Holographic 3-point function at one-loop

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Common Trends in Gauge Fields, Strings and Integrable Systems

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Agnese Bissi, Troels Harmark and MO, arXiv:1112.5075
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Plan of talk:

- **Introduction**

- **Gauge Theory Side**

  Tree level computation
  One loop computation

- **String Theory Side**

  The Frolov-Tseytlin limit
  One loop computation

- **Comparison & Comments**

Introduction

Integrability has been the driving force behind the recent progress in the study of the spectral problem in the AdS/CFT correspondence between $\mathcal{N}$=4 SYM theory and type IIB string theory on AdS_5XS^5.

Dictionary between gauge theory and string theory quantities:

string tension	$T_{\mathrm{str}} = \dfrac{R^2}{4\pi\alpha'}$	with	$R^4 = \lambda(\alpha')^2$
string coupling	$g_s = \dfrac{\lambda}{N}$	't Hooft coupling	$\lambda = \dfrac{g_{\mathrm{YM}}^2 N}{4\pi^2}$

The spectral problem consists of determining the exact spectrum of gauge theory operators in the planar limit and match this to the spectrum of string theory states.

However, to solve completely $\mathcal{N}$=4 SYM theory in the planar limit one should also know the set of all 3-point correlation functions.

To have a full understanding of the AdS/CFT correspondence in the planar limit one should be able to compute the 3-point functions on both the gauge theory and string theory sides, and match the two sides, possibly with the aid of integrability.

However, this is not an easy task.

On the gauge theory side, the computation of 3-point functions is much more difficult than 2-point functions.

On the string theory side, one needs to understand the vertex operators of string states in type IIB string theory on $AdS_5 X S^5$.

Given three gauge-invariant operators $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ with definite scaling dimensions Δ_1, Δ_2 and Δ_3 in $\mathcal{N} = 4$ SYM theory, which is a conformal field theory, 3-point correlation functions are of the form

$$\boxed{\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_3 - x_1|^{\Delta_3+\Delta_1-\Delta_2}}}$$

To compute the full 3-point correlation function it is thus enough to compute the coefficient C_{123}. In the planar limit, we are only interested in the leading part of C_{123} which goes like 1/N.

In the planar limit, we are only interested in the leading part of C_{123} which goes like 1/N.

The overall goal is to compute the coefficient C_{123} on the gauge theory side and on the string theory side and find a way to interpolate the results

Recently, progress on computing C_{123} have been made on the string theory side by considering the special case of a **3-point function with two heavy (semi-classical) operators and one light chiral primary operator**, starting with the work of **[Zarembo, arXiv:1008.1059]** and **[Costa, Monteiro, Santos, Zoakos, arXiv:1008.1070]**.

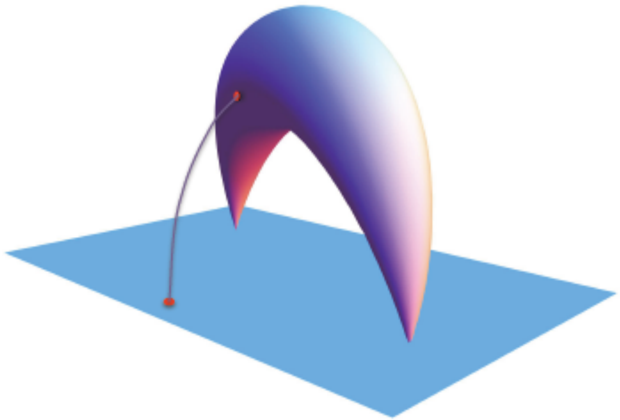
There are also attempts to extend this set up to the more general case of 3-point functions involving **3 heavy operators** **[Klose, McLoughlin, arXiv:1106.0495] [Janik, Wereszczynski, arXiv:1109.6262], [Buchbinder, Tseytlin, ArXiv:1110.5621], [Kazama, Komatsu, arXiv:1109.3949]**.

Since the paper of Zarembo, there is been a lot of interest on this subject

Zarembo, arXiv:1008.1059.
Costa, Monteiro, Santos, Zoakos, arXiv:1008.1070.
Roiban and Tseytlin, arXiv:1008.4921.
Grossardt, Plefka, arXiv:1007.2356.
Hernandez, arXiv:1011.0408.
Arnaudov, Rashkov, arXiv:1011.4669.
Georgiou, arXiv:1011.5181.
Escobedo, Gromov, Sever, Vieira, arXiv:1012.3293.
Bak, Chen, Wu, arXiv:1103.2024.
Arnaudov, Rashkov, Vetsov, arXiv:1103.6145.
Hernandez, arXiv:1104.1160.
Bai, Lee, Park, arXiv:1104.1896.
Ahn, Bozhilov, arXiv:1105.3084.
Klose, McLoughlin, arXiv:1106.0495.
Arnaudov, Rashkov, arXiv:1106.0859,
Arnaudov, Rashkov, arXiv:1106.4298.
Michalcik, Rashkov, Schimpf, arXiv:1107.5795.
Gromov, Sever, Vieira, arXiv:1111.2349.
Russo, Tseytlin, arXiv:1012.2760.
Bissi, Kristjansen, Young, Zoubos, arXiv:1103.4079.
Escobedo, Gromov, Sever, Vieira, arXiv:1104.5501.
Georgiou, arXiv:1107.1850.
Janik, Wereszczynski, arXiv:1109.6262.
Kazama, Komatsu, arXiv:1110.3949.
Buchbinder, Tseytlin, arXiv:1110.5621.
Georgiou, Gili, Grossardt, Plefka, arXiv:1201.0992.

