Nonlocality: Aspects and Consequences
Stockholm, June 28, 2012

Gauge theories on canonically deformed spaces

Michael Wohlgenannt (Vienna)

Content

Introduction

- NC coordinates
- UV/IR mixing in scalar NCQFT
- Grosse-Wulkenhaar model

NC gauge theory

- Induced gauge model
- On a curved background

Aspects of Gravity

Open questions

Introduction

We consider 4D deformed Euclidean space:

$$[\hat{x}^i, \hat{x}^j] = \mathrm{i}\theta^{ij}(\hat{x})\,,$$

$\theta^{ij} = -\theta^{ji} = \mathrm{const},$

- $\theta^{ij}(\hat{x}) = const$, canonical
- $\theta^{ij}(\hat{x}) = \kappa c^{ij}_k \hat{x}^k$, kappa
- $\theta^{ij}(\hat{x}) = (\frac{1}{q}\hat{R}^{ij}_{kl} - \delta^i_l \delta^j_k)\hat{x}^k \hat{x}^l$, quantum groups

Introduction

Let us choose canonical relations. Can be represented as

$$[x^i \stackrel{\star}{,} x^j] = \mathrm{i}\theta^{ij} \,,$$

with Weyl-Moyal $\star$-product

$$f \star g\,(x) = \mathrm{e}^{\frac{\mathrm{i}}{2}\Theta^{ij}\partial_i^x \partial_j^y} f(x)\, g(y)\Big|_{y\to x} \,.$$

$\Rightarrow$ non-local product

Introduction

NC scalar ϕ^4

$$S = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\,\partial_\mu\phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi \right)$$

Feynman rules:

- propagator $G(p) = \frac{1}{p^2+m^2}$
- vertex function $\Gamma(p_1,\ldots,p_4) = \lambda\delta^{(4)}(p_1+p_2+p_3+p_4)\mathrm{e}^{-\mathrm{i}\sum_{i<j} p_i\Theta p_j}$

Introduction

2-point tadpole

$$\Pi(\Lambda, p) \propto \int d^4k \frac{2 + \cos k\tilde{p}}{k^2 + m^2} = \Pi^{UV}(\Lambda) + \Pi^{IR}(\Lambda, p)$$

with the IR-divergent non-planar part

$$\Pi^{IR} \sim \frac{1}{\tilde{p}^2}$$

$\tilde{p}_\mu = \Theta_{\mu\nu} p_\nu$;
not yet a problem: $\int d^4p\, \tilde{\phi}(p) \frac{1}{\tilde{p}^2} \tilde{\phi}(-p)$
but higher loop insertions yields: $\int d^4p\, \tilde{\phi}(p) \frac{1}{(\tilde{p}^2)^n} \tilde{\phi}(-p)$
UV/IR mixing destroys renormalizability.

Introduction

2 different strategies to cure UV/IR mixing:

1 - Adding an oscillator potential (Grosse, Wulkenhaar 03, 05):

$$S = \int d^D x \left(\frac{1}{2} \phi \star [\tilde{x}_\nu, [\tilde{x}^\nu, \phi]_\star]_\star + \frac{\Omega^2}{2} \phi \star \{\tilde{x}^\nu, \{\tilde{x}_\nu, \phi\}_\star\}_\star + \frac{\mu^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right)(x) ,$$

where $\tilde{x}_\nu = \theta^{-1}_{\nu\alpha} x^\alpha$ and $i\partial_\mu f = [\tilde{x}_\mu, f]_\star$

2 - Adding a non-local term (Gurau, Magnen, Rivasseau, Tanasa 08):

$$S_{nl} = \int d^4 p \frac{a}{2} \phi(p) \frac{1}{\tilde{p}^2} \phi(-p)$$

Introduction

Properties of the oscillator model:

- Langmann-Szabo duality

- no UV/IR mixing due to oscillator term; propagator given by the Mehler kernel - IR damping implemented

$$K_M(p,q) = \frac{\omega^3}{8\pi^2} \int_0^\infty \frac{d\alpha}{\sinh^2 \alpha} \mathrm{e}^{-\frac{\omega}{4}(p-q)^2 \coth \frac{\alpha}{2} - \frac{\omega}{4}(p+q)^2 \tanh \frac{\alpha}{2}}$$

- theory perturbatively renormalisable, vanishing β function (Rivasseau et al. 2006)

- Oscillator term can be interpreted as coupling of the scalar field to the curvature of a NC background (Buric, MW 08)

Introduction

- Moreover, there are hints that this model can be constructed non-perturbatively, at least at the self-dual point $\Omega = 1$ (Grosse, Wulkenhaar 2012).

  They obtain a non-linear equation for the function G alone, at $\lambda \neq 0$.

  Then, two- and four-point functions can be expressed entirely in terms of G.

