c=1): $\langle \phi(z)\phi(w)\rangle = -\ln(z-w)$

u(1) current: $J(z) = i\partial\phi(z)$

primary fields: vertex operators $V_\beta =: e^{i\beta\phi(z)}:$

"electron" operator:

$V_e(z) =: e^{i\sqrt{m}\phi(z)}:$

"quasi-hole" operator:

$V_q(w) =: e^{i/\sqrt{m}\phi(w)}:$

$$\langle V_e(z_1)\dots V_e(z_N)\, V_q(w_1)\dots V_q(w_M)\rangle \sim \prod_{i<j} z_{ij}^m \prod_{i<j} w_{ij}^{1/m} \prod_{i,j}(z_i - w_j)$$

2d CFT and the Fractional Quantum Hall Effect

Ising CFT ⟷ wave function for Moore-Read state [Moore, Read, 91]

Z_k parafermions ⟷ wave functions for Read-Rezayi states

$\Psi = \Phi_{12}$ electron

$\sigma = \Phi_{21}$ quasihole

The electron eigenfunction is **monovalued**:

Moore-Read wavefunction $\sim$ $\prod_{i<j} z_{ij}^2 \, \langle \Psi(z_1) \ldots \Psi(z_N) \rangle = \prod_{i<j} z_{ij}^2 \; \mathrm{Pf} \left(\frac{1}{z_{ij}} \right)$

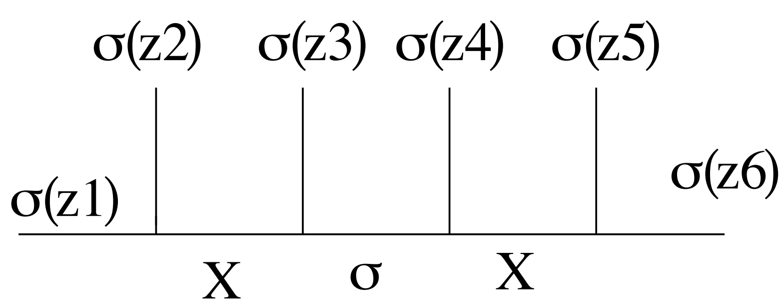
with **clustering properties** (it vanishes when a cluster of 3 particles come together):

eigenfunction of a three-body Hamiltonian $H = \sum_{i \neq j \neq k} \delta^{(2)}(x_i - x_j) \delta^{(2)}(x_j - x_k)$

2d CFT and the Fractional Quantum Hall Effect

quasihole wave-function: $\sim \quad \Psi(z)_a \equiv \prod_{i<j} z_{ij}^{3/8} \langle \sigma(z_1) \ldots \sigma(z_M) \rangle_a$ **[Nayak, Wilczek, 96]**

it is **multivalued**, with the $2^{M/2-1}$ conformal blocks corresponding to the different fusion channels



$X = \mathrm{I}$ or $X = \Psi$.

→ wave-function with non-abelian braiding properties

2d CFT and the Calogero-Sutherland model

Trigonometric CS model: set of N commuting Hamiltonians for N particles on a circle:

$$H_1^g = \mathcal{P} = \sum_{i=1}^{N} z_i \partial_i$$

$$H_2^g = H^g = \sum_{i=1}^{N} (z_i\partial_i)^2 - g(g-1) \sum_{i\neq j} \frac{z_i z_j}{z_{ij}^2} \qquad z_j = e^{2i\pi x_j/L}$$

$$H_3^g = \sum_{i=1}^{N} (z_i\partial_i)^3 + \frac{3}{2} g(1-g) \sum_{i\neq j} \frac{z_i z_j}{z_{ij}^2} (z_i\partial_i - z_j\partial_j).$$

...