+ the ones I forgot to write down…..

In the case of **2 heavy operators and 1 light operator**, it is possible to compute the 3-point function using a prescription that employs the classical string world-sheet corresponding to the 2-point function of the heavy operators. This prescription rests on the validity of the probe approximation for the supergravity state dual to the light chiral primary operator.



Picture taken from [Zarembo, arXiv:1008.1059]

Building on this, a **weak/strong coupling match** for this type of 3-point functions was found in **[Escobedo, Gromov, Sever, Vieira, arXiv:1104.5501]** where **the tree-level part of a 3-point function in gauge theory has been matched to the corresponding 3-point function on the string side**, taking the so-called Frolov-Tseytlin limit **[Frolov, Tseytlin, hep-th/0306143], [Kruczenski, hep-th/0311203]**.

Our goal:

Motivated by this result and by the fact that there has been indications that this match should also hold at higher loops **[Russo, Tseytlin, arXiv:1012.2760]**, we decided to further explore this matching by computing the one-loop correction on both sides of the correspondence.

Gauge Theory Side

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_3-x_1|^{\Delta_3+\Delta_1-\Delta_2}}$$

We can write $C_{123} = C_{123}^{(0)} + \lambda' C_{123}^{(1)} + \mathcal{O}(\lambda'^2)$ $\lambda' = \lambda/J^2$

We use renormalized operators ➔ $C_{123}^{(1)}$ is the scheme independent part of the one-loop coefficient.

The gauge theory operators for which we compute the 3-point function are single trace operators made out of complex scalars of $\mathcal{N}$=4 SYM theory. More precisely

$\mathcal{O}_1(x_1)$	$J_1 + j\ \bar{Z}$	$J_2 - j\ \bar{X}$
$\mathcal{O}_2(x_2)$	$J_1\ Z$	$J_2\ X$
$\mathcal{O}_3(x_3)$	$j\ Z$	$j\ \bar{X}$

$J = J_1 + J_2$

$j \ll J$

each operator is in an SU(2) sector of $\mathcal{N}$=4 SYM theory.

Note that this gives a **non-extremal** 3-point function for any j≠0

⬇

$\Delta_1 \neq \Delta_2 + \Delta_3$

The 2 semiclassical operators are described using coherent states.

Kruczenski showed that semiclassical operators in the SU(2) sector of $\mathcal{N}$=4 SYM theory can be described using the Landau-Lifshitz model. **[Kruczenski, hep-th/0311203]**

$$\boxed{\begin{aligned}\mathcal{O}_1(x_1) &= \mathcal{N}_1 \bar{\mathbf{u}}_{i_1}\left(\frac{k+1}{l}\right)\bar{\mathbf{u}}_{i_2}\left(\frac{k+2}{l}\right)\cdots\bar{\mathbf{u}}_{i_L}\left(\frac{k}{l}\right) : \mathrm{Tr}(\bar{W}^{i_1}\bar{W}^{i_2}\ldots\bar{W}^{i_J}) : (x_1)\\ \mathcal{O}_2(x_2) &= \mathcal{N}_2 \mathbf{v}^{j_1}\left(\frac{k+1}{l}\right)\mathbf{v}^{j_2}\left(\frac{k+2}{l}\right)\cdots\mathbf{v}^{j_L}\left(\frac{k}{l}\right) : \mathrm{Tr}(W_{j_1}W_{j_2}\cdots W_{j_J}) : (x_2)\end{aligned}}$$

with $\quad W^i = (Z, X)\ ,\ \ \bar{W}_i = (\bar{Z}, \bar{X})\ ,\ \ l \equiv \frac{J}{2\pi}$

$\mathbf{u}(u_1,u_2)$ and $\mathbf{v}(v_1,v_2)$ correspond for each site of the single trace operators to coherent states in the spin ½ repr. of SU(2). They satisfy $\bar{\mathbf{u}}\bullet\mathbf{u}=1$

$\sigma = k/l$ and the functions **u** and **v** are periodic in σ with period 2π and they take values in C^2.

