Shape of the Anomaly

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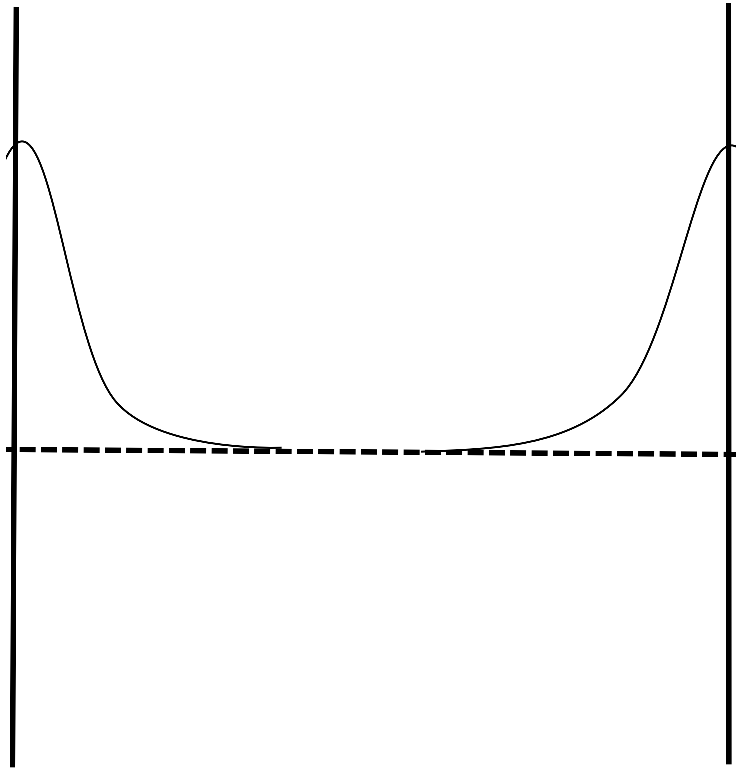
Fermilab

Compactification and Chiral Fermions

$$\int_I d^4x \, \overline{\psi}_L i \not{D}_L \psi_L$$



$$\int_{II} d^4x \, \overline{\psi}_R i \not{D}_R \psi_R$$

How does anomaly cancellation work in extra dimensions?

Lightning Review of Chiral Anomaly

$$S = \int d^4x\, \overline{\psi}_L (i\partial\!\!\!/ - A\!\!\!/_L)\psi_L - (1/4)F_{\mu\nu}F^{\mu\nu}$$

$$\partial_\mu F^{\mu\nu} = -j_L^\nu \,, \qquad j_{\mu L} = \overline{\psi}_L \gamma_\mu \psi_L .$$

The antisymmetry of $F_{\mu\nu}$ in turn forces:

$$\partial_\mu j_L^\mu = 0$$

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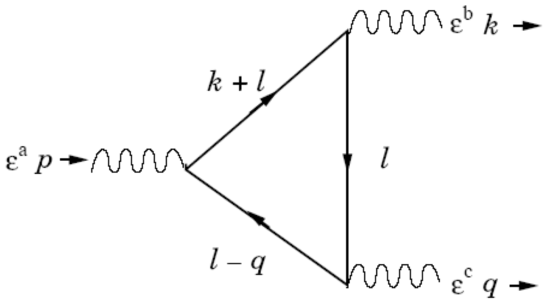
$$\partial_\mu F^{\mu\nu} = -j^\nu_L\ , \qquad j_{\mu L} = \overline{\psi}_L \gamma_\mu \psi_L .$$

But - massless spinor loop is anomalous:

$$\partial_\mu j^\mu_L = -\frac{1}{24\pi^2} dA_L dA_L$$



$$dAdA = \epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu \partial_\rho A_\sigma$$

Gauge invariance is destroyed by the anomaly, and as a consequence, under the gauge transformation,

$$\psi_L \to e^{i\theta}\psi_L \qquad A_L \to A_L - \partial\theta$$

the action shifts by the anomaly:

$$S \to S + \frac{1}{24\pi^2}\int d^4x\, \theta dA_L dA_L$$

$$dAdA = \epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu \partial_\rho A_\sigma$$

A single Weyl fermion coupled to a gauge field is incompatible with quantum mechanics

We can define massless electrodynamics as

$$S = \int d^4x\, \overline{\psi}_L(i\partial\!\!\!/ - A\!\!\!/)\psi_L + \overline{\psi}_R(i\partial\!\!\!/ - A\!\!\!/)\psi_R$$

The gauge anomalies cancel between the L and R spinors. Defining $j = j_L + j_R$ and $j^5 = j_R - j_L$ we have consistent anomalies:

"consistent" anomalies:

$$\partial_\mu j^\mu = 0 \qquad \partial_\mu j^{\mu 5} = \frac{1}{12\pi^2} dAdA$$

and we might be tempted to conclude that the physical axial anomaly is $1/24\pi^2 F_{\mu\nu}\tilde{F}^{\mu\nu}$ (note $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$). This is false, however, for a subtle reason, as we now see.

Let us introduce a background auxilliary axial vector field B_μ coupled to the axial current as $B_\mu j^{\mu 5}$. This is a gauge invariant "perturbation" on the theory. We write the action as:

$$S = \int d^4x\, \overline{\psi}_L(i\partial\!\!\!/ - A\!\!\!/ + B\!\!\!/\,)\psi_L + \overline{\psi}_R(i\partial\!\!\!/ - A\!\!\!/ - B\!\!\!/\,)\psi_R \quad (12)$$

We see that $-\delta S/\delta B_\mu = j^5_\mu$ and $-\delta S/\delta A_\mu = j_\mu$.

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We see that $-\delta S/\delta B_\mu = j^5_\mu$ and $-\delta S/\delta A_\mu = j_\mu$.

Indeed, with $A_L = A - B$ and $A_R = A + B$

$$\partial_\mu j^\mu = \frac{1}{6\pi^2} dAdB \qquad \partial_\mu j^{\mu 5} = \frac{1}{12\pi^2}\left(dAdA + dBdB\right)$$

Add a Counterterm: (Adler,Bardeen)

$$S' = \frac{1}{6\pi^2} \int d^4x \; ABdA$$

variation wrt B or A the counterterm adds corrections to the vector and axial currents:

$$-\frac{\delta S'}{\delta A_\mu} \;=\; \delta j^\mu = -\frac{1}{3\pi^2}\epsilon_{\mu\nu\rho\sigma}B^\nu\partial^\rho A^\sigma + \frac{1}{6\pi^2}\epsilon_{\mu\nu\rho\sigma}A^\nu\partial^\rho B^\sigma$$

$$-\frac{\delta S'}{\delta B_\mu} \;=\; \delta j^{5\mu} = \frac{1}{6\pi^2}\epsilon_{\mu\nu\rho\sigma}A^\nu\partial^\rho A^\sigma$$

"covariant anomalies"

The full currents, $\tilde{j} = j + \delta j$, now satisfy:

$$\partial_\mu \tilde{j}^\mu \;=\; 0\,, \qquad \partial_\mu \tilde{j}^{5\mu} = \frac{1}{4\pi^2}\left(dAdA + \frac{1}{3}dBdB\right).$$

Mass term:

$$\partial^\mu \widetilde{j}_\mu = 0$$
$$\partial^\mu \widetilde{j}^5_\mu - 2im\overline{\psi}\gamma^5\psi = \frac{1}{4\pi^2}(dAdA + \frac{1}{3}dBdB)$$

$m^2 \to \infty$

In this limit we find that the fermionic current divergences decouple. The anomaly is now carried entirely by the mass-term contributions

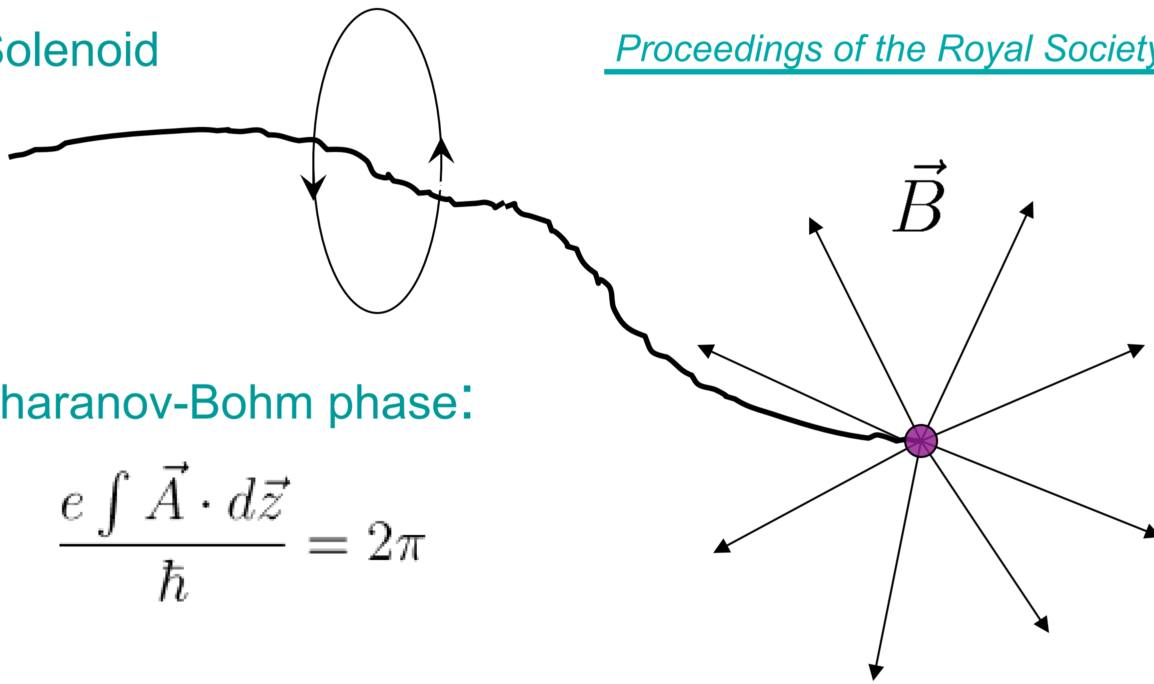
$$-\langle 0|\, 2im\overline{\psi}\gamma^5\psi\, |\gamma_1, \gamma_2\rangle = \langle 0|\, \frac{1}{4\pi^2}(dAdA + \frac{1}{3}dBdB)\, |\gamma_1, \gamma_2\rangle$$

$$\langle 0|\, \partial^\mu \widetilde{j}_\mu\, |\gamma_1, \gamma_2\rangle = \langle 0|\, \partial^\mu \widetilde{j}^5_\mu\, |\gamma_1, \gamma_2\rangle = 0 \qquad m^2 \to \infty$$

Bosonic Origin of Anomaly: Dirac Monopole

P.A.M. Dirac, "Quantized Singularities in the Electromagnetic Field", *Proceedings of the Royal Society*, **A133** (1931) pp 60-72.

Solenoid



Aharanov-Bohm phase:

$$\frac{e\int \vec{A}\cdot d\vec{z}}{\hbar} = 2\pi$$

$$\int \vec{B}\cdot d\vec{S} = 4\pi g$$

Stoke's Theorem: $\int \vec{B}\cdot d\vec{S} = \int \vec{A}\cdot d\vec{z}$ $\Rightarrow$ $\boxed{g = \frac{\hbar}{2e}}$

Electric charge quantization

Turn on Chern-Simons term:

$$\int d^3\vec{x}\; c\, \epsilon_{ijk} A^i \partial^j A^k$$

Turn on Chern-Simons term:

$$\int d^3\vec{x}\; c\, \epsilon_{ijk} A^i \partial^j A^k$$

$$\partial^2 A_i - \partial_i \partial_j A^j = c\, \epsilon_{ijk} \partial^j A^k \quad \Longrightarrow \quad \nabla^2 \vec{A} = c\vec{B}$$

$$\nabla^2(\vec{\nabla} \times \vec{A}) = c\vec{\nabla} \times \vec{B} = c\nabla^2 \vec{A} \qquad \boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$$

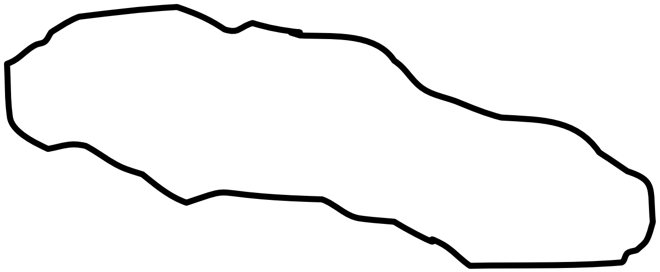
$$\Longrightarrow \quad \nabla^2 \vec{B} = c^2 \vec{B}$$

Chern-Simons term is a mass term in D=3

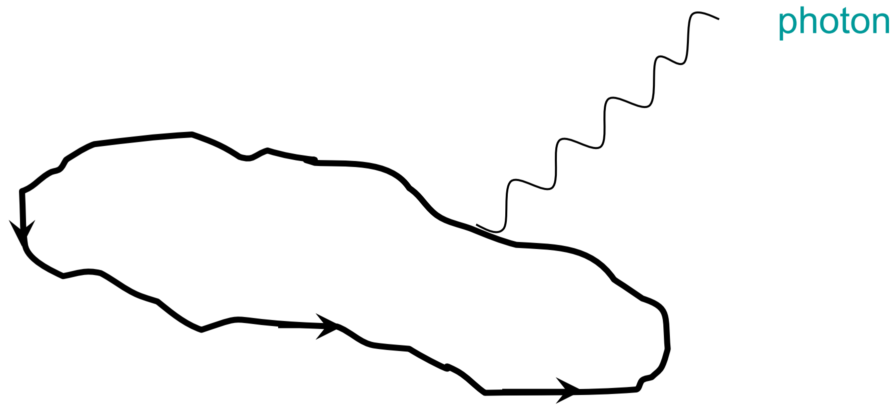
But: There are no Yukawa-Monopole Solutions with CS term!!!