Two different **boundary conditions** for the wave functions:

$\Psi^+(z) = \Delta^g(z) F^+(z)$ or $\Psi^-(z) = \Delta^{1-g}(z) F^-(z)$

$$\Delta^\gamma(z) = \prod_{i<j} (z_i - z_j)^\gamma$$

2d CFT and the Calogero-Sutherland model

Eigenfunctions with abelian monodromy:

$$\Psi_\lambda^+(z) = \Delta^g(z) J_\lambda^{1/g}(z) \qquad \Psi_\lambda^-(z) = \Delta^{1-g}(z) J_\lambda^{1/(1-g)}(z)$$

Jack polynomials: eigenfunctions of the Hamiltonian $\qquad \alpha^{-1} = g \quad \text{or} \quad 1-g$

$$\mathcal{H}^\alpha = \sum_{i=1}^N (z_i \partial_i)^2 + \frac{1}{\alpha} \sum_{i<j}^N \frac{z_i + z_j}{z_{ij}} (z_i \partial_i - z_j \partial_j)$$

$$\mathcal{E}_\lambda^\alpha = \sum_i^N \lambda_i \left[\lambda_i + \frac{1}{\alpha}(N+1-2i) \right]$$

characterized by **partitions** λ with λ_i integers: $\lambda_1 \geq ... \geq \lambda_N \geq 0$

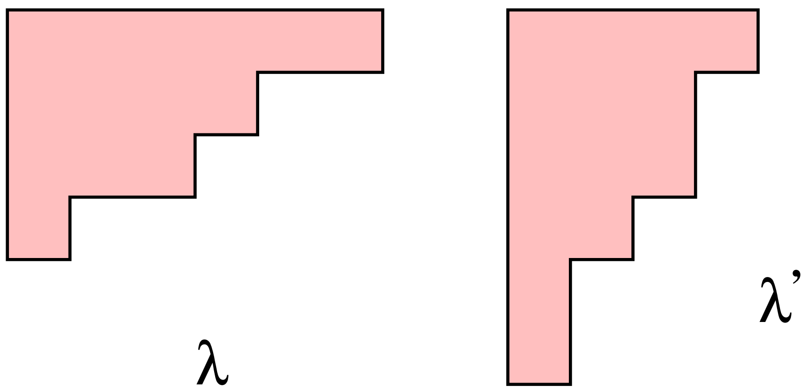
2d CFT and the Calogero-Sutherland model

Duality $g \to 1/g$: [Stanley 89; Macdonald 88; Gaudin 92]

$$\left[\mathcal{H}^{1/g} + g\,\mathcal{H}^{g} + C(N,M)\right] \prod_{i=1}^{N}\prod_{i=1}^{M}(1+z_i w_j) = 0$$

$$\prod_{i,j}(1+z_i w_j) = \sum_{\lambda} J_{\lambda}^{1/g}(z) J_{\lambda'}^{g}(w)$$

Dual partitions:



Apply this property to the state:

$$\langle V_e(z_1)\ldots V_e(z_N)\,V_q(w_1)\ldots V_q(w_M)\rangle \sim \prod_{i<j} z_{ij}^{m} \prod_{i<j} w_{ij}^{1/m} \prod_{i,j}(z_i - w_j)$$

States with non-abelian monodromy and Virasoro theories

Virasoro models with central charge : $$c = 1 - 6\frac{(g-1)^2}{g}$$ Ising: g=4/3

Degenerate field with dimensions : $$\Delta_{(r|s)} = \frac{1}{4}\left(\frac{r^2-1}{g} + (s^2-1)g + 2(1-rs)\right)$$

Two second-level degenerate fields : $$\left(L_{-1}^2 - gL_{-2}\right)\Phi_{(1|2)} = 0\,, \qquad \left(L_{-1}^2 - \frac{1}{g}L_{-2}\right)\Phi_{(2|1)} = 0$$

When inserted in correlation function, the null-vector conditions translate into differential equations:

$$\mathcal{O}^g(z)\langle\Phi_{(1|2)}(z)\Phi_{\Delta_1}(z_1)\dots\Phi_{\Delta_N}(z_N)\rangle = 0$$

with

$$\mathcal{O}^g(z) = \frac{\partial^2}{\partial z^2} - g\left(\sum_{j=1}^{N}\frac{\Delta_i}{(z-z_j)^2} + \frac{1}{z-z_j}\frac{\partial}{\partial z_j}\right)$$