$\mathcal{O}_1$ and $\mathcal{O}_2$ are semi-classical operators with J>>1 $\longrightarrow$ they are slowly varying with σ

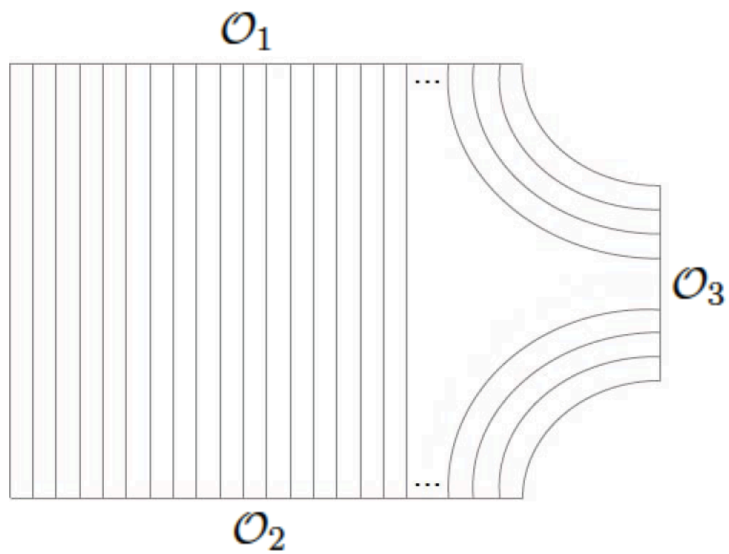
The third operator is a light chiral primary operator of the form

$$\boxed{\mathcal{O}_3(x_3) = \mathcal{N}_3 : \mathrm{Tr}(\mathrm{sym}(\bar{X}^j Z^j)) : (x_3)}$$

Tree-level computation

The tree-level part has already been computed in **[Escobedo, Gromov, Sever, Vieira, arXiv:1104.5501]**

Our convention for the tree-level 3-point diagram is that we contract the j first letters of O_1 with O_3 and the rest is then contracted with O_2. Also, we contract the j first letters of O_2 with O_3 and the rest with O_1 . Then we sum over all possible choices of the first site.



$$\mathcal{O}_1(x_1) = \mathcal{N}_1 \bar{\mathbf{u}}_{i_1}(\frac{k+1}{l})\, \bar{\mathbf{u}}_{i_2}(\frac{k+2}{l}) \cdots \bar{\mathbf{u}}_{i_L}(\frac{k}{l}) : \mathrm{Tr}(\bar{W}^{i_1}\bar{W}^{i_2}\cdots \bar{W}^{i_J}) : (x_1)$$

$$\mathcal{O}_2(x_2) = \mathcal{N}_2 \mathbf{v}^{j_1}(\frac{k+1}{l})\, \mathbf{v}^{j_2}(\frac{k+2}{l}) \cdots \mathbf{v}^{j_L}(\frac{k}{l}) : \mathrm{Tr}(W_{j_1}W_{j_2}\cdots W_{j_J}) : (x_2)$$

$$\mathcal{O}_3(x_3) = \mathcal{N}_3 : \mathrm{Tr}(\mathrm{sym}(\bar{X}^j Z^j)) : (x_3)$$

$\mathcal{O}_1(x_1)$	$J_1 + j\ \bar{Z}$	$J_2 - j\ \bar{X}$
$\mathcal{O}_2(x_2)$	$J_1\ Z$	$J_2\ X$
$\mathcal{O}_3(x_3)$	$j\ Z$	$j\ \bar{X}$

$J = J_1 + J_2$

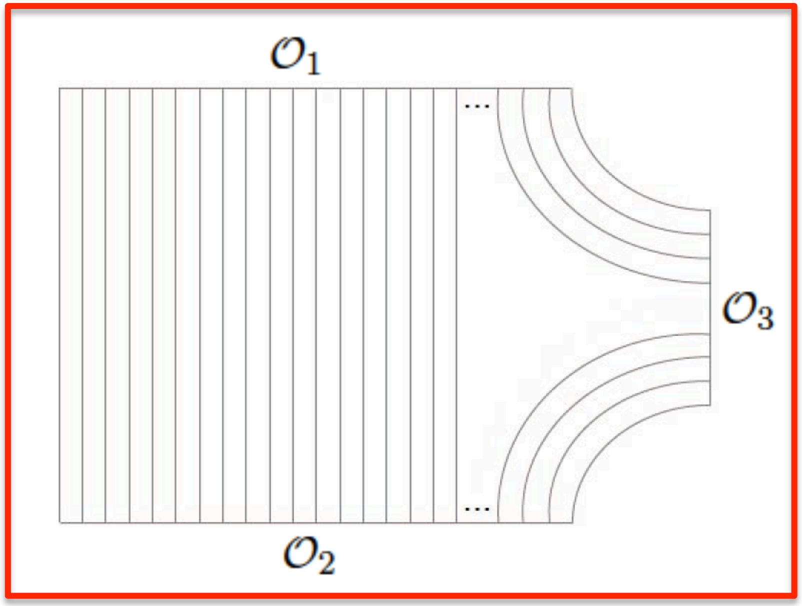
J large

$j \ll J$

Define

$$B \equiv \prod_{m=1}^{J} \bar{\mathbf{u}}(\frac{m}{l}) \cdot \mathbf{v}(\frac{m}{l})$$

Note that B does not depend on the choice of k, but only on **ū** and **v**



$\bar{\mathbf{u}}=\bar{\mathbf{u}}(\bar{u}^1,\bar{u}^2)$ and $\mathbf{v}=\mathbf{v}(v_1,v_2)$

From the tree-level contractions we get

$$\sum_k A(k) = B \sum_k \prod_{m=k+1}^{k+j} \frac{\bar{u}^1(\frac{m}{l})v_2(\frac{m}{l})}{\bar{\mathbf{u}}(\frac{m}{l}) \cdot \mathbf{v}(\frac{m}{l})}$$

(disregarding propagators, combinatoric factors, etc.)