Discard the Monopole

Consider solenoid as a closed string:



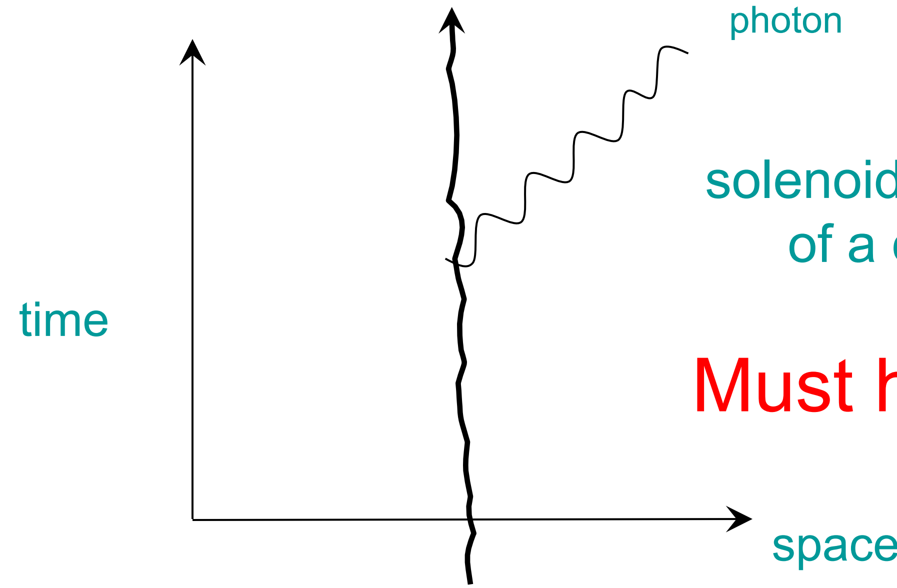
Solenoid loop becomes a physical current loop when Chern-Simons term is present



$$\int d^3\vec{x}\; c\; \epsilon_{ijk} A^i \partial^j A^k \quad \Rightarrow \quad \int d^3x\; A^i J_i$$

$$J_i = c\; \epsilon_{ijk} \partial^j A^k$$

Consider Minkowski space D=1+2



solenoid is now a world-line of a charged particle

Must have charge e!

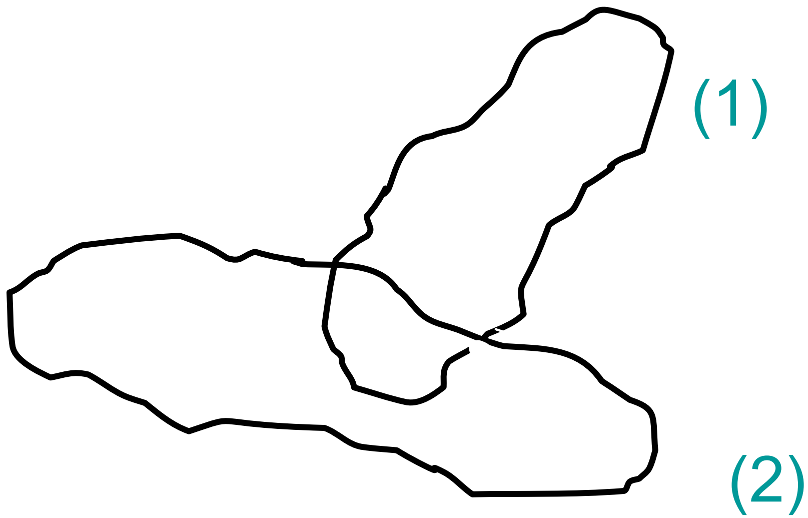
$$\int d^3\vec{x}\; c\; \epsilon_{ijk} A^i \partial^j A^k \Rightarrow c\int ds\; A^i J_i = 2\times\left(\mathbf{c}\right)\times\left(\frac{2\pi}{e}\right)\int dx^\mu\cdot A_\mu$$

$$=\boxed{e}\int dx^\mu \cdot A_\mu \Rightarrow$$

Chern-Simons term coefficient is determined by demanding world-line charge is e $\Rightarrow$

$$\boxed{c=\frac{e^2}{4\pi}}$$

Gauss linking of two solenoid strings:



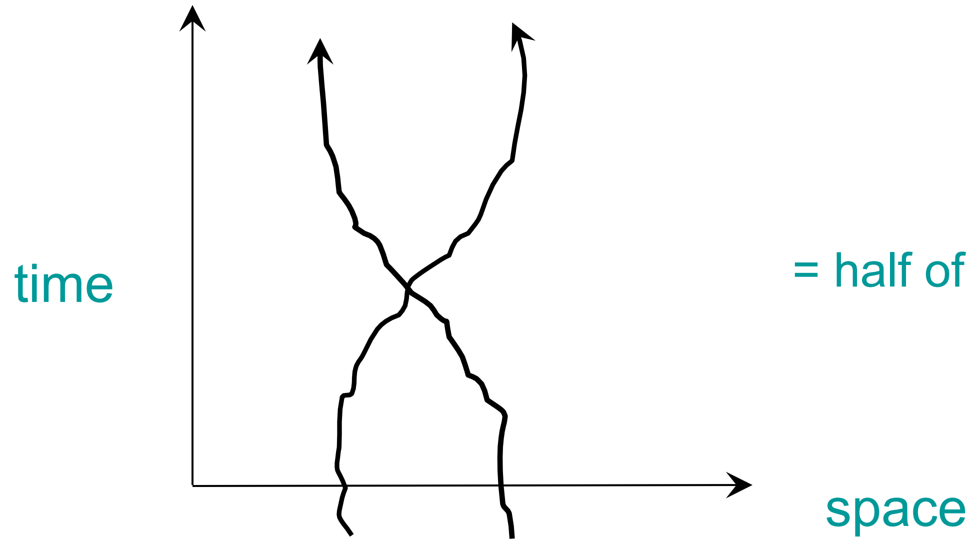
$$\int d^3\vec{x}\; c\; \epsilon_{ijk} A^i \partial^j A^k \quad \Rightarrow \quad c\int A_1 dA_2 + A_2 dA_1 = 2c \times \left(\frac{2\pi}{e}\right)\left(\frac{2\pi}{e}\right)$$

$$= 2\pi$$

$$\Rightarrow \quad \boxed{c = \frac{e^2}{4\pi}}$$

Chern-Simons term coefficient is determined by demanding action under linking shifts by multiple of 2π

Exchange two “solenoid particles”



= half of a Gauss linking

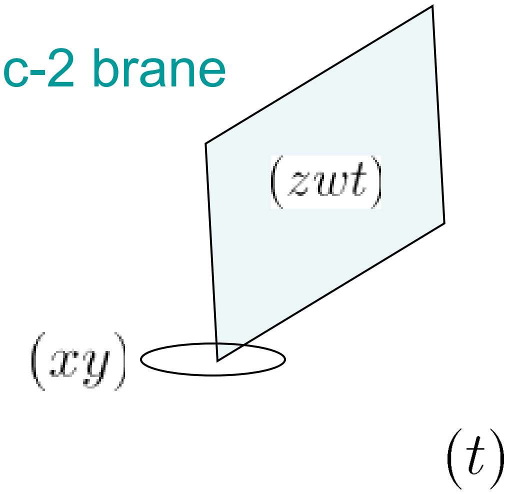
$$Nc \int d^3x \, \epsilon_{ijk} A^i \partial^j A^k \quad \Rightarrow \quad N\pi$$

solenoid is a fermion for odd N

solenoid is an anyon for arbitrary c

D=5 QED: Generalized Dirac Solenoid Construction:

Dirac-2 brane

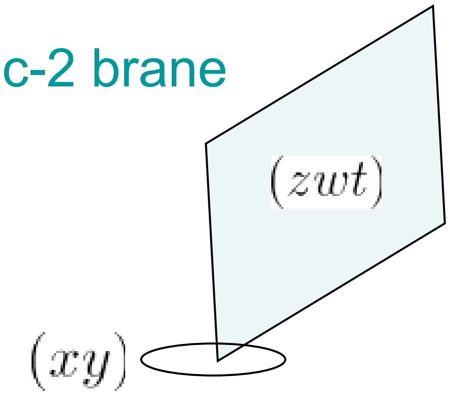


Dirac Branes

generalization of solenoid:
"stack" of current loops in xy plane
yields zwt Dirac-2 brane carrying:

$$\tilde{F}^{zwt} = \epsilon_{xyzwt} F^{xy}$$

D=5 QED: Generalized Dirac Solenoid Construction:



Dirac Branes

generalization of solenoid:
A “stack” of current loops in
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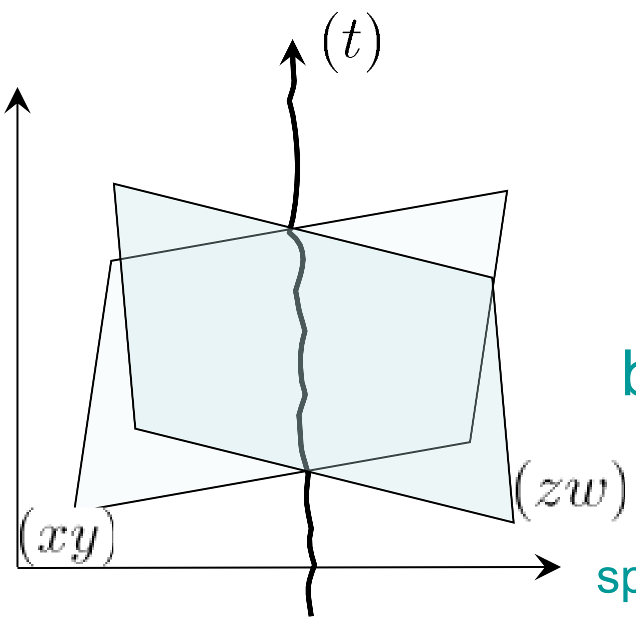
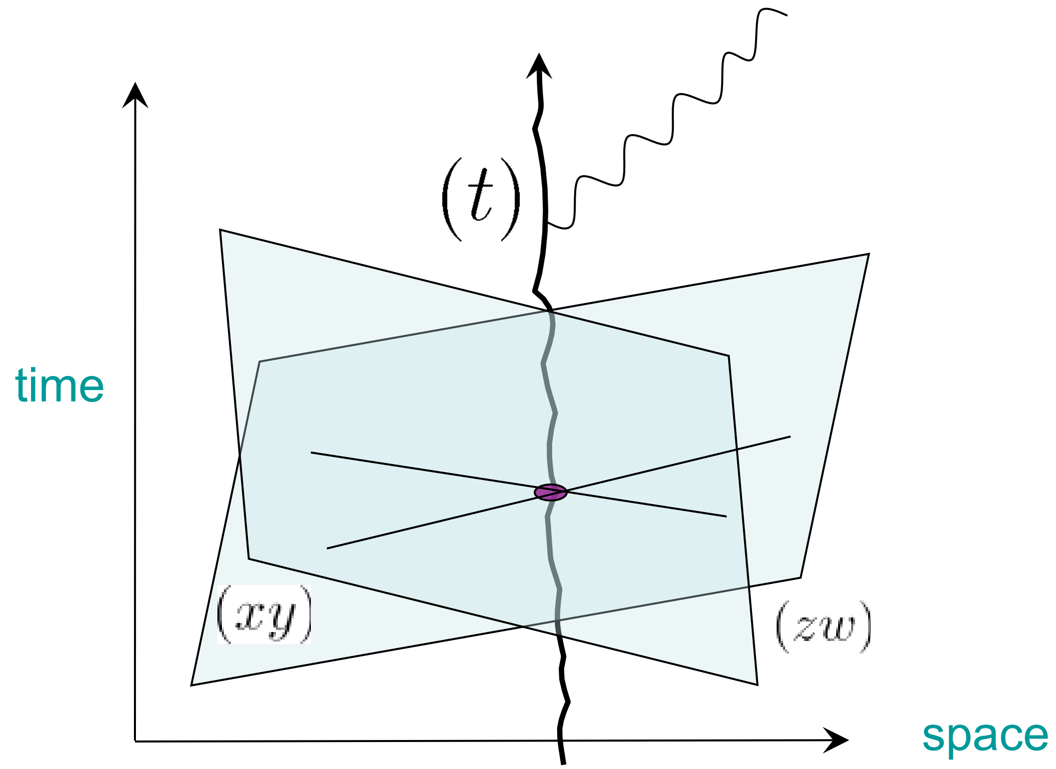
$$\tilde{F}^{zwt} = \epsilon_{xyzwt} F^{xy}$$



intersection of two Dirac-2 branes is a world-line with current

$$J_t = \epsilon_{xyzwt} F^{xy} F^{zw}$$

D=5 QED: Generalized Dirac Solenoid Construction:



Turn on Chern-Simons term:

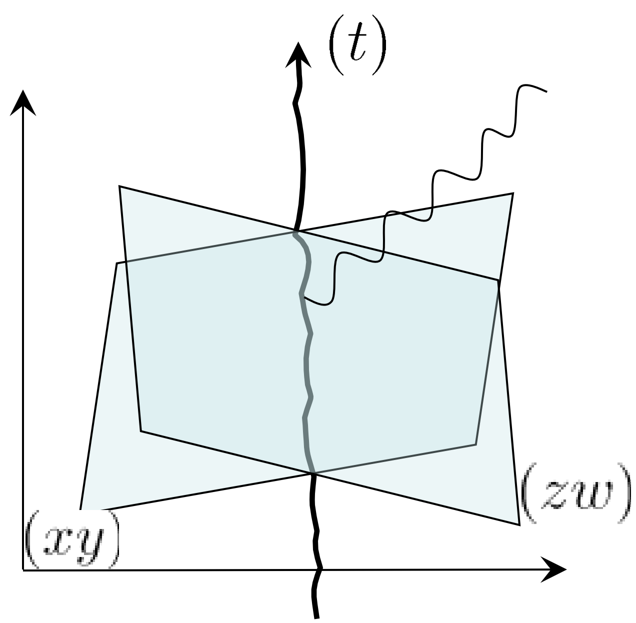
$$c\,\epsilon_{ABCDE}A^A\partial^B A^C\partial^D A^E$$

world-line becomes charged!

Compute Chern-Simons Coefficient:

$$c\,\epsilon_{ABCDE}\int d^5x \;\; < \Phi_{xy}\Phi_{wz}, A_\mu | A^A \partial^B A^C \partial^D A^E | \Phi_{xy}\Phi_{wz} >$$

$$= 3\cdot 2\, c \int dx^0\, A_0^\gamma \int dxdy\, \partial_x A_y \int dzdw\, \partial_z A_w$$



$$= 3\cdot 2\, c \int dx^0\, A_0^\gamma \left(\frac{2\pi}{e}\right)\left(\frac{2\pi}{e}\right)$$

$$\equiv e \int dx^0\, A_0^\gamma$$

$$\boxed{c = \frac{e^3}{24\pi^2}}$$

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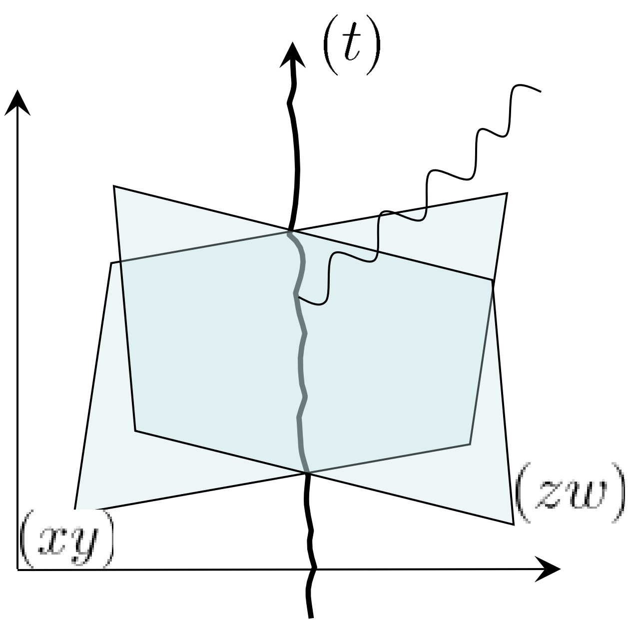

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$$\equiv e \int dx^0\, A_0^\gamma$$

$$\boxed{c = \frac{e^3}{24\pi^2}}$$

Recall: $\partial_\mu j_L^\mu = -\frac{1}{24\pi^2} dA_L dA_L$

Generalize to any odd D:

$$c\,\epsilon_{ABC...DE}\int d^5x\ \ < \Phi_{xy}\Phi_{wz}, A_\mu|A^A\partial^B A^C...\partial^D A^E|\Phi_{xy}\Phi_{wz} >$$

$$= [((D+1)/2)!]\,c\,\left(\frac{2\pi}{e}\right)^{(D-1)/2}\int\ A_0 dx^0$$

Demanding that this yield the defining electric charge e determines the Chern-Simons term coefficient:

$$c = \frac{e^{(D+1)/2}}{[((D+1)/2)!]\,(2\pi)^{(D-1)/2}}$$

= Consistent Anomaly Coefficient in D-1;

add counterterm and obtain covariant anomaly in D:

$$\tilde{c} = \frac{2}{(2\pi)^{D/2}(D/2)!}$$

consistent with Frampton and Kephart PRD, 28, 1010

Compare Adler's result: $\partial(\tilde{j}_5) = \frac{1}{8\pi^2}F\tilde{F} = \tilde{c}\,(dV dV)$

QED in D=5:

$$S_{fermion} = \int_{-\infty}^{+\infty} dy \int d^4x\, \overline{\psi}(x,y) \left(i\partial\!\!\!/ + A\!\!\!/\,(x,y) - \gamma^5 \partial_y + i\gamma^5 A_5(x,y) + m(y) \right) \psi(x,y)$$

gauge transformations

$$\psi(x,y) \to e^{i\theta(x,y)} \psi(x,y)$$

$$A_\mu(x,y) \to A_\mu(x,y) + \partial_\mu \theta(x,y)$$

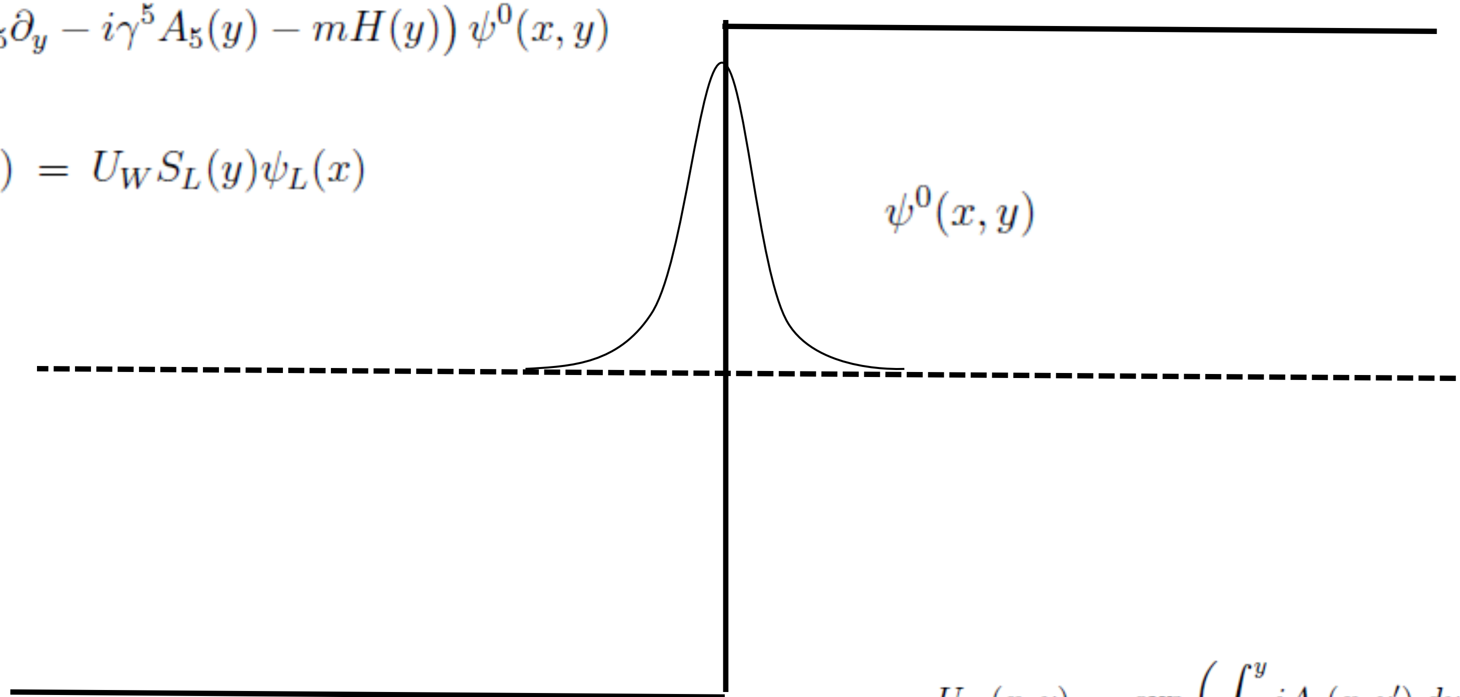
$$A_5(x,y) \to A_5(x,y) + \partial_y \theta(x,y)$$

Fermionic zero mode

$m(y) = mH(y) \qquad H(y) = 1, \text{ for } y \geq 0; \qquad H(y) = -1 \text{ for } y < 0;$

$$0 = \left(\gamma_5 \partial_y - i\gamma^5 A_5(y) - mH(y)\right) \psi^0(x,y)$$

$$\psi^0(x,y) \;=\; U_W S_L(y) \psi_L(x)$$



$$U_W(x,y) \;=\; \exp\left(\int_0^y iA_5(x,y')\,dy'\right)$$

$$Z_L^{1/2}(y) \;=\; \frac{1}{\sqrt{2m}} \exp\left(-\int_0^y m(y')\,dy'\right) = \frac{1}{\sqrt{2m}} \exp\left(-m|y|\right)$$

$$\int_{-\infty}^{+\infty} dy\; Z_L(y,y_0) = 1$$

gauge transformation property on $\psi_L(x)$

$$\psi^0(x,y) \rightarrow e^{i\theta(x,y)}\psi^0(x,y)$$
$$\psi_L(x) \rightarrow e^{i\theta(x,0)}\psi_L(x)$$

Truncating on the the zero-mode fermion field, the effective action is then:

$$S_f \;=\; \int_{\infty}^{\infty} dy\; d^4x\; \overline{\psi}_L(x)\,(i\partial\!\!\!/ + B\!\!\!/\,(x,y))\,\psi_L(x)\; Z_L(y)$$

$B_\mu(x,y)$ is a gauge transformed vector potential, given by:

$$B_\mu(x,y) = A_\mu(x,y) - \partial_\mu \int_0^y A_5(x,y')\; dy'$$

$$B_5(x,y) = A_5(x,y) - \partial_y \int_0^y A_5(y')dy' = 0$$

Theory is anomalous on the brane:

$$
\begin{aligned}
\delta S_f &= -\int d^4x\, \theta(x)\partial_\mu J_L^\mu \\
&= \frac{1}{24\pi^2}\int d^4x\, \theta(x)\epsilon^{\mu\nu\rho\lambda}\partial_\mu \overline{B}_{L\nu}(x)\partial_\rho \overline{B}_{L\lambda}(x) \\
&= \frac{1}{24\pi^2}\int d^4x\, \theta(x) d\overline{B}_L d\overline{B}_L
\end{aligned}
$$

$$
\overline{B}_{\mu L} = \int_0^a Z_L(y) B_\mu(x,y)\, dy
$$

Can we fix this with the Chern-Simons Term?

$$
\partial_\mu J_L^\mu = -\frac{1}{48\pi^2} F_{B_L\mu\nu}\widetilde{F}^{\mu\nu}_{B_L}\,, \qquad J_L^\mu = \overline{\psi}^0\gamma^\mu\psi_L^0\,, \qquad \widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}.
$$

Chern-Simons Term

$$S_{CS} = c\int d^5x\, \epsilon^{ABCDE} A_A \partial_B A_C \partial_D A_E$$

$$A_A \to A_A + \partial_A \theta$$

$$\delta S_{CS} = c\int d^4x\, \theta(x)\, \epsilon^{\mu\nu\rho\lambda}\partial_\mu B_\nu(x,0)\partial_\rho B_\lambda(x,0) - c\int d^4x\, \theta(x)\, \epsilon^{\mu\nu\rho\lambda}\partial_\mu B_\nu(x,\infty)\partial_\rho B_\lambda(x,\infty)$$

The behavior on the surface at infinity leads to a difficulty here. Technically, the zero-mode gauge fields, those with no y-momentum, are constants in y, so $\int \theta dB^0 dB^0(0) = \int \theta dB^0 dB^0(\infty)$. Hence the Chern-Simons term anomaly for the zero-modes, B^0, cancels between the $y = 0$ and $y = \infty$ boundaries with our chosen CS term.