States with non-abelian monodromy and Virasoro theories

The correlators of second order degenerate fields

$\langle \Phi_{(2|1)}(w_1) \cdots \Phi_{(2|1)}(w_M) \rangle^a$ and $\langle \Phi_{(1|2)}(z_1) \cdots \Phi_{(1|2)}(z_N) \rangle^b$

are groundsates of the CS model with non-abelian monodromy **[Cardy, 04]**

How to characterize the excited states above these ground states?

States with non-abelian monodromy and Virasoro theories

The correlators of second order degenerate fields

$$\langle \Phi_{(2|1)}(w_1) \cdots \Phi_{(2|1)}(w_M) \rangle^a \qquad \text{and} \qquad \langle \Phi_{(1|2)}(z_1) \cdots \Phi_{(1|2)}(z_N) \rangle^b$$

are groundsates of the CS model with non-abelian monodromy **[Cardy, 04]**

How to characterize the excited states above these ground states?

consider the mixed correlators dressed by an extra u(1) component
[Bernevig, Estienne, Santachiara, 10]

$$\mathcal{F}^{a,b}_{M,N}(w;z) \equiv \langle \Phi_{(2|1)}(w_1) \cdots \Phi_{(2|1)}(w_M) \Phi_{(1|2)}(z_1) \cdots \Phi_{(1|2)}(z_N) \rangle_{a,b} \prod_{i<j}^{M} w_{ij}^{2\widetilde{h}} \prod_{i,j} (w_i - z_j)^{1/2} \prod_{1 \le i<j}^{N} z_{ij}^{2h}$$

Duality :

$$\left[h^{\alpha}(z) + g\, h^{\widetilde{\alpha}}(w) \right] \mathcal{F}^{a,b}_{M,N}(w;z) = 0 \qquad \alpha^{-1} = 1 - g\,, \qquad \widetilde{\alpha}^{-1} = 1 - g^{-1}$$

$$h^{\alpha}(z) \equiv \mathcal{H}^{\alpha}(z) - \mathcal{E}_0^{\alpha} + \left(\frac{N-2}{\alpha} - 1 \right) [\mathcal{P}(z) - \mathcal{P}_0] - \frac{NM(M-2)}{4}$$

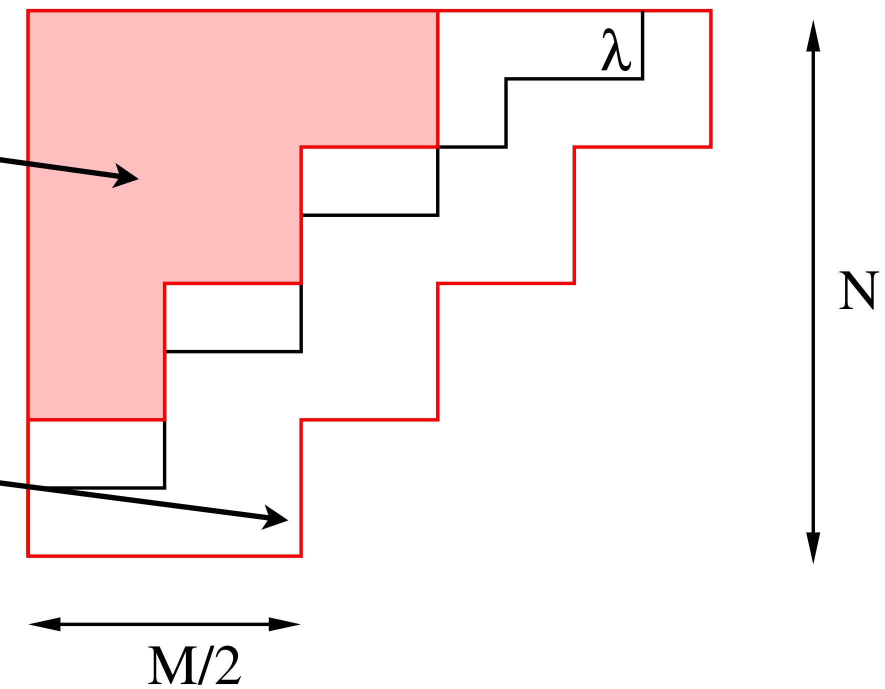
States with non-abelian monodromy and Virasoro theories