Since **ū** and **v** vary slowly and j << J, the difference for **ū** and **v** at two different values of σ can be estimated using a Taylor expansion. We find

$$\sum_k A(k) = B \sum_k \left(\frac{(\bar{u}^1 v_2)(\frac{k}{l})}{(\bar{\mathbf{u}} \cdot \mathbf{v})(\frac{k}{l})}\right)^j \left(1 + \frac{j(j+1)}{2\,l}\left(\frac{(\bar{u}^1 v_2)'(\frac{k}{l})}{(\bar{u}^1 v_2)(\frac{k}{l})} - \frac{(\bar{\mathbf{u}} \cdot \mathbf{v})'(\frac{k}{l})}{(\bar{\mathbf{u}} \cdot \mathbf{v})(\frac{k}{l})}\right) + \cdots\right)$$

In our limit J>>1 therefore we find

$$\sum_k A(k) \simeq B \sum_k \left(\frac{(\bar{u}^1 v_2)(\frac{k}{l})}{(\bar{\mathbf{u}} \cdot \mathbf{v})(\frac{k}{l})} \right)^j \simeq B\, J \int_0^{2\pi} \frac{d\sigma}{2\pi} \left(\frac{(\bar{u}^1 v_2)(\sigma)}{(\bar{\mathbf{u}} \cdot \mathbf{v})(\sigma)} \right)^j$$

now that using the fact that $\mathcal{O}_3$ is a small operator, $\mathcal{O}_2$ can be regarded as being more or less the complex conjugate of $\mathcal{O}_1$.This also means that we consider the approximation **v≈u** which gives B=1. We thus have

$$B \equiv \prod_{m=1}^{J} \bar{\mathbf{u}}(\frac{m}{l}) \cdot \mathbf{v}(\frac{m}{l})$$

$$\boxed{C_{123}^{(0)} = \frac{\mathcal{N}_3}{N} \sum_k A(k) = \frac{1}{N} \frac{j!J}{\sqrt{(2j-1)!}} \int_0^{2\pi} \frac{d\sigma}{2\pi} (\bar{u}_1 u_2)^j}$$

with $\mathcal{N}_3 = \frac{j!}{\sqrt{(2j-1)!}}$

Comments:

- ✓ Our result coincides with the one obtained in **[Escobedo, Gromov, Sever, Vieira, arXiv:1104.5501]** .
- ✓ Among all the possible terms in $\mathcal{O}_3$, only $\mathrm{Tr}\left(\bar{X}^j Z^j\right)$ contributes to the result
- ✓ We made the assumption that $\mathcal{O}_2$ is almost the complex conjugate of $\mathcal{O}_1$. It is however important how the 2 semiclassical operators differ from each other. In **[Escobedo, Gromov, Sever, Vieira, arXiv:1104.5501]** it was shown that at this order the approximation is valid.

One-loop computation

Two types of contributions:

- **One-loop diagrams** with 2 legs in one of the operators and the other 2 legs in 2 different operators.
- The functions **u** and **v** receive corrections at O(λ) from considering the **2-loop contribution to the effective σ-model description**.

Two-loop correction to eigenstate

The origin of these type of corrections is that our computation should be thought os as the first correction in an all-order series in powers of λ.

These corrections have been neglected in earlier studies of 3-point functions of gauge theory operators in $\mathcal{N}$=4 SYM theory, as pointed out in **[Beisert, Kristjansen, Plefka, Semenoff, Staudacher, hep-th/0208178], [Kristjansen, arXiv:1012.3997]].**

While in general it is rather complicated to take into account this contribution, it actually becomes very easy for the set of operators that we are considering.

This is due to the enormous simplification that one has by using a coherent state representation for the gauge theory operators. This is also the reason that made the computation of the leading order contribution to C_{123} possible **[Escobedo, Gromov, Sever, Vieira, arXiv:1104.5501]** .