Chern-Simons Term in A5=0 Gauge

$$S_{CS} = -2c\int_0^\infty dy \int d^4x\, \epsilon^{\mu\nu\rho\lambda} B_\mu(x,y)\partial_y B_\nu(x,y)\partial_\rho B_\lambda(x,y)$$

[We check that the residual gauge transformation generates the anomaly as before. With $\delta B_\mu(x,y) = \partial_\mu\theta(x)$ (note, $\partial_y\theta(x) = 0$), we have:

$$\begin{aligned}
\delta S_{CS} &= -2c\int_0^\infty dy\, d^4x\, \epsilon^{\mu\nu\rho\lambda}\, \partial_\mu\theta(x)\, \partial_y B_\nu(x,y)\, \partial_\rho B_\lambda(x,y) \\
&= c\int_0^\infty dy\, d^4x\, \theta(x)\, \epsilon^{\mu\nu\rho\lambda}\, \partial_y(\partial_\mu B_\nu \partial_\rho B_\lambda) \\
&= c\int d^4x\, \theta(x)\, \epsilon^{\mu\nu\rho\lambda}\, \partial_\mu B_\nu \partial_\rho B_\lambda \Big|_0^\infty \\
&= c\int d^4x\, \theta(x)(dB(\infty)dB(\infty) - dB(0)dB(0))
\end{aligned} \tag{26}$$

The problem of the thickness of the zero-mode

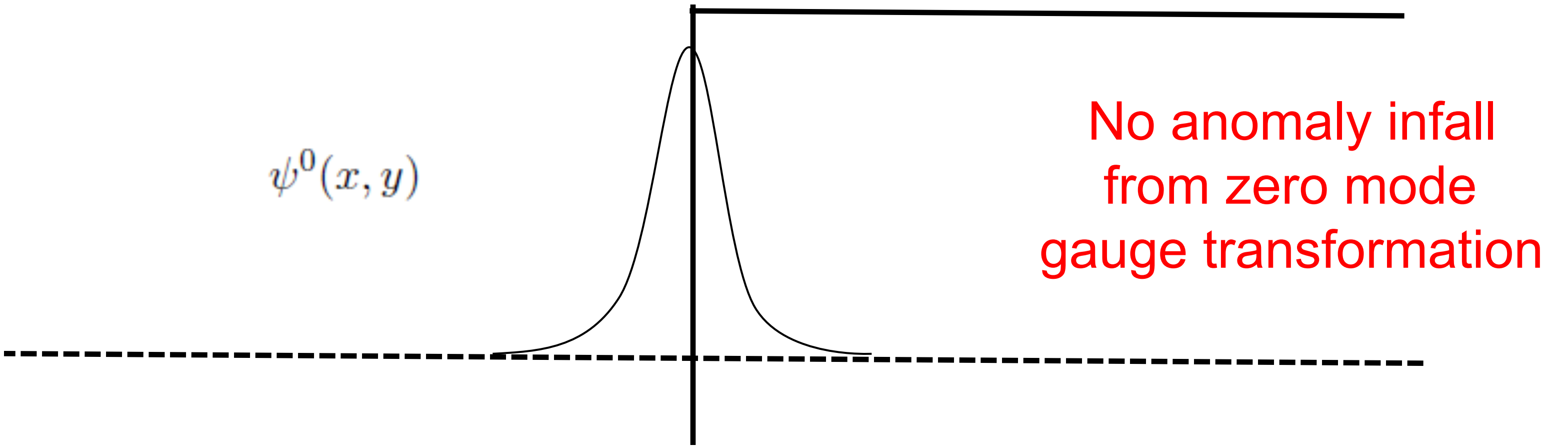
Our procedure is to replace S_{CS} by a modified S_{ZCS} given by:

$$S_{CS} \to S_{ZCS} \equiv -c\int_{-\infty}^{\infty} dy\; Z_L(y)\int_y^{\infty} dy'\;\int d^4x\;\epsilon^{\mu\nu\rho\lambda} B_\mu(x,y)\partial_y(B_\nu(x,y))\partial_\rho \overline{B}_\lambda(x)$$

$$\begin{aligned}
\delta S_{CS} &= -c\int_{-\infty}^{\infty} dy\; Z_L(y)\int_y^{\infty} dy'\;\int d^4x\;\epsilon^{\mu\nu\rho\lambda}\partial_\mu\theta(x)\partial_{y'}(B_\nu(x,y')\partial_\rho\overline{B}_\lambda(x)) \\
&= -c\int_{-\infty}^{\infty} dy\; Z_L(y)\int d^4x\;\epsilon^{\mu\nu\rho\lambda}\theta(x)\partial_\mu B_\nu(x,y)\partial_\rho\overline{B}_{L\lambda}(x) \\
&= -c\int d^4x\;\epsilon^{\mu\nu\rho\lambda}\theta(x)\partial_\mu(\overline{B}_{L\nu}(x) - B^0_\mu(x))\partial_\rho\overline{B}_{L\lambda}(x) \\
&= -c\int d^4x\; d(\overline{B}_{L\nu}(x) - B^0_\mu(x))d\overline{B}_{L\lambda}(x)
\end{aligned}$$

Note that we have had to treat the $\partial_y B$ term carefully, removing the zero mode B^0. This latter result identically cancels the anomaly for $Z_L(y) = \delta(y)$ of eq.(16), except for the $dB^0 dB^0$ piece. This also cancels the anomalies for a general $Z_L(y)$ as would occur in the action S_f of eq.(10).

Single isolated zero mode theory can't be fixed:



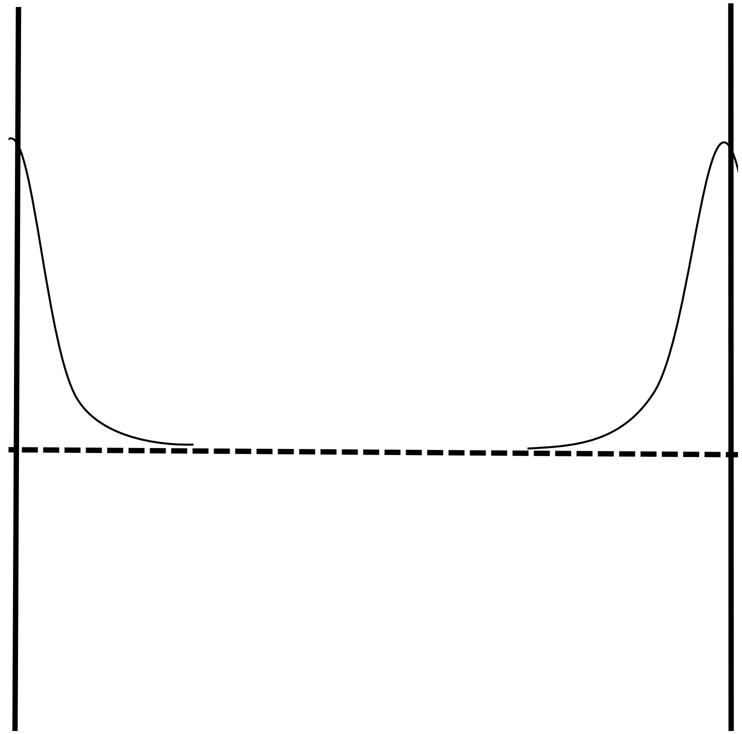
$$S_{CS} = c\int d^5x\, \epsilon^{ABCDE} A_A \partial_B A_C \partial_D A_E$$

$$A_A(x_\mu, y) \rightarrow A_A(x_\mu, y) + \partial_A \theta(x_\mu, y)$$

But, for zero mode: $\partial_y \theta(x) = 0$ $\delta S_{CS} = 0$

Compactification on the Orbifold, S_1/Z_2

$$\psi_L^0(x,y) \;=\; U_W Z_L^{1/2}(y)\psi_L(x)$$



$$\psi_R^0(x,y) \;=\; U_W Z_R^{1/2}(y)\psi_R(x)$$

$$0 = \left(\gamma_5\partial_y - i\gamma^5 A_5(y) - m\right)\psi^0(x,y)$$

$$U_W(x,y) \;=\; \exp\left(\int_0^y iA_5(x,y')\,dy'\right)$$

$$Z_L^{1/2}(y) \;=\; N\exp\left(-my\right)$$

$$Z_R^{1/2}(y) \;=\; N\exp\left(-m(a-y)\right)$$

The effective fermionic action, truncating on the zero-mode fermion fields is now:

$$S_f = \int_0^a dy\, d^4x \left[\overline{\psi}_L(x)\,(i\partial\!\!\!/ + B\!\!\!/\,(x,y))\,\psi_L(x)\; Z_L(y) \right.$$
$$\left. +\overline{\psi}_R(x)\,(i\partial\!\!\!/ + B\!\!\!/\,(x,y))\,\psi_R(x)\; Z_R(y)\right]$$

$$\delta B_\mu(x,y) = \partial_\mu \theta(x,0) \equiv \partial_\mu \theta(x)$$

$$\overline{B}_{\mu L} = \int_0^a Z_L(y) B_\mu(x,y)\, dy \qquad \overline{B}_{\mu R} = \int_0^a Z_R(y) B_\mu(x,y)\, dy$$

Consider $m \to \infty$: **localized L (R) fermions at** $y=0$ $(y=1)$.

$$Z_L(y) \to \delta(y) \qquad Z_R(y) \to \delta(y-a)$$

$$S_f = \int d^4x\, \overline{\psi_L}(i\partial\!\!\!/ + \overline{B}\!\!\!/_L)\psi_L(x) + \int d^4x\, \overline{\psi_R}(i\partial\!\!\!/ + \overline{B}\!\!\!/_R)\psi_R(x)$$

$$\overline{B}_L = B(x,0) = B^0(x) + B^1(x,0) \qquad \overline{B}_R = B(x,a) = B^0(x) + B^1(x,a)$$

Under a gauge transformation, $\theta(x,y)$, the fermionic action shifts as:

$$\begin{aligned}\delta S_f &= -\int d^4x\, \theta(x)\partial_\mu J^\mu_L + \int d^4x\, \theta(x)\partial_\mu J^\mu_R \\ &= \frac{1}{24\pi^2}\int d^4x\, \theta(x)(dB_L dB_L - dB_R dB_R)\end{aligned}$$

Enter the Chern-Simons Term:

$$S_{CS} = -2c \int_0^a dy \int d^4x \, \epsilon^{\mu\nu\rho\lambda} B_\mu(x,y) \partial_y B_\nu(x,y) \partial_\rho B_\lambda(x,y)$$

$$B_\mu(x,y) = A_\mu(x,y) - \partial_\mu \int_0^y A_5(x,y') \, dy'$$

$$B_5(x,y) = A_5(x,y) - \partial_y \int_0^y A_5(y') dy' = 0$$

$$\delta S_{CS} = c \int d^4x \, \theta(x) \, \epsilon^{\mu\nu\rho\lambda} \partial_\mu B_\nu(x,0) \partial_\rho B_\lambda(x,0) - c \int d^4x \, \theta(x) \, \epsilon^{\mu\nu\rho\lambda} \partial_\mu B_\nu(x, a) \partial_\rho B_\lambda(x, a)$$