Consequence of duality:

$$\mathcal{F}^{a,b}_{M,N}(w;z) = \sum_{\lambda} P^{\widetilde{\alpha},a}_{\lambda'}(w)\, P^{\alpha,b}_{\lambda}(z)$$

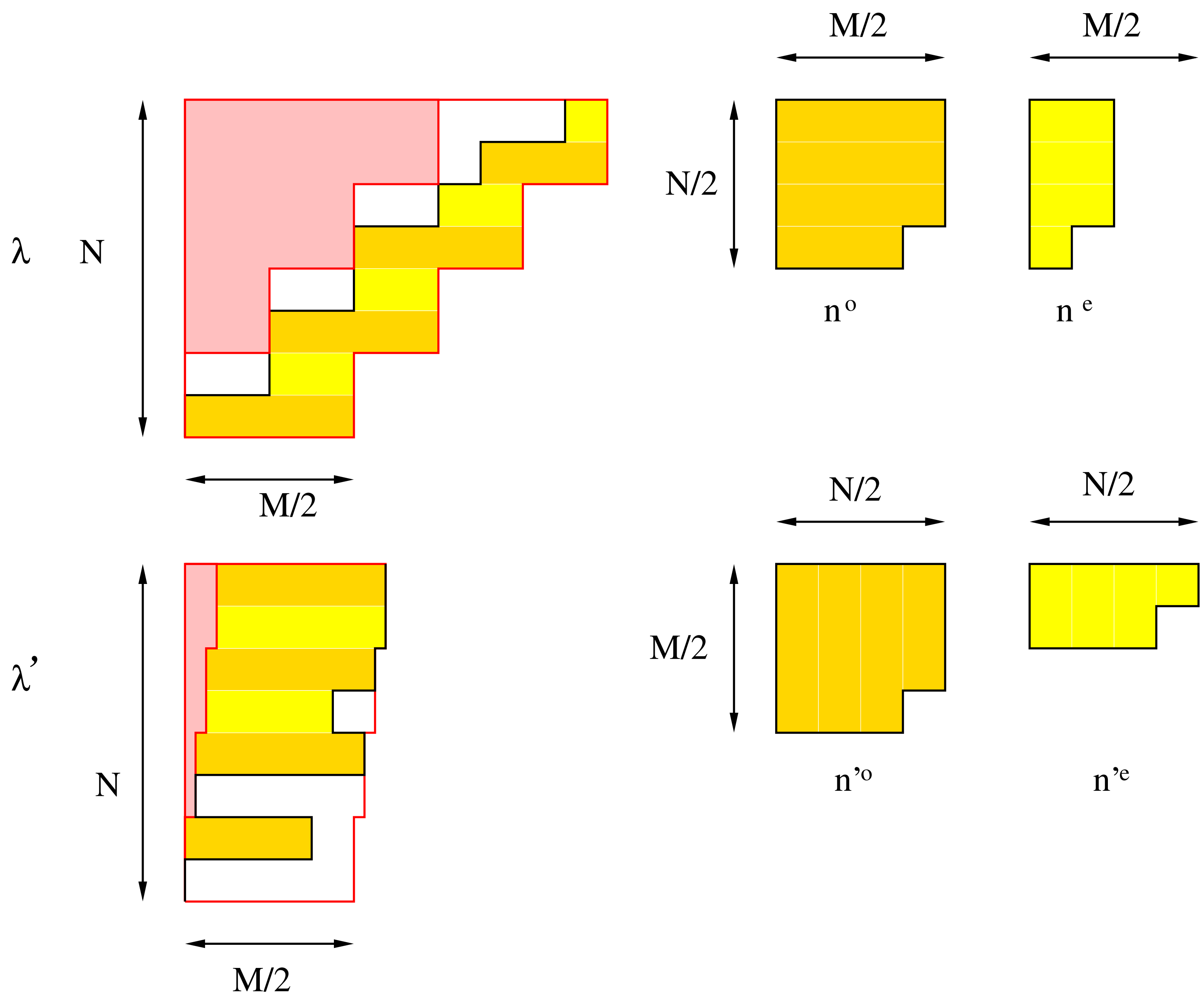
smallest "partition" $\lambda^0_{2i-1} = \lambda^0_{2i} \;=\; 2h(N-2i)$