The 2 semiclassical operators are described using the Landau-Lifshitz model. [Kruczenski, hep-th/0311203]

In the semiclassical approx., using the Landau-Lifshitz model, one can for example compute the energy of an operator such as $\mathcal{O}_1$ and, up to one-loop, one gets

$$E \simeq J\left(1+\frac{\lambda'}{2}\int_0^{2\pi}\frac{d\sigma}{2\pi}\partial_\sigma\bar{\mathbf{u}}\cdot\partial_\sigma\mathbf{u}\right)$$

In the computation of the tree-level part of C_{123}, the coherent state function **u** was solution of the EOM of the Landau-Lifshitz model up to one loop. Therefore here, to compute the one–loop correction to C_{123}, **u** should satisfy the EOM of the Landau-Lifshitz model up to 2 loops. We write

$$\mathbf{u} = \mathbf{u}^{(0)} + \lambda'\mathbf{u}^{(1)} + \lambda'^2\mathbf{u}^{(2)} + \mathcal{O}(\lambda'^3)$$

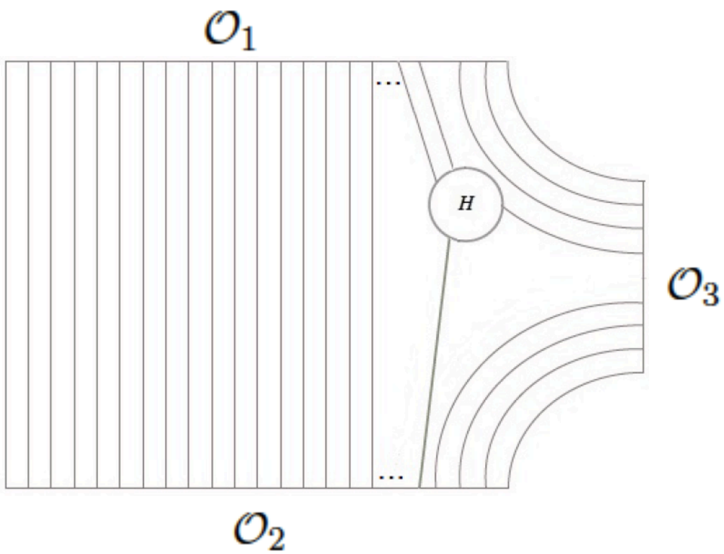
Thus, the contribution of the first type of corrections to the one-loop part of C_{123}, is computed by substituting the u given above in the expression

$$\boxed{C_{123} = \frac{1}{N}\frac{j!J}{\sqrt{(2j-1)!}}\int_0^{2\pi}\frac{d\sigma}{2\pi}(\bar{u}_1 u_2)^j}$$

and extracting the contribution of order λ’.

One-loop diagrams

The other type of correction contributing at one-loop comes form the insertion of the one-loop Hamiltonian with two legs in one of the operators and the other two legs in two different operators.



We compute these corrections using the prescription given in theory [Okuyama, Tseng, hep-th/0404190], [Roiban, Volovich, hep-th/0407140], [Alday, Gava, Narain, hep-th/0502186] which gives

$$\boxed{\tilde{C}^{(1)}_{123} = \frac{1}{32\pi^2} B \sum_{k=1}^{J} \left(\frac{(\bar{u}^1 v_2)(\frac{k}{l})}{(\bar{\mathbf{u}} \cdot \mathbf{v})(\frac{k}{l})} \right)^j \left(f^1_{23}(k) + f^2_{31}(k) + f^3_{12}(k)\right)}$$

where B is again given by $B \equiv \prod_{m=1}^{J} \bar{\mathbf{u}}(\frac{m}{l}) \cdot \mathbf{v}(\frac{m}{l})$

f^1_{23} is the constant referring to the 3-point diagram with 2 contractions in O_1 and 1 contraction each with O_2 and O_3 and so on.

For a given k we have

$$f^1_{23}(k) = -\frac{\bar{u}^{i_1}(\frac{k+j+1}{l})\bar{u}^{i_2}(\frac{k+j}{l})v_{j_1}(\frac{k+j+1}{l})\delta^1_{j_2}}{(\bar{u}\cdot v)(\frac{k+j+1}{l})\bar{u}^3(\frac{k+j}{l})}\mathcal{H}^{j_1j_2}_{i_1i_2} - \frac{\bar{u}^{i_1}(\frac{k+1}{l})\bar{u}^{i_2}(\frac{k}{l})\delta^1_{j_1}v_{j_2}(\frac{k}{l})}{\bar{u}^3(\frac{k+1}{l})(\bar{u}\cdot v)(\frac{k}{l})}\mathcal{H}^{j_1j_2}_{i_1i_2}$$

$$f^2_{31}(k) = -\frac{\delta^{i_1}_2\bar{u}^{i_2}(\frac{k+j+1}{l})v_{j_1}(\frac{k+j}{l})v_{j_2}(\frac{k+j+1}{l})}{v_1(\frac{k+j}{L})(\bar{u}\cdot v)(\frac{k+j+1}{L})}\mathcal{H}^{j_1j_2}_{i_1i_2} - \frac{\bar{u}^{i_1}(\frac{k}{L})\delta^{i_2}_2 v_{j_1}(\frac{k}{l})v_{j_2}(\frac{k+1}{l})}{(\bar{u}\cdot v)(\frac{k}{l})v_1(\frac{k+1}{l})}\mathcal{H}^{j_1j_2}_{i_1i_2}$$