Completely cancels the anomaly

Chern-Simons term:

Maintains anomaly free theory

Locks T-parity to space-time parity

Implies new interactions
amongst bulk KK-mode photons

The orbifold mode expansion

$$\begin{aligned} A^0_\mu(x,y) &= \sqrt{\frac{1}{R}}\tilde{e}A^0_\mu(x) \\ A_\mu(x,y) &= \sum_{n=1}^{\infty}(-1)^n\sqrt{\frac{2}{R}}\tilde{e}\cos(n\pi y/R)A^n_\mu(x) \\ A_5(x,y) &= \sum_{n=1}^{\infty}(-1)^{n+1}\sqrt{\frac{2}{R}}\tilde{e}\sin(n\pi y/R)A^n_5(x) \end{aligned}$$

$$S_1 = -\frac{1}{4\tilde{e}^2}\int_0^R dy\int d^4x\ F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\sum_n\int d^4x\ F^n_{\mu\nu}F^{n\mu\nu}$$

$$S_2 = \frac{1}{2\tilde{e}^2}\int_0^R dy\int d^4x\ F_{\mu5}F^{\mu5} = \frac{1}{2}\sum_{n=1}M_n^2\int d^4x\ B^n_\mu B^{n\mu}$$

$$M_n = n\pi/R\,; \qquad B^n_\mu = A^n_\mu + \frac{1}{M_n}\partial_\mu A^n_5\,; \qquad F^n_{\mu\nu} \equiv \partial_\mu B^n_\nu - \partial_\nu B^n_\mu .$$

"Stueckelberg fields."

$$e = \tilde{e}/\sqrt{R} \equiv e_0 \qquad\qquad e' = \sqrt{2}\tilde{e}/\sqrt{R} = \sqrt{2}e \equiv e_n \quad (n \neq 0)$$

$$
\begin{aligned}
S_{CS} &= \frac{1}{24\pi^2}\int_0^R dy \int d^4x\; \epsilon^{\mu\nu\rho\sigma}(\partial_y B_\mu) B_\nu F_{\rho\sigma} \\
&\equiv \frac{1}{12\pi^2}\sum_{nmk}\int d^4x\; (e_n e_m e_k) c_{nmk} (B^n_\mu B^m_\nu \widetilde{F}^{k\mu\nu})
\end{aligned}
$$

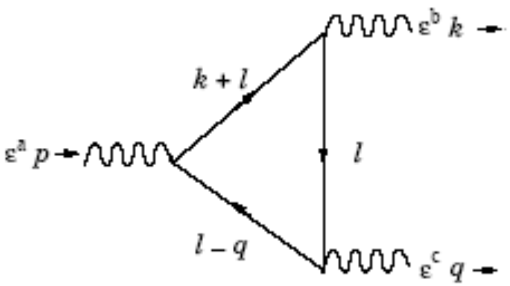
$$
\begin{aligned}
c_{nmk} &= (-1)^{(k+n+m)} \int_0^1 dz\; \partial_z [\cos(n\pi z)] \cos(m\pi z) \cos(k\pi z) \\
&= \frac{n^2(k^2+m^2-n^2)\left[(-1)^{(k+n+m)}-1\right]}{(n+m+k)(n+m-k)(n-k-m)(n-m+k)}
\end{aligned}
$$

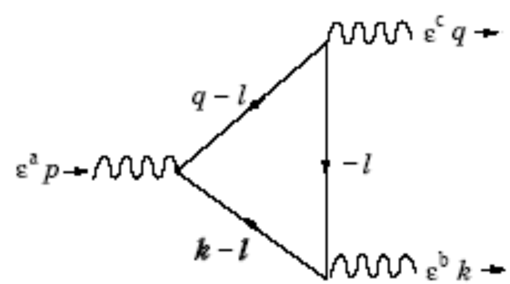
$$
\begin{aligned}
c_{nm0} &= c_{n0m} = -\frac{n^2}{n^2-m^2}\left[(-1)^{n+m}-1\right] \\
c_{0nm} &= c_{000} = 0 \\
c_{n00} &= \left[1-(-1)^n\right].
\end{aligned}
$$

Give Zero Mode Mass m
D=4 Effective Theory in large m limit

$$S_{tree} = \int d^4x\, [\frac{1}{12\pi^2}\sum_{nmk}\bar{c}_{nmk}B^n_\mu B^m_\nu \widetilde{F}^{k\mu\nu} - \frac{1}{4e^2}F^0_{\mu\nu}F^{0\mu\nu} - \frac{1}{4e'^2}\sum_{n\geq 1}F^n_{\mu\nu}F^{n\mu\nu} + \sum_{n=0}\frac{1}{2e_n^2}M_n^2 B^n_\mu B^{n\mu}]$$

Integrate out the Fermions:
Dirac Determinant effective interactions





Dirac Determinant effective interaction:

$$\mathcal{O}_3 = -\frac{1}{12\pi^2}\epsilon^{\mu\nu\rho\sigma}\sum_{nmk}(e_n e_m e_k)a_{nmk}B^n_\mu B^m_\nu \partial_\rho B^k_\sigma$$

$$a_{nmk} = \frac{1}{2}(1-(-1)^{n+m+k})(-1)^{m+k}$$

This operator is equivalent to $(-1/6\pi^2)\epsilon_{\mu\nu\rho\sigma}A^\mu V^\nu \partial^\rho V^\sigma$

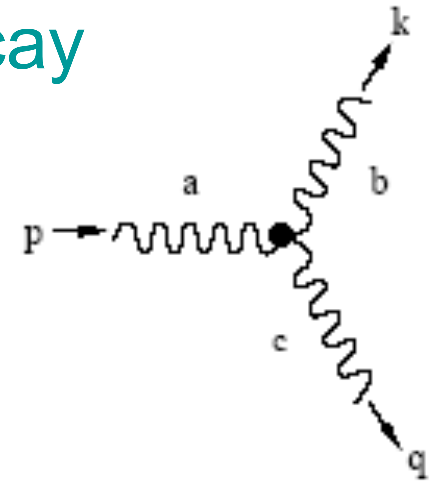
Dirac Determinant effective interaction equivalent To Bardeen’s counterterm: consistent -> covariant

Full Effective Theory

$$\bar{c}_{nmk} = c_{nmk} - a_{nmk} \qquad \text{(massive spinors)}$$

$$\bar{c}_{nmk} = \left[(-1)^{(k+n+m)} - 1\right] \left(\frac{n^2(k^2+m^2-n^2)}{(n+m+k)(n+m-k)(n-k-m)(n-m+k)} + \frac{1}{2}(-1)^{m+k} \right)$$

Compute KK-Mode decay



Feynman rule for a vertex $B^a \to B^b + B^c$:

$$T_{CS} = -\frac{ee'^2}{12\pi^2}\left[(-\bar{c}_{abc} + \bar{c}_{bac} + \bar{c}_{bca} - \bar{c}_{cba})[B] + (\bar{c}_{acb} - \bar{c}_{cab} + \bar{c}_{bca} - \bar{c}_{cba})[A]\right]$$

$$[A] = \epsilon^{\mu\nu\rho\sigma}\epsilon^a_\mu \epsilon^b_\nu \epsilon^\gamma_\rho k^\sigma \qquad [B] = \epsilon^{\mu\nu\rho\sigma}\epsilon^a_\mu \epsilon^b_\nu \epsilon^\gamma_\rho q^\sigma$$

A. Decay of KK-mode to KK-mode plus γ

c = photon

$$T_{CS} = -\frac{ee'^2}{12\pi^2}\left[(-\bar{c}_{ab0} + \bar{c}_{ba0} + \bar{c}_{b0a} - \bar{c}_{0ba})[B] + (\bar{c}_{a0b} - \bar{c}_{0ab} + \bar{c}_{b0a} - \bar{c}_{0ba})[A]\right]$$

$$= \frac{ee'^2}{2\pi^2}\left(\frac{M_b^2}{M_a^2 - M_b^2} - \frac{1}{2}((-1)^b - 1)\right)[B].$$

Gauge invariant in photon

$$\left.\begin{aligned}\Gamma_{1^-\to 1^+\gamma} &= \frac{2\alpha^3}{3\pi^3}\left(\frac{M_a^3}{M_b^2}\right)\\[2ex] \Gamma_{1^+\to 1^-\gamma} &= \frac{2\alpha^3}{3\pi^3}M_b\end{aligned}\right\} \qquad M_a^2 >> M_b^2$$

$$\Gamma_{1^\pm\to 1^\mp\gamma} = \frac{2\alpha^3}{3\pi^3}\Delta M \qquad \Delta M = M_a - M_b << M_a$$

$$\Gamma_{1^-\to 1^+\gamma} = \frac{\alpha(0)\alpha'^2(M_a)}{6\pi^3}\left(\frac{M_a}{R^2M_b^2}\right) \qquad \Gamma_{1^+\to 1^-\gamma} = \frac{\alpha(0)\alpha'^2(M_a)}{6\pi^3}\left(\frac{M_b}{R^2M_a^2}\right)$$

B. Zero Mode + Zero Mode → KK-Mode Vanishes

$$T_{CS} = -\frac{ee'^2}{12\pi^2}[(-\bar{c}_{a00} + \bar{c}_{0a0} + \bar{c}_{00a} - \bar{c}_{00a})[B] + (\bar{c}_{a00} - \bar{c}_{0a0} + \bar{c}_{00a} - \bar{c}_{00a})[A]]$$

= 0 gauge invariance (Landau-Yang theorem)

The Problem: What happens with thick zero modes?

$$S_f = \int_0^a dy \int d^4x \left[\overline{\psi}_L(x)(i\partial\!\!\!/ + B\!\!\!/_L(x,y))\psi_L(x)Z_L(y) + \overline{\psi}_R(x)(i\partial\!\!\!/ + B\!\!\!/_R(x,y))\psi_R(x)Z_R(y)\right]$$

Define the "smeared" gauge fields:

$$\overline{B}_L(x) = \int_0^a dy\ Z_L(y)\ B(x,y)$$

$$\overline{B}_R(x) = \int_0^a dy\ Z_R(y)\ B(x,y)$$

Under the residual zero-mode gauge transformation we have:

$$\delta S_f = \frac{1}{24\pi^2} \int d^4x\ \theta(x)\epsilon_{\mu\nu\rho\lambda}(\partial^\mu \overline{B}_L{}^\nu \partial^\rho \overline{B}_L^\lambda - \partial^\mu \overline{B}_R{}^\nu \partial^\rho \overline{B}_R^\lambda)$$

The "shaped" anomaly;

The anomaly acquires shape due to the smearing!

Construction of a CS term that works with thick fermions

(Hill, Martin, Stavenga, in progress):

define:

$$\overline{B}_V(x) = \overline{B}_L(x) + \overline{B}_R(x) = \int_0^1 dy (Z_L(y) + Z_R(y))\; B(x,y)$$

then guess:

$$S'_{CS} = c \int d^4x \int_0^a \; dy_2\, dy_1\; Z_R(y_2)\; Z_L(y_1)\; \left(\int_{y_1}^{y_2} \epsilon_{\mu\nu\rho\lambda} B^\mu(x,y) \partial_y B^\nu(x,y) \partial^\rho \overline{B}_V^\lambda(x) \right)$$

Perform a gauge transformation:

$$= 2c \int d^4x \int_0^a dy_2\, dy_1\, Z_R(y_2) Z_L(y_1) \int_{y_1}^{y_2} \epsilon_{\mu\nu\rho\lambda} \partial^\mu \theta(x) \partial_y B^\nu(x,y) \partial^\rho B_V^\lambda(x)$$
$$= 2c \int d^4x \int_0^a dy_2\, dy_1\, Z_R(y_2) Z_L(y_1) \int_{y_1}^{y_2} \epsilon_{\mu\nu\rho\lambda} \partial_y (\theta(x) \partial^\mu B^\nu(x,y) \partial^\rho B_V^\lambda(x))$$
$$= 2c \int d^4x \int_0^a dy_2\, dy_1\, Z_R(y_2) Z_L(y_1)\, \theta(x) \epsilon_{\mu\nu\rho\lambda} (\partial^\mu B^\nu(x,y_2) \partial^\rho B_V^\lambda(x) - \partial^\mu B^\nu(x,y_1) \partial^\rho B_V^\lambda(x))$$
$$= c \int d^4x\, \theta(x)\, \epsilon_{\mu\nu\rho\lambda} \left(\partial^\mu \overline{B}_R^\nu(x) \partial^\rho (\overline{B}_L^\lambda(x) + \overline{B}_R^\lambda(x)) - \partial^\mu \overline{B}_L^\nu(x) \partial^\rho (\overline{B}_L^\lambda(x) + \overline{B}_R^\lambda(x)) \right)$$

$$\delta S'_{CS} = c \int d^4x\, \theta(x)\, \epsilon_{\mu\nu\rho\lambda} \left(\partial^\mu \overline{B}_R^\nu(x) \partial^\rho \overline{B}_R^\lambda(x) - \partial^\mu \overline{B}_L^\nu(x) \partial^\rho \overline{B}_L^\lambda(x) \right)$$

Cancels the shaped anomaly !