largest "partition" $\Lambda^0_i \;=\; \lambda^0_i + \dfrac{M}{2}$



2h is not integer in general

An excited state λ is characterized by **two partitions** n_e and n_o (reminiscent of AGT conjecture):



$$b(n) \equiv 2\sum_i (i-1)n_i = \sum_j n'_j(n'_j - 1)$$

$$|n| \equiv \sum_i n_i = |n'|$$

$$\mathcal{E}^{\alpha}_{\lambda} = [b(n'^o) + b(n'^e)] - g\,[b(n^o) + b(n^e)] + ((1-g)N - M + g)(|n^o| + |n^e|) + 2(g-1)|n^e| + \mathcal{E}^{\alpha}_{(M/2)^N}$$

Bosonized version of the CS Hamiltonian

Heisenberg algebra : $[a_n, a_m] = n\delta_{n+m,0}$

Virasoro algebra : $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}.$

Feigin-Fuchs representation : $L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} : b_{n-m}b_m : -\alpha_0(n+1)b_n$

$$2\alpha_0 = \sqrt{\frac{2}{g}} - \sqrt{2g}$$

basis for the Hilbert space:

$$a_{-m_1}\ldots a_{m_l}L_{-n_1}\ldots L_{-n_k}|q\rangle$$

Degenerate fields dressed by u(1) vertex operators :

$$V(z) \equiv \Phi_{12}(z)\, e^{i\sqrt{\frac{g}{2}}\phi(z)}\ , \qquad \widetilde{V}(w) \equiv \Phi_{21}(w)\, e^{i\frac{1}{\sqrt{2g}}\phi(w)}$$

Bosonized version of the CS Hamiltonian

Consider generic correlation functions :

$$f_\mu^+(z_1, z_2, \cdots, z_N) = \langle \mu | V(z_1) V(z_2) \cdots V(z_N) | P \rangle$$
$$f_\mu^-(z_1, z_2, \cdots, z_N) = \langle P | V(z_1) V(z_2) \cdots V(z_N) | \mu \rangle$$

$|\mu\rangle$: generic state (primary or descendant)

$|P\rangle$: primary state

Translate the CS action on states :

$$H_n^g f_\mu^\pm(z_1, z_2, \cdots, z_N) = \sum_\nu \left[I_{n+1}^\pm(g) \right]_{\mu,\nu} f_\nu^\pm(z_1, z_2, \cdots, z_N)$$

$$I_3^{(\pm)}(g) = 2(1-g) \sum_{m \geq 1} m a_{-m} a_m \pm \sqrt{2g} \sum_{m \neq 0} a_{-m} L_m \pm \sqrt{\frac{g}{2}} \left(\sum_{m,k \geq 1} a_{-m-k} a_m a_k + a_{-m} a_{-k} a_{m+k} \right)$$

$$I_n^{(\pm)}(g) \propto I_n^{(\mp)}(1/g)$$

[Alba, Fateev, Litvinov, Tarnopolsky,10]
for Liouville $\quad g \to -g$

Bosonized version of the CS Hamiltonian

- rotate the boson basis: $c_m = \frac{1}{\sqrt{2}}(a_m + b_m)$, $\widetilde{c}_m = \frac{1}{\sqrt{2}}(a_m - b_m)$