$$f^3_{12}(k) = -\frac{\bar{u}^{i_1}(\frac{k+j}{l})\delta^{i_2}_2\delta^1_{j_1}v_{j_2}(\frac{k+j}{l})}{\bar{u}^3(\frac{k+j}{l})v_1(\frac{k+j}{l})}\mathcal{H}^{j_1j_2}_{i_1i_2} - \frac{\delta^{i_1}_2\bar{u}^{i_2}(\frac{k+1}{l})v_{j_1}(\frac{k+1}{l})\delta^1_{j_2}}{\bar{u}^3(\frac{k+1}{l})v_1(\frac{k+1}{l})}\mathcal{H}^{j_1j_2}_{i_1i_2}$$

with

$$\mathcal{H}^{j_1j_2}_{i_1i_2} = 2(I-P)^{j_1,j_2}_{i_1,i_2}, \ , \quad I^{j_1j_2}_{i_1i_2} = \delta^{j_1}_{i_1}\delta^{j_2}_{i_2} \ , \quad P^{j_1j_2}_{i_1i_2} = \delta^{j_2}_{i_1}\delta^{j_1}_{i_2}$$

From our choice of the operator $\mathcal{O}_3$, one can see that $\boxed{f^3_{12} = 0.}$

$$\mathcal{O}_3(x_3) = \mathcal{N}_3 : \mathrm{Tr}(\mathrm{sym}(\bar{X}^jZ^j)) : (x_3)$$

This is because, among all the states, only $\mathrm{Tr}\left(\bar{X}^jZ^j\right)$ and $\mathrm{Tr}\left(\bar{X}^{j-1}Z\bar{X}Z^{j-1}\right)$ contribute with a relative minus sign.

We can compute

$$f_{23}^1(k) = -2\left[2 - \frac{\bar{u}^1(\frac{k+j+1}{l})\bar{\mathbf{u}}(\frac{k+j}{l})\mathbf{v}(\frac{k+j+1}{l})}{(\bar{\mathbf{u}}\cdot\mathbf{v})(\frac{k+j+1}{l})\bar{u}^1(\frac{k+j}{l})} - \frac{\bar{\mathbf{u}}(\frac{k+1}{l})\mathbf{v}(\frac{k}{l})\bar{u}^1(\frac{k}{l})}{\bar{u}^1(\frac{k+1}{l})(\bar{\mathbf{u}}\cdot\mathbf{v})(\frac{k}{l})}\right]$$

Again, using that $\bar{\mathbf{u}}$ and $\mathbf{v}$ vary slowly and that j<<J, we can estimate the difference of $\bar{\mathbf{u}}$ and $\mathbf{v}$ at two different values of σ using a Taylor expansion.

Moreover we also use that $\mathbf{u}\approx\mathbf{v}$. We get

$$f_{23}^1(k) \sim -2\left[\frac{1}{l^2}\bar{\mathbf{u}}'\cdot\mathbf{u}' - \frac{j}{l^2}\left(\frac{\bar{u}^{1\prime}(\frac{k}{l})}{\bar{u}^1(\frac{k}{l})}\right)'\right] \quad f_{31}^2(k) \sim -2\left[\frac{1}{l^2}\bar{\mathbf{u}}'\cdot\mathbf{u}' - \frac{j}{l^2}\left(\frac{u_2'(\frac{k}{l})}{u_2(\frac{k}{l})}\right)'\right] \quad l \equiv \frac{J}{2\pi}$$

Putting all these results together and summing over k we obtain

$$\tilde{C}_{123}^{(1)} = -\frac{\lambda'}{2\,N}\frac{j!J}{\sqrt{(2j-1)!}}\int_0^{2\pi}\frac{d\sigma}{2\pi}(\bar{u}_1u_2)^j\left(\partial_\sigma\bar{\mathbf{u}}\cdot\partial_\sigma\mathbf{u} - \frac{j}{4}\partial_\sigma^2(\log(\bar{u}_1u_2))\right)$$

Combining this with the result for the leading order term with the wave function $\bar{u}$ solution of the EOMs up to two loops, we arrive at the final expression for the 3 point function coefficient

$$\boxed{C_{123} = \frac{1}{N}\frac{j!J}{\sqrt{(2j-1)!}}\int_0^{2\pi}\frac{d\sigma}{2\pi}(\bar{u}_1u_2)^j\left[1 - \frac{\lambda'}{2}\left(\partial_\sigma\bar{\mathbf{u}}\cdot\partial_\sigma\mathbf{u} - \frac{j}{4}\partial_\sigma^2(\log(\bar{u}_1u_2))\right)\right] + \mathcal{O}(\lambda'^2)}$$