- Minkowski space: Rank 2, i.e. commutative time (Grosse, MW 2012)

  Analytic continuation of the 1-loop contributions

  Fixed point, i.e. $\beta_\lambda = 0$, as in Euclidean case with full rank

Remark on localization

Grosse, Lechner 2007:

Consider free NC scalar fields on NC Minkowski space. Some localisation is indeed present:

$$[\Phi_W(x), \Phi_{\tilde{W}}(y)] = 0\,,$$

if the wedges $(W + x)$ spacelike to $(\tilde{W} + y)$

$$\mathcal{W}_0 := \{\Lambda W_1 : \Lambda \in \mathcal{L}\}\,,$$

where W_1 is the reference wedge

$$W_1 := \{x \in \mathbb{R}^4 : x_1 > |x_0|\}\,.$$

Induced NC gauge theory

gauge invariant action

$$S = \int d^D x \left(\frac{1}{2} \phi \star [\tilde{X}_\nu, [\tilde{X}^\nu, \phi]_\star]_\star + \frac{\Omega^2}{2} \phi \star \{\tilde{X}^\nu, \{\tilde{X}_\nu, \phi\}_\star\}_\star + \frac{\mu^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right)(x) \,,$$

where $\tilde{X}_\mu = \tilde{x}_\mu + A_\mu$ are covariant coordinates, and

$$\begin{aligned} \phi &\mapsto u^* \star \phi \star u \,, \\ A_\mu &\mapsto \mathrm{i} u^* \star \partial_\mu u + u^* \star A_\mu \star u \,, \\ \tilde{X}_\mu &\mapsto u^* \star \tilde{X}_\mu \star u \,. \end{aligned}$$

Induced gauge theory

heat kernel expansion

$$\Gamma^{\epsilon}_{1l}[\phi] = -\frac{1}{2}\int_{\epsilon}^{\infty}\frac{dt}{t}\,\mathrm{Tr}\left(e^{-tH} - e^{-tH^0}\right) \; .$$

where we use the effective potential

$$\frac{\delta^2 S}{\delta\phi^2} \equiv H = \frac{2}{\theta}H^0 + V$$

H^0 field independent; field dependent terms contained in V.
The method is not manifestly gauge invariant, contributions from different orders need to add up to a gauge invariant result.

Induced NC gauge theory

The action contains an oscillator term

$$\frac{\Omega^2}{2}\phi \star \{\tilde{X}^\nu, \{\tilde{X}_\nu, \phi\}_\star\}_\star \,.$$

This term is crucial, it alters the free theory.

Therefore, we expand around the free action $-\Delta + \Omega^2\tilde{x}^2$ rather than $-\Delta$.
Seeley-de Witt coefficients cannot be used!! (e.g. Vassilevich 04)

$$Tr e^{-tH} \simeq \sum_n t^{\frac{n-4}{2}} \int_M d^4x \sqrt{g}\, a_n(x, H)$$

Induced NC gauge theory

Induced gauge action (de Goursac, Wallet, Wulkenhaar 07; Grosse, MW 07)

$$
\begin{aligned}
S \quad = \quad & \int d^4x \bigg\{ \frac{3}{\theta}(1-\rho^2)(\tilde{\mu}^2-\rho^2)(\tilde{X}_\nu \star \tilde{X}^\nu - \tilde{x}^2) \\
& \qquad + \frac{3}{2}(1-\rho^2)^2 \big((\tilde{X}_\mu \star \tilde{X}_\mu)^{\star 2} - (\tilde{x}^2)^2\big) - \frac{\rho^4}{4} F_{\mu\nu} F_{\mu\nu} \bigg\},
\end{aligned}
$$

where $F_{\mu\nu} = -i[\tilde{x}_\mu, A_\nu]_\star + i[\tilde{x}_\nu, A_\mu]_\star - i[A_\mu, A_\nu]_\star$
$\tilde{X}_\mu = \tilde{x}_\mu + A_\mu, \;\; \rho = \frac{1-\Omega^2}{1+\Omega^2}, \; \tilde{\mu}^2 = \frac{m^2\theta}{1+\Omega^2}$

Induced NC gauge theory

- $\Omega \to 0$ $(\rho \to 1)$: usual NCYM
- $\Omega \to 1$ $(\rho \to 0)$: obtain interesting matrix models
- non-trivial vacuum

Induced NC gauge theory

- For 1-loop calculation introduce an x-dependent gauge fixing in order to eliminate terms linear in A:

$$S_{gf} = \int d^4xs(-i\bar{c}\star f - \frac{\alpha}{2}\bar{c}\star B)$$

with

$$f = \tilde{x}_\mu A_\mu + \beta\tilde{x}^2 + \gamma$$

$\alpha = \frac{1}{2}$, $\beta = \frac{\alpha}{2g}$, $\gamma = \frac{\kappa\alpha}{2g}$