Conclusion:

The Fermionic Zero-Mode Theory:

$$S_f = \int_0^a dy \int d^4x \left[\overline{\psi}_L(x)(i\partial\!\!\!/ + B\!\!\!/_L(x,y))\psi_L(x)Z_L(y) + \overline{\psi}_R(x)(i\partial\!\!\!/ + B\!\!\!/_R(x,y))\psi_R(x)Z_R(y)\right]$$

is gauge invariant with the modified Chern-Simons term:

$$S'_{CS} = c \int d^4x \int_0^a dy_2\, dy_1\, Z_R(y_2)\, Z_L(y_1) \left(\int_{y_1}^{y_2} \epsilon_{\mu\nu\rho\lambda} B^\mu(x,y) \partial_y B^\nu(x,y) \partial^\rho \overline{B}_V^\lambda(x) \right)$$

where:

$$\overline{B}_L(x) = \int_0^a dy\; Z_L(y)\; B(x,y)$$

$$\overline{B}_R(x) = \int_0^a dy\; Z_R(y)\; B(x,y)$$

$$\overline{B}_V = \overline{B}_L(x) + \overline{B}_R(x)$$

Conclusion:

The modified Chern-Simons term:

$$S'_{CS} = c \int d^4x \int_0^a \, dy_2 \, dy_1 \, Z_R(y_2) \, Z_L(y_1) \, \left(\int_{y_1}^{y_2} \epsilon_{\mu\nu\rho\lambda} B^\mu(x,y) \partial_y B^\nu(x,y) \partial^\rho \overline{B}_V^\lambda(x) \right)$$

Implies modified KK-mode transitions

We believe this is derivable by carefully integrating out KK-modes above the zero-mode (CTH, Martin, Stavenga).

II. Yang-Mills in D=5

(a) "quarks" on branes; gauge theory of flavor
compactify with A_5 zero-mode -> mesons $f_{pi} = 1/R$;
Wilson line mass term <-> chiral condensate

(b) Chiral delocalization requires a Chern-Simons term;
anomaly matching, quantization

(c) Chern-Simons term implies bulk and
holographic interactions amongst KK-modes
effective D=4 interaction,

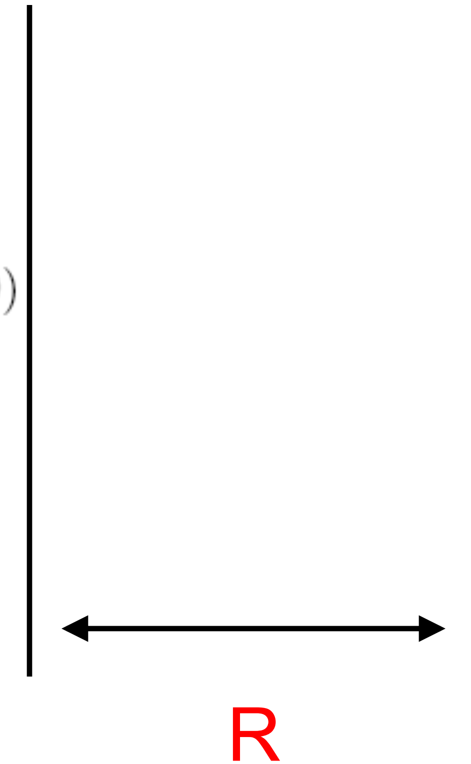
(d) Large m_q limit -> Fermionic Dirac determinant modifies
effective interaction; maintains gauge invariance

(e) Obtain effective interaction: holographic part is
the full Wess-Zumino-Witten term.

QED in D=5:

$$\int_I d^4x\,\overline{\psi}_L i\not{D}_L\psi_L$$

$$D_{L\mu} = \partial_\mu - iA_\mu(x_\mu, 0)$$



$$\int_{II} d^4x\,\overline{\psi}_R i\not{D}_R\psi_R$$

$$D_{R\mu} = \partial_\mu - iA_\mu(x_\mu, R)$$

Dirac mass:

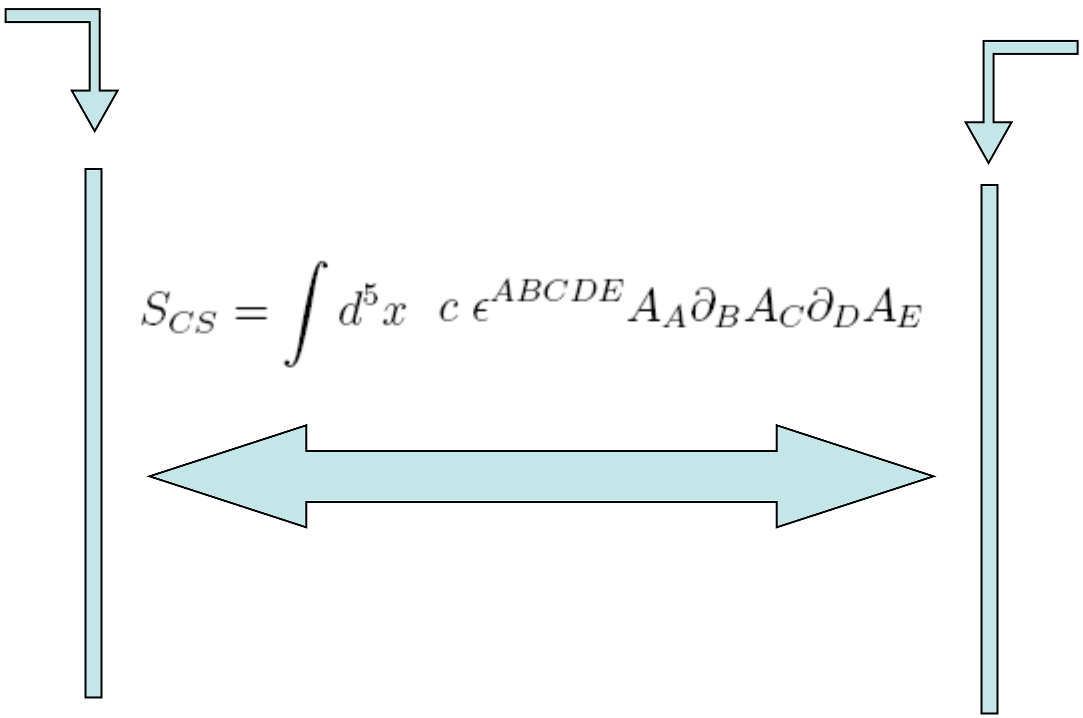
$$m\overline{\psi}_L(x_\mu, 0)W\psi_R(x_\mu, R) + h.c.$$

Wilson Line:

$$W = \exp(i\int_0^R A_5(x_\mu, x_5)dx^5)$$

QED in D=5 requires Chern-Simons term:

$$L_{CS} = c\,\epsilon^{ABCDE} A_A \partial_B A_C \partial_D A_E = \frac{c}{4}\epsilon^{ABCDE} A_A F_{BC} F_{DE}$$



$$S_{CS} = \int d^5x \;\; c\,\epsilon^{ABCDE} A_A \partial_B A_C \partial_D A_E$$

$$S_{CS} \;\rightarrow\; S_{CS} + \frac{c}{4}\int_{II} d^4x\;\theta(R)\;\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}(R) - \frac{c}{4}\int_{I} d^4x\;\theta(0)\;\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}(0)\;.$$

Anomaly Cancellation Condition:

$$S_{branes} \to S_{branes} + \frac{1}{48\pi^2}\int_I d^4x\, \theta(x_\mu, 0) F^{\mu\nu}\widetilde{F}_{\mu\nu}(0) \; - \; \frac{1}{48\pi^2}\int_{II} d^4x\, \theta(x_\mu, R) F^{\mu\nu}\widetilde{F}_{\mu\nu}(R)$$

$$S_{CS} \to S_{CS} - \frac{c}{2}\int_I d^4x\, \theta(x_\mu, 0) F^{\mu\nu}\widetilde{F}_{\mu\nu} + \frac{c}{2}\int_{II} d^4x\, \theta(x_\mu, R) F^{\mu\nu}\widetilde{F}_{\mu\nu}$$

Consistent Anomalies: $c = \dfrac{1}{24\pi^2}$

Summary of Anomalies:

W. A. Bardeen, PR 184, 1848 (199)

Consistent Anomalies: | **Consistent $L = V - A$ and $R = V + A$ Forms:**

(1) Pure Massless Weyl Spinors ($p_i \cdot p_j >> m^2$):

$$\partial^\mu \overline{\psi}\gamma_\mu \psi_L = -\frac{1}{48\pi^2} F_{L\mu\nu}\tilde{F}_L^{\mu\nu}$$
$$\partial^\mu \overline{\psi}\gamma_\mu \psi_R = \frac{1}{48\pi^2} F_{R\mu\nu}\tilde{F}_R^{\mu\nu}$$

$$\partial^\mu \overline{\psi}\gamma_\mu \psi = \frac{1}{12\pi^2} F_{V\mu\nu}\tilde{F}_A^{\mu\nu}$$
$$\partial^\mu \overline{\psi}\gamma_\mu \gamma^5 \psi = \frac{1}{24\pi^2}(F_{V\mu\nu}\tilde{F}_V^{\mu\nu} + F_{A\mu\nu}\tilde{F}_A^{\mu\nu})$$

(2) Heavy Massive Weyl Spinors ($p_i \cdot p_j << m^2$):

$$\partial^\mu \overline{\psi}\gamma_\mu \psi_L + im(\overline{\psi}_L \psi_R - \overline{\psi}_R \psi_L) = -\frac{1}{48\pi^2} F_{L\mu\nu}\tilde{F}_L^{\mu\nu}$$
$$\partial^\mu \overline{\psi}_R\gamma_\mu \psi_R + im(\overline{\psi}_R \psi_L - \overline{\psi}_L \psi_R) = \frac{1}{48\pi^2} F_{R\mu\nu}\tilde{F}_R^{\mu\nu}$$

$$\partial^\mu \overline{\psi}\gamma_\mu \psi = \frac{1}{12\pi^2} F_{V\mu\nu}\tilde{F}_A^{\mu\nu}$$
$$\partial^\mu \overline{\psi}\gamma_\mu \gamma^5 \psi - 2im\overline{\psi}\gamma^5\psi = \frac{1}{24\pi^2}(F_{V\mu\nu}\tilde{F}_V^{\mu\nu} + F_{A\mu\nu}\tilde{F}_A^{\mu\nu})$$

$$\partial^\mu \overline{\psi}\gamma_\mu \psi_L = \frac{1}{48\pi^2}(F_{L\mu\nu}\tilde{F}_R^{\mu\nu} + F_{R\mu\nu}\tilde{F}_R^{\mu\nu})$$
$$\partial^\mu \overline{\psi}\gamma_\mu \psi_R = -\frac{1}{48\pi^2}(F_{L\mu\nu}\tilde{F}_R^{\mu\nu} + F_{L\mu\nu}\tilde{F}_L^{\mu\nu})$$

$$\partial^\mu \overline{\psi}\gamma_\mu \psi = \frac{1}{12\pi^2} F_{V\mu\nu}\tilde{F}_A^{\mu\nu}$$
$$\partial^\mu \overline{\psi}\gamma_\mu \gamma^5 \psi = -\frac{1}{12\pi^2}(F_{V\mu\nu}\tilde{F}_V^{\mu\nu})$$

$$im\overline{\psi}\gamma^5\psi \rightarrow -\frac{1}{48\pi^2}[F_{L\mu\nu}\tilde{F}_L^{\mu\nu} + F_{R\mu\nu}\tilde{F}_R^{\mu\nu} + F_{L\mu\nu}\tilde{F}_R^{\mu\nu}]$$

Anomalies (cont'd): [HEP-TH 0601155]

Covariant Forms:

Add a term to the lagrangian of the form $(1/6\pi^2)\epsilon_{\mu\nu\rho\sigma}A^\mu V^\nu \partial^\rho V^\sigma$. The currents are now modified to $\tilde{J} = J + \delta J$ and $\tilde{J}^5 = J^5 + \delta J^5$

$$\begin{aligned} \frac{\delta S'}{\delta V_\mu} &= \delta J^\mu = -\frac{1}{3\pi^2}\epsilon_{\mu\nu\rho\sigma}A^\nu \partial^\rho V^\sigma + \frac{1}{6\pi^2}\epsilon_{\mu\nu\rho\sigma}V^\nu \partial^\rho A^\sigma \\ \frac{\delta S'}{\delta A_\mu} &= \delta J^{5\mu} = \frac{1}{6\pi^2}\epsilon_{\mu\nu\rho\sigma}V^\nu \partial^\rho V^\sigma \end{aligned}$$

(1) Pure Massless Weyl Spinors ($p_i \cdot p_j >> m^2$):

$$\begin{aligned} \partial^\mu \tilde{J}_\mu &= 0 \\ \partial^\mu \tilde{J}^5_\mu &= \frac{1}{8\pi^2}(F_{V\mu\nu}\tilde{F}^{\mu\nu}_V + \frac{1}{3}F_{A\mu\nu}\tilde{F}^{\mu\nu}_A) \end{aligned}$$

(2) Heavy Massive Weyl Spinors ($p_i \cdot p_j << m^2$):

$$\begin{aligned} \partial^\mu \tilde{J}_\mu &= 0 \\ \partial^\mu \tilde{J}^5_\mu - 2im\overline{\psi}\gamma^5\psi &= \frac{1}{8\pi^2}(F_{V\mu\nu}\tilde{F}^{\mu\nu}_V + \frac{1}{3}F_{A\mu\nu}\tilde{F}^{\mu\nu}_A) \end{aligned}$$

$$\begin{aligned} \partial^\mu \tilde{J}_\mu &= 0 \\ \partial^\mu \tilde{J}^5_\mu &= 0 \end{aligned}$$

Summary: Technically natural QED in D=5

Bulk: $D_A = \partial_A - iA_A \,, \qquad F_{AB} = i[D_A, D_B] \,, \qquad L_0 = -\frac{1}{4\tilde{e}^2} F_{AB} F^{AB}$

$$\int_I d^4x \, \overline{\psi}_L i \not{D}_L \psi_L$$

$$\int_{II} d^4x \, \overline{\psi}_R i \not{D}_R \psi_R$$

$$D_{L\mu} = \partial_\mu - iA_\mu(x_\mu, 0)$$

Orbifold

$$D_{R\mu} = \partial_\mu - iA_\mu(x_\mu, R)$$

$$S_{CS} = \int d^5x \; \frac{1}{24\pi^2} \; \epsilon^{ABCDE} A_A \partial_B A_C \partial_D A_E$$

$$m\overline{\psi}_L(x_\mu, 0) W \psi_R(x_\mu, R) + h.c.,$$

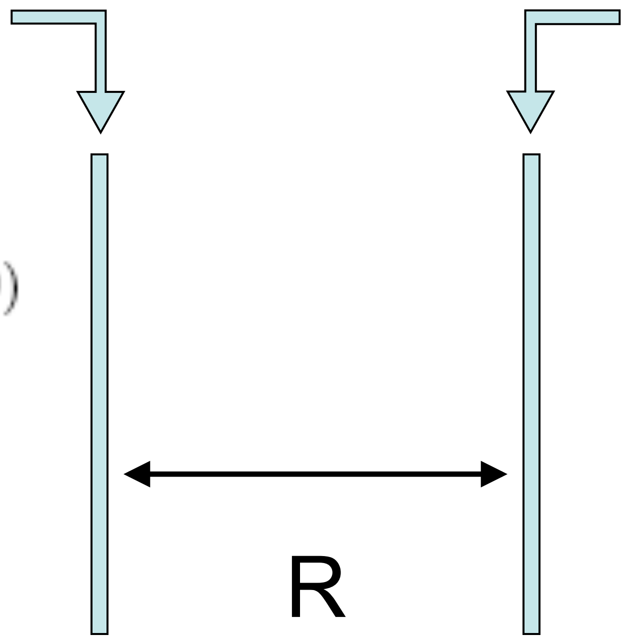
$$W = \exp(i \int_0^R A_5(x_\mu, x_5) dx^5)$$

Yang-Mills gauge theory of quark flavor in D=5:

Bulk: $$D_A = \partial_A - iB_A \qquad B_A = B_A^a T^a,$$

$$G_{AB} = i[D_A, D_B] = \partial_A B_B - \partial_B B_A - i[B_A, B_B].$$

$$\int_I d^4x\, \overline{\psi}_L i \not{D}_L \psi_L \qquad\qquad \int_{II} d^4x\, \overline{\psi}_R i \not{D}_R \psi_R$$

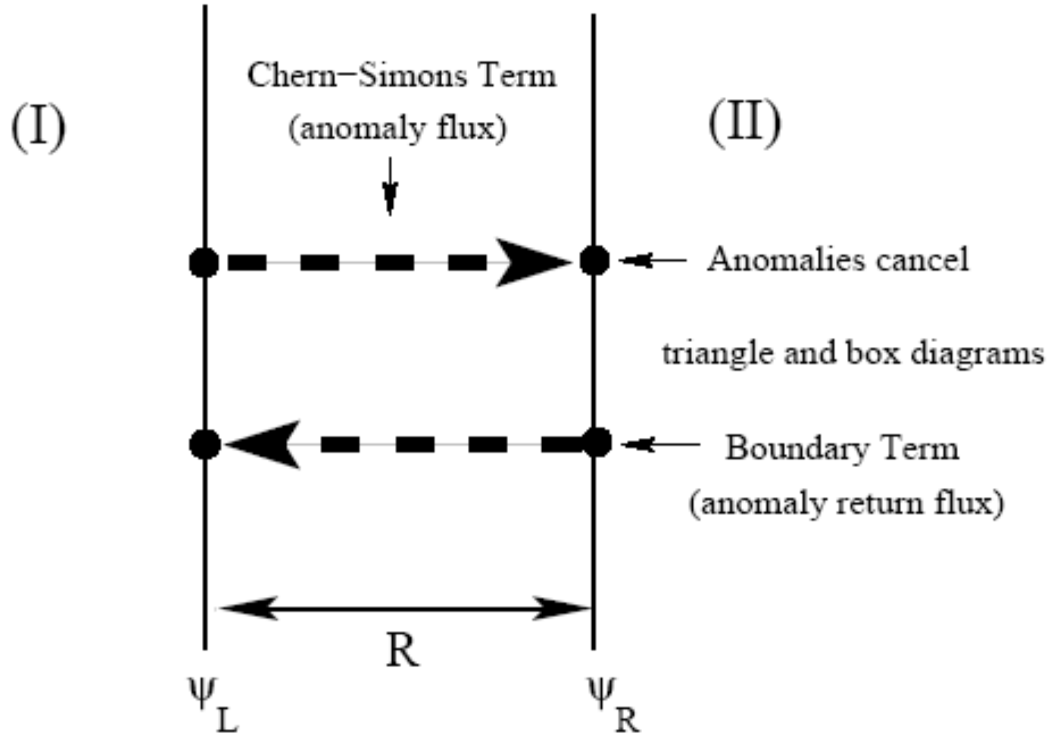


$$D_{L\mu} = \partial_\mu - iB_\mu(x_\mu, 0) \qquad\qquad D_{R\mu} = \partial_\mu - iB_\mu(x_\mu, R)$$

$$m\overline{\psi}_L(x_\mu, 0) W \psi_R(x_\mu, R) + h.c. \qquad W = P\exp\left(-i\int_0^R dx^5 B_5\right) = \exp(2i\tilde{\pi}/f_\pi)$$

Generic compactification; include B_5 zero-mode

Derivation of the full Wess-Zumino-Witten term directly from the Yang-Mills theory



Theory Requires Chern-Simons Term:

$$\mathcal{L}_{CS} = c\epsilon^{ABCDE}\,\mathrm{Tr}\Big(A_A\partial_B A_C\partial_D A_E - \frac{3i}{2}A_A A_B A_C\partial_D A_E - \frac{3}{5}A_A A_B A_C A_D A_E\Big)$$

$$= \frac{c}{4}\epsilon^{ABCDE}\,\mathrm{Tr}\big(A_A G_{BC} G_{DE} + iA_A A_B A_C G_{DE} - \frac{2}{5}A_A A_B A_C A_D A_E\big)\ .$$

Gauge transformation: $A_A \rightarrow V(A_A + i\partial_A)V^\dagger$ where: $V = \exp(i\theta^a T^a)$

$$\delta S_{CS} = c\epsilon^{\mu\nu\rho\sigma}\theta^a\,\mathrm{Tr}[T^a(\partial_\mu A_\nu\partial_\rho A_\sigma - \frac{i}{2}(\partial_\mu A_\nu A_\rho A_\sigma - A_\mu\partial_\nu A_\rho A_\sigma + A_\mu A_\nu\partial_\rho A_\sigma)]\Big|_0^R$$

Consistent Anomaly; To cancel against fermion anomalies: $c = \dfrac{N_c}{24\pi^2}\ .$

Transforming to Axial Gauge, $B_5 \to 0$

$$V(x^\mu, y) = P \exp\left(-i \int_0^y dx^5 \; B_5^0(x^\mu, x^5)\right)$$

$$\psi'_L = \psi_L, \qquad \psi'_R = V(R)\psi_R$$

$$\tilde{B}_\mu(x^\mu, y) = V(B_\mu + i\partial_\mu)V^\dagger \qquad \tilde{B}_5(x^\mu, y) = V(B_5 + i\partial_y)V^\dagger = 0$$

$$\overline{\psi}(i\partial\!\!\!/ + B\!\!\!/\,)\psi = \overline{\psi}'(i\partial\!\!\!/ + \tilde{B\!\!\!/}\,)\psi' \qquad \overline{\psi}_L W \psi_R = \overline{\psi}'_L \psi'_R \qquad W \to V(0) W V^\dagger(R) = 1.$$

B is now a tower of vector mesons comingled
with the spin-0 mesons; must extract the physical mesons:

Redefinition: $\tilde{B}_\mu(x^\mu, y) = \tilde{U}(y)(A_\mu(x^\mu, y) + i\partial_\mu)\tilde{U}^\dagger(y) \qquad \tilde{U} = \exp(2i\tilde{\pi}y/f_\pi)$

Compactification Decomposition of Chern-Simons Term:

$$S_{CS} = c\int d^5x\,\epsilon^{ABCDE}\,\mathrm{Tr}\Big(B_A\partial_B B_C\partial_D B_E - \frac{3i}{2}B_A B_B B_C\partial_D B_E - \frac{3}{5}B_A B_B B_C B_D B_E)\Big)$$

$$= \frac{c}{2}\,\mathrm{Tr}\int d^4x\int_0^R dy\,\Big[(\partial_5 B_\mu)K^\mu + \frac{3}{2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}(B_5 G_{\mu\nu}G_{\rho\sigma})\Big],$$

$$K^\mu \equiv \epsilon^{\mu\nu\rho\sigma}\,(iB_\nu B_\rho B_\sigma + G_{\nu\rho}B_\sigma + B_\nu G_{\rho\sigma})\,.$$

CTH and Zachos

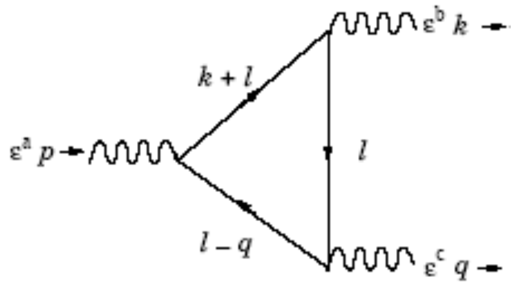
Axial Gauge:

$$S_{CS} = \frac{c}{2}\epsilon^{\mu\nu\rho\sigma}\int d^4x\int_0^R dy\;\mathrm{Tr}\Big[\partial_y\tilde{B}_\mu\Big(i\tilde{B}_\nu\tilde{B}_\rho\tilde{B}_\sigma + G(\tilde{B})_{\nu\rho}\tilde{B}_\sigma + \tilde{B}_\nu G(\tilde{B})_{\rho\sigma}\Big)\Big]$$

$$= \frac{c}{2}\,\mathrm{Tr}\int d^4x\int_0^R dy\,(\partial_y\tilde{B})(2d\tilde{B}\tilde{B} + 2\tilde{B}d\tilde{B} - 3i\tilde{B}^3)$$