String Theory Side

Our starting point is the sigma-model for type IIB string theory on AdS_5XS^5

$$ds^2 = R^2 \left[-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, (d\Omega_3')^2 + d\zeta^2 + \sin^2\zeta\, d\alpha^2 + \cos^2\zeta\, (d\Omega_3)^2 \right]$$

$$F_{(5)} = 2R^4 \left[\cosh\rho\, \sinh^3\rho\, dt\, d\rho\, d\Omega_3' + \sin\zeta\, \cos^3\zeta\, d\zeta\, d\alpha\, d\Omega_3 \right]$$

with

$$(d\Omega_3)^2 = d\psi^2 + \cos^2\psi d\phi_1^2 + \sin^2\psi d\phi_2^2 = d\psi^2 + d\phi_-^2 + d\phi_+^2 + 2\cos(2\psi) d\phi_- d\phi_+$$

$$2\phi_\pm = \phi_1 \pm \phi_2$$

In particular, we can focus on the $\mathcal{R}XS^3$ part of the background.
The metric is

$$ds^2 = R^2 \left[-dt^2 + \frac{1}{4}(d\Omega_2)^2 + \left(d\phi_+ + \frac{1}{2}\sin\theta d\varphi \right)^2 \right] , \quad (d\Omega_2)^2 = d\theta^2 + \cos^2\theta d\varphi^2$$

where we introduced the new angles $\theta \equiv 2\psi - \frac{\pi}{2}$, $\varphi \equiv 2\phi_-$

The computation on the string side is done using the prescription of [Zarembo, arXiv: 1008.1059] which in our notation reads

$$\boxed{C_{123} = c_j \frac{\sqrt{\lambda}}{N} \int_{-\infty}^{+\infty} d\tau_e \int_0^{2\pi} \frac{d\sigma}{2\pi} \frac{(\bar{U}_1 U_2)^j}{\cosh^{2j}(\frac{\tau_e}{\kappa})} \left[\frac{2}{\kappa^2 \cosh^2(\frac{\tau_e}{\kappa})} - \frac{1}{\kappa^2} - \partial_a \bar{\mathbf{U}} \cdot \partial^a \mathbf{U} \right]}$$

τ_e is the Euclidean time and the constant c_j is $\quad c_j = \frac{(2j+1)!}{2^{2j+2} j! \sqrt{(2j-1)!}}$

U(τ,σ) is a complex vector that parametrizes the embedding of type IIB string on S^5

$$\mathbf{U}(\sigma,\tau) = e^{i\tau/\kappa} \mathbf{u}(\sigma,\tau) \qquad \mathbf{u}(\sigma,\tau) = (u_1(\sigma,\tau), u_2(\sigma,\tau))$$

As in the gauge theory side, also here we have that $\bar{\mathbf{u}} \bullet \mathbf{u} = 1$

κ is a constant related to the gauge choice.

We work in the **Frolov-Tseytlin limit** which in our notation is [Frolov, Tseytlin, hep-th/0306143], [Kruczenski, hep-th/0311203].

$$\boxed{\kappa \to 0 \ , \quad \frac{1}{\kappa} \partial_\tau \mathbf{u} \ \ \text{fixed}, \quad \partial_\sigma \mathbf{u} \ \ \text{fixed}}$$

The holographic dual of the 3-point function computed on the gauge theory side corresponds to a string which is point-like in AdS and is moving with angular momentum J non-trivially in S^5

The charges are

$$E = i\partial_t \ , \quad J \equiv J_1 + J_2 = -i\partial_{\phi_+}$$

It is convenient to introduce the light-cone coordinates

$$\boxed{x^+ = \lambda' t \ , \quad x^- = \phi_+ - t}$$

where we rescaled t with λ’ and λ’ goes to zero.

The charges now are

$$\boxed{i\partial_+ = H = \frac{E-J}{\lambda'} \ , \quad -i\partial_- = J}$$

and the metric becomes

$$ds^2 = R^2\Big[\frac{1}{4}(d\Omega_2)^2 + \Big(2\frac{dx^+}{\lambda'} + dx^- + \omega\Big)(dx^- + \omega)\Big] \ , \quad \omega = \frac{1}{2}\sin\theta d\varphi$$

The bosonic sigma model Lagrangian and Virasoro constraints are

$$\mathcal{L} = -\frac{1}{2} h^{\alpha\beta} G_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu$$

$$G_{\mu\nu}(\partial_\alpha x^\mu \partial_\beta x^\nu - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma x^\mu \partial_\delta x^\nu) = 0$$

where $h^{\alpha\beta} = \sqrt{-\det\gamma}\gamma^{\alpha\beta}$ with $\gamma_{\alpha\beta}$ being the world-sheet metric.

Our gauge choice is

$$x^+ = \kappa\tau$$

$$2\pi p_- = \frac{\partial\mathcal{L}}{\partial\partial_\tau x^-} = \text{const.}\ , \quad \frac{\partial\mathcal{L}}{\partial\partial_\sigma x^-} = 0$$

Then the idea is to solve perturbatively the Virasoro constraints and from that compute the gauge fixed Lagrangian.

We can then determine the constant κ and to leading order in λ' we find

$$J = \int_0^{2\pi} d\sigma p_- = \frac{R^2\kappa}{\lambda'}$$

which, using $R^4 = \lambda(\alpha')^2$ gives $\boxed{\kappa = \sqrt{\lambda'}.}$ We see that $\kappa \to 0.$

This gives an expansion in κ which parallels the one in λ‘ on the gauge theory side.