- 1-loop calculation, additional counter terms arise, also terms linear in A again

$$\tilde{x}^2\tilde{x}_\mu A_\mu\,,\ \tilde{x}_\mu A_\mu\,,\ (\tilde{x}_\mu A_\mu)^2$$

Curved gauge model

Truncated Heisenberg algebra:

$$[\mu\hat{x}^1, \mu\hat{x}^2] = i\epsilon(1 - \bar{\mu}\hat{x}^3)\,,$$
$$[\mu\hat{x}^1, \bar{\mu}\hat{x}^3] = i\epsilon(\mu\hat{x}^2\bar{\mu}\hat{x}^3 - \bar{\mu}\hat{x}^3\mu\hat{x}^2)\,,$$
$$[\mu\hat{x}^2, \bar{\mu}\hat{x}^3] = -i\epsilon(\mu\hat{x}^1\bar{\mu}\hat{x}^3 - \bar{\mu}\hat{x}^3\mu\hat{x}^1)\,,$$

limit $\bar{\mu} \to 0$ leads to 2D Heisenberg algebra

This algebra has non-vanishing curvature, computed using frame formalism of J. Madore and M. Buric.

We are interested in the 2-dimensional limit, $\hat{x}^3 \to 0$, where the curvature survives.

$$S = \int d^2x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{M^2}{2} \phi^2 + \frac{\xi}{2} R \phi^2 + \frac{\lambda}{4!} \phi^{\star 4} \right)$$

where $R = \frac{15\mu^2}{2} - 8\mu^4(x^2 + y^2)$.

Curved gauge model

Differential calculus remains 3-dimensional (as e.g. for the fuzzy sphere) $\{A_1, A_2, A_3\} \to \{A_1, A_2, \phi\}$.

Action proposed by Buric, Grosse, Madore 10

$$
\begin{aligned}
S &= \int d^2x \Big((1-\alpha^2) F_{12}^{*2} - 2(1-\alpha^2)\mu\, F_{12} \star \phi + (5-\alpha^2)\mu^2\phi^2 \\
&\quad +4i\alpha F_{12} \star \phi^{\star 2} + (D_i\phi)^2 - \alpha^2 \{p_i + A_i \overset{\star}{,} \phi\}^2 \Big)
\end{aligned}
$$

Curved gauge model

- 1-loop calculations leads to (logarithmic) IR divergences only (Buric, Dimitrijevic, Radovanovic, MW 2012):

$$\int d^2\phi\,,\int d^2x\tilde{x}_\mu\star A_\mu$$

$$\int d^2xA_\mu\star A_\mu\,,\int d^2x\phi\star\phi\,,\int d^2x\{\tilde{x}_\mu\stackrel{\star}{,}A_\mu\}\star\phi$$

- no UV divergences

Aspects of gravity

$$g_l(x) \star f(x) \star g_l^{-1}(x) = f(x+l)\,,$$

with

$$g_l = e_\star^{-il^k \theta_{kj}^{-1} x^j}$$

Aspects of gravity

Emergent gravity (Steinacker et al. 2008/09, Yang 2008/09)

Reinterpretation of UV/IR mixing in terms of gravity.

Starting point

$$S = Tr\frac{1}{2}g_{ab}[X^a, \Phi][X^b, \Phi]\,,$$

for $n \times n$ matrices with $[X^a, X^b] = i\theta^{ab}(X)$

In the semiclassical limit, this can be written as

$$S \sim \int d^4x \sqrt{|G_{\mu\nu}|}G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\,,$$

with $G^{\mu\nu} = \theta^{\mu\alpha}\theta^{\nu\beta}g_{\alpha\beta}$ and $[X^\mu, \Phi] \sim i\theta^{\mu\nu}(x)\partial_\nu\phi$.

Aspects of gravity

- Expand around Weyl-Moyal vacuum

$$X^a = Y^a - \bar\theta^{ab} A_a \,,$$

where $[Y^a, Y^b] = i\bar\theta^{ab} = const.$

Therefore e.g.

$$i\theta^{ab}(x) = i\bar\theta^{ab} - i\bar\theta^{ac}\bar\theta^{bd}F_{cd}$$

- compute 1-loop effective action using Seeley-de Witt coefficients and express the effective coupling in terms of the field strength

- second approach: compute 1-loop effective action for NC $U(1)$ gauge theory by calculating Feynman graphs explicitly

$\Rightarrow$ both approaches coincide

Concluding remark

- Similar to the scalar case, overcome 1-loop order and find a renormalization scheme to all orders

  algebraic renormalization

  multiscale analysis

Questions

- How can one extract the gravity degrees of freedom from NC $U(1)$ gauge theory?
- Is a better localization than on wedges possible in NC field theory?