[AFLT, 10; Belavin and Belavin, 11]

- introduce the one-component bosonised CS Hamiltonians:

$$\mathcal{I}_3^{\pm}(c;g) = (1-g)\sum_{m>0} m c_{-m} c_m \pm \sqrt{g} \sum_{m,k>0} (c_{-m-k} c_m c_k + c_{-m} c_{-k} c_{m+k})$$

[Jevicki, 91]
[Awata, Matsuo, Odake, Shiraishi, 95]

then the integrals of motion in Virasoro x u(1) have the triangular structure:

$$I_3^{+}(g) = \mathcal{I}_3^{+}(c;g) + \mathcal{I}_3^{+}(\widetilde{c};g) + \sqrt{2g}\,(b_0 - \alpha_0)(\mathcal{I}_2(c) - \mathcal{I}_2(\widetilde{c})) + 2(1-g)\sum_{m>0} m c_{-m} \widetilde{c}_m$$

also
[Maulik, Okounkov, unpublished]
[Shou, Wu, Yu, 11]

Constructing the eigenstates of the CS Hamiltonian

At g=1 the eigensstates can be constructed with the help of Schur polynomials
+ reflection condition

$$|n^o, n^e; q\rangle = |n^e, n^o; -q\rangle$$

$$c_{-n} \sim p_n = \sum_i x_i^n$$

$$|n^o, n'^e; q\rangle = S_{n^o}(c) S_{n^e}(\widetilde{c})|q\rangle + S_{n^o}(\widetilde{c}) S_{n^e}(c)| - q\rangle$$

$$b_0|q\rangle = q|q\rangle$$

At arbitrary g, use the Jack basis:

$$|n^o, n^e; q\rangle = J^{1/g}_{n^o}(c)\ J^{1/g}_{n^e}(\widetilde{c})\ |q\rangle + \dots$$

$|n^o, n^e; q\rangle$ is the basis used by AFLT to compute the matrix elements $\dfrac{\langle \nu_i | V_{\Delta_{i-1}}(1) | \mu_{i+1}\rangle}{\langle \nu_i | \mu_{i+1}\rangle}$

and prove the AGT conjecture

Parafermionic and WA_{k-1} theories

The same construction extends to WA_{k-1} algebras (differential equations, duality, bosonisation)

k-1 bosons x u(1) component: ⟶ k- component CS Hamiltonian

$$I_3^{(\pm)}(g) = 2(1-g)\sum_{m\geq 1} m a_{-m} a_m \pm \sqrt{2g}\sum_{m\geq 1}(a_{-m}L_m + L_{-m}a_m) \pm \sqrt{\frac{g}{2}}\left(\sum_{m,k\geq 1} a_{-m-k}a_m a_k + a_{-m}a_{-k}a_{m+k}\right) \pm \widetilde{W}_0$$

after basis rotation in the space of bosons:

$$I_3^{+}(g) = \sum_{j=1}^{k} \mathcal{I}^{\pm}(c^j; g) + 2(1-g)\sum_{j<l}\sum_{m\geq 1} m : c^j_{-m} c^l_m : + \quad \ldots$$

⟶ additive spectrum depending on **k partitions** (~ AGT conjecture for U(k) theories)

[see also Fateev, Litvinov, arXiv1109.4042]

Conclusions

- We have learned how to characterize the states of the Vir x H CFT, or WA_{k-1} x H, in terms of CS integrals of motion

- AFLT: this basis gives an efficient way to compute matrix elements of the fields (representation of the conformal blocks)

- Similar structure in the FQHE (different physics)

open questions

- Theory of non-polynomial CS eigenfunctions?

- How to systematically generate the integrals of motion (transfer matrix?) in CFT?
  [Maulik, Okounkov, unpublished] in 4d gauge theory context

- Relation with the integrable structure uncovered by **[Bazhanov, Lukyanov, Zamolodchikov, 94-98]** (no Heisenberg factor)?

...