Let's assume that we solved perturbatively the sigma model up to 2 loops.

[Kruczenski, Ryzhov, Tseytlin, hep-th/0403120], [Minahan, Tirziu, Tseytlin, hep-th/0509071, hep-th/0510080]

We use this to compute the one-loop correction to the holographic 3-point function for 2 semiclassical and 1 chiral primary operators

Taking the Frolov-Tseytlin limit we have

$$C_{123} = c_j \frac{\sqrt{\lambda}}{N} \int_{-\infty}^{+\infty} d\tau_e \int_0^{2\pi} \frac{d\sigma}{2\pi} \frac{(\bar{u}_1 u_2)^j}{\cosh^{2j}(\frac{\tau_e}{\kappa})} \left[\frac{1}{\kappa^2 \cosh^2(\frac{\tau_e}{\kappa})} - \partial_\sigma \bar{\mathbf{u}} \cdot \partial_\sigma \mathbf{u} + \mathcal{O}(\kappa^2) \right]$$

The integrals over τ_e are of two types

$$\rightarrow \quad I_0 = \int_{-\infty}^{+\infty} d\tau_e \frac{1}{\cosh^{2j+2}(\frac{\tau_e}{\kappa})} = \kappa \frac{2^{2j+1}\,(j!)^2}{(2j+1)!}$$

$$\rightarrow \quad I_1 = \int_{-\infty}^{+\infty} d\tau_e \frac{1}{\cosh^{2j}(\frac{\tau_e}{\kappa})} = I_0 \frac{2j+1}{2j}$$

We can use this to write our final result for the holographic 3-point function, which in terms of gauge theory variables read

$$\boxed{C_{123} = \frac{J}{N} \frac{j!}{\sqrt{(2j-1)!}} \int_0^{2\pi} \frac{d\sigma}{2\pi} (\bar{u}_1 u_2)^j \left[1 - \lambda' \frac{2j+1}{2j} \partial_\sigma \bar{\mathbf{u}} \cdot \partial_\sigma \mathbf{u} \right]}$$

Comparisons & ....

Gauge theory side

$$C_{123} = \frac{J}{N} \frac{j!}{\sqrt{(2j-1)!}} \int_0^{2\pi} \frac{d\sigma}{2\pi} (\bar{u}_1 u_2)^j \left[1 - \frac{\lambda'}{2} \left(\partial_\sigma \bar{\mathbf{u}} \cdot \partial_\sigma \mathbf{u} - \frac{j}{4} \partial_\sigma^2 (\log(\bar{u}_1 u_2)) \right) \right]$$

String theory side

$$C_{123} = \frac{J}{N} \frac{j!}{\sqrt{(2j-1)!}} \int_0^{2\pi} \frac{d\sigma}{2\pi} (\bar{u}_1 u_2)^j \left[1 - \lambda' \frac{2j+1}{2j} \partial_\sigma \bar{\mathbf{u}} \cdot \partial_\sigma \mathbf{u} \right]$$

Perfect matching for the tree-level part
but
disagreement at one-loop

Note that a matching for the tree-level part has also been observed for operators in the SL(2) sector of N=4 SYM theory in **[Georgiou, arXiv:1107.1850]**

…. Comments

- ✓ With our current understanding of the AdS/CFT correspondence, there is no reason to expect a matching.

  It is nevertheless worth considering whether the mismatch we find is due to overlooked subtleties on the gauge theory or string theory side.

- ✓ On the gauge theory side we used the prescription of **[Okuyama, Tseng, hep-th/0404190], [Roiban, Volovich, hep-th/0407140], [Alday, Gava, Narain, hep-th/0502186]**

  This seems a physically sound prescription, as it consists in computing all the 3-point diagrams involving all 3 operators using the one-loop Hamiltonian and summing them up. However, it would be prudent to validate further this prescription by making checks for explicit examples, such as in **[Georgiou, Gili, Grossardt, Plefka, arXiv:1201.0992]**

- ✓ Moreover, one needs to address the subtlety that the two semi-classical states which are approximated with the same coherent state have to be slightly different due to conservation of the R charges. **[Escobedo, Gromov, Sever, Vieira, arXiv:1104.5501]** .

- ✓ On the string theory side we used the prescription of **[Zarembo, arXiv:1008.1059]**

Conclusion

The study of the 3-point functions in the planar limit of the AdS/CFT correspondence is a highly fascinating new avenue to follow and that it would be very interesting if the techniques of integrability could be extended to this as well.

With this in hand, one could possibly understand how the 3-point coefficients can interpolate from weak to strong coupling.

This is clearly an interesting problem that deserves further investigation also in view of the fact that a similar comparison between the weak and the strong coupling result in the case of 2-point correlation functions was crucial in establishing a connection between the two opposite regimes.

A similar study in the case of 3-point correlation functions would be important in deriving an all loop result.