Form notation: $G(\tilde{B}) = 2d\tilde{B} - 2i\tilde{B}^2.$

Integrate out the Fermions:
Dirac Determinant effective interactions



+ …

Dirac Determinant effective interaction
Is the Bardeen Counterterm as in QED:

$$S_{boundary} = -\frac{c}{2}\int \mathrm{Tr}\,(\frac{1}{2}(G_R\tilde{B}_R + \tilde{B}_R G_R)\tilde{B}_L - \frac{1}{2}(G_L\tilde{B}_L + \tilde{B}_L G_L)\tilde{B}_R$$
$$+i\tilde{B}_R^3\tilde{B}_L - i\tilde{B}_L^3\tilde{B}_R - \frac{i}{2}(\tilde{B}_R\tilde{B}_L)^2)$$

"Boundary term"

Notation:

$$\tilde{B}_\mu = \tilde{A}_\mu - i\alpha_\mu \qquad \tilde{A}_\mu = \tilde{U} A_\mu \tilde{U}^\dagger$$

$$\tilde{A}_{L\mu} = A_{L\mu} = A_\mu(x^\mu, 0) \qquad \tilde{A}_{R\mu} = U A_\mu(x^\mu, R) U^\dagger$$

$$\alpha_\mu = -\tilde{U}\partial_\mu \tilde{U}^\dagger \qquad \beta_\mu = U^\dagger \partial_\mu U = U^\dagger \alpha_\mu U \qquad \tilde{U} = \exp(2i\tilde{\pi} y / f_\pi)$$

$$S_{CS} \Rightarrow \frac{c}{2} \mathrm{Tr} \int d^4x \, dy \, [-i(\partial_y \alpha) + (\partial_y \tilde{A})]$$
$$\times (2d\tilde{A}\tilde{A} - 2i\alpha^2\tilde{A} - 2id\tilde{A}\alpha - 4\alpha^3 + 2\tilde{A}d\tilde{A} - 2i\tilde{A}\alpha^2 - 2i\alpha d\tilde{A}$$
$$-3i\tilde{A}^3 - 3\alpha\tilde{A}^2 - 3\tilde{A}\alpha\tilde{A} - 3\tilde{A}^2\alpha + 3i\alpha^2\tilde{A} + 3i\alpha\tilde{A}\alpha + 3i\tilde{A}\alpha^2 + 3\alpha^3)$$

$$S_{boundary} \Rightarrow \frac{c}{2} \int \mathrm{Tr} \, [(dA_L A_L + A_L dA_L) U A_R U^\dagger - (dA_R A_R + A_R dA_R) U^\dagger A_L U$$
$$-i(dA_L A_L + A_L dA_L)\alpha - A_L^3 \alpha - A_L \alpha^3 + iA_R^3 U^\dagger A_L U - iA_L^3 U A_R U^\dagger$$
$$-i(dA_R dU^\dagger A_L U - dA_L dU A_R U^\dagger) - (A_R U^\dagger A_L U A_R \beta + A_L U A_R U^\dagger A_L \alpha)$$
$$+\frac{i}{2} A_L \alpha A_L \alpha + \frac{i}{2} U A_R U^\dagger A_L U A_R U^\dagger A_L - i(A_L U A_R U^\dagger \alpha^2 - A_R U^\dagger A_L U \beta^2)]$$

We first isolate the term:

$$S_{CS0} \;=\; i\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\alpha)\alpha^3$$

$$\partial_y\alpha = \partial_y\tilde{U}d\tilde{U}^\dagger = \frac{2i}{Rf_\pi}\tilde{U}d\tilde{\pi}\tilde{U}^\dagger \qquad\qquad \alpha \approx \frac{2iy}{f_\pi}d\tilde{\pi} - \frac{2y^2}{f_\pi^2}[\tilde{\pi}, d\tilde{\pi}] + ...$$

$$\begin{aligned} S_{CS0} \;&=\; -\frac{2N_c}{3\pi^2 f_\pi^5}\int d^4x\; dy y^4\,\mathrm{Tr}(\tilde{\pi}d\tilde{\pi}d\tilde{\pi}d\tilde{\pi}d\tilde{\pi}) + ... \\ &=\; -\frac{2N_c}{15\pi^2 f_\pi^5}\int d^4x\,\mathrm{Tr}(\tilde{\pi}d\tilde{\pi}d\tilde{\pi}d\tilde{\pi}d\tilde{\pi}) + ... \end{aligned}$$

$$
\begin{aligned}
S_{CS\ \alpha^3\tilde{A}} &= -i\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\alpha)(-2id\tilde{A}\alpha - 2i\alpha d\tilde{A} - 2i\alpha^2\tilde{A} - 2i\tilde{A}\alpha^2 + 3i(\alpha^2\tilde{A} + \alpha\tilde{A}\alpha + \tilde{A}\alpha^2)) \\
&\quad -\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\tilde{A})[\alpha^3]
\end{aligned} \tag{42}
$$

Note that, upon integrating in $D = 4$ by parts:

$$
\mathrm{Tr}\int(\partial_y\alpha)(d\tilde{A}\alpha + \alpha d\tilde{A}) = 2\,\mathrm{Tr}\int(\partial_y\alpha)(\alpha\tilde{A}\alpha)
$$

Thus, we can immediately write:

$$
\begin{aligned}
S_{CS\ \alpha^3\tilde{A}} &= -i\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\alpha)(i\alpha^2\tilde{A} + i\tilde{A}\alpha^2 - i\alpha\tilde{A}\alpha) - \frac{c}{2}\,\mathrm{Tr}\int d^4x dy(\partial_y\tilde{A})[\alpha^3] \\
&= \frac{c}{2}\,\mathrm{Tr}\int d^4x\int_0^1 dy\ \partial_y(\alpha^3\tilde{A})
\end{aligned}
$$

If we now explicitly perform this integral we obtain:

$$
S_{CS\ \alpha^3\tilde{A}} = -\frac{c}{2}\,\mathrm{Tr}(A_R\beta^3)
$$

where use has been made $\mathrm{Tr}(\alpha^3\tilde{A}_R) = \mathrm{Tr}(\alpha^3 U A_R U^\dagger) = \mathrm{Tr}(U^\dagger\alpha^3 U A_R) = \mathrm{Tr}(\beta^3 A_R) = -\mathrm{Tr}(A_R\beta^3)$. We see the operational parity asymmetry of our gauge tranformation leads to the absence of a corresponding parity conjugate term, $-\mathrm{Tr}(A_L\alpha^3)$. As mentioned above, this term will come from the boundary term, and the overall final result will be parity symmetric.

Obtain the full Wess-Zumino-Witten Term

$$\tilde{S} = S_{CS} + S_{boundary} \quad = S_{WZW} + S_{bulk}$$

$$\begin{aligned}
S_{WZW} \;=\; & S_{CS0} + \frac{N_c}{48\pi^2}\mathrm{Tr}\int d^4x[-(A_L\alpha^3 + A_R\beta^3) - (A_L^3\alpha + A_R^3\beta) \\
& -i((dA_LA_L + A_LdA_L)\alpha + dA_RA_R + A_RdA_R)\beta) + \frac{i}{2}[(A_L\alpha)^2 - (A_R\beta)^2] \\
& -i(A_L^3UA_RU^\dagger - A_R^3U^\dagger A_LU) \\
& +(dA_LA_L + A_LdA_L)UA_RU^\dagger - (dA_RA_R + A_RdA_R)U^\dagger A_LU \\
& -i(dA_RdU^\dagger A_LU - dA_LdUA_RU^\dagger) - (A_LUA_RU^\dagger A_L\alpha + A_RU^\dagger A_LUA_R\beta) \\
& +\frac{i}{2}UA_RU^\dagger A_LUA_RU^\dagger A_L - i(A_LUA_RU^\dagger\alpha^2 - A_RU^\dagger A_LU\beta^2)]
\end{aligned}$$

$$\tilde{S}_{CS0} \;=\; -\frac{2N_c}{15\pi^2 f_\pi^5}\int d^4x\,\mathrm{Tr}(\tilde{\pi}d\tilde{\pi}d\tilde{\pi}d\tilde{\pi}d\tilde{\pi}) + \ldots$$

in complete agreement with Kaymakcalan, Rajeev and Schechter

Effective brane (holographic) interaction

Normal Derivation of WZW term:

- promote full theory of mesons to D=5.
- In D=5, a certain manifestly chirally invariant and topologically interesting Chern-Simons term occurs, which is included into the theory.
- Compactify the fifth dimension with the Chern-Simons term, back into to D=4, resulting in the Wess-Zumino term.
- Perform gauge transformations upon the resulting object, and infer how to ``integrate in'' the gauge fields by brute force and some guess work.

introduce counterterm: $$\mathcal{O} = f \int AV dV ... dV$$

$$\begin{aligned} \delta j &= \frac{\delta}{\delta V}\mathcal{O} = -\frac{1}{2} f \ (DAdV...dV - (D-2)V dAdV..dV) \\ \delta j_5 &= \frac{\delta}{\delta A}\mathcal{O} = f \ V dV...dV \end{aligned}$$

$$\partial(j + \delta j) = (Dc - f) \, dAdV..dV +$$

$$\partial(j_5 + \delta j_5) = 2\left(c + \frac{f}{2}\right) \ (dV...dV + ...)$$

We demand that the vector current is conserved, whence:

$$\partial(j + \delta j) = 0 \qquad \partial(j_5 + \delta j_5) = 2c\left(1 + \frac{D}{2}\right) \ (dV...dV + ...)$$

$$
\begin{aligned}
S_{bulk} \;=\;& -i\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\alpha)(\tilde{U}(3dAA+3AdA-4iA^3)\tilde{U}^\dagger) \\
&+\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\tilde{A})[\tilde{U}(2dAA+2AdA-3iA^3)\tilde{U}^\dagger]
\end{aligned}
$$

$$
\partial_y\alpha = \frac{2i}{f_\pi}\tilde{U}(d\tilde{\pi})\tilde{U}^\dagger \qquad\qquad \partial_y\tilde{A} = \partial_y\tilde{U}A\tilde{U}^\dagger = \frac{2i}{f_\pi}\tilde{U}([\tilde{\pi},A])\tilde{U}^\dagger
$$

$$
S_{bulk} \;=\; -\frac{3c}{2f_\pi}\int d^4x\int_0^1 dy\,\mathrm{Tr}(\tilde{\pi}GG)+\frac{c}{2}\int d^4x\int_0^1 dy\,\mathrm{Tr}(\partial_yA)(2dAA+2AdA-3iA^3)) \tag{6}
$$

Effective bulk interaction

Suppose we don't integrate out the quarks?

Parity symmetric redefinition field: $\tilde{U}(y) = \exp\left(\frac{2i\tilde{\pi}(y-1/2)}{f_\pi}\right)$

$$\tilde{B}_L = \xi A_L \xi^\dagger - j_L \qquad \tilde{B}_R = \xi^\dagger A_L \xi - j_R$$

chiral currents $\quad j_L = i\xi d\xi^\dagger \qquad j_R = -i\xi^\dagger d\xi$

$$\begin{aligned} S &= S_{CS0} + S'_{WZW} + S_{bulk} \\ &+ \int_I d^4x\, \overline{\psi}_L(i\partial\!\!\!/ + \xi A\!\!\!/_L \xi^\dagger - j_L)\psi_L + \int_{II} d^4x\, \overline{\psi}_R(i\partial\!\!\!/ + \xi^\dagger A\!\!\!/_R \xi - j_R)\psi_R \end{aligned}$$

$$\begin{aligned} S'_{WZW} &= -\frac{c}{2}\mathrm{Tr}(A_R j_R^3 + A_L j_L^3) - \frac{c}{2}\mathrm{Tr}(A_R^3 j_R + A_L^3 j_L) - i\frac{c}{4}\mathrm{Tr}(A_R j_R A_R j_R - A_L j_L A_L j_L) \\ &-i\frac{c}{2}\mathrm{Tr}[(dA_R A_R + A_R dA_R) j_R + (dA_L A_L + A_L dA_L) j_L] \qquad (67 \end{aligned}$$

Effective theory with unintegrated massless fermions

Envisioned applications:

(1) Little Higgs Theories.

(2) RS Models

(3) A WZW Term for the Goldstone-Wilczek Current

(4) Skyrme/instanton baryogenesis/b+L violation
in extra dimensional theories
(which is what started this project).

(5) Technical tool to analyze/generalize WZW term structure

(6) Can apply to D=3 YM matching to D=2;

(7) Any D+1 to D.

(8) …

$$U = e$$

$$\overline{T}^{+} = \frac{ee'^2}{2\pi^2}\left(\frac{M_a^2}{M_a^2 - M_b^2}\right)[B] \qquad 1^- \to 1^+ + \gamma$$

$$\overline{T}^{-} = \frac{ee'^2}{2\pi^2}\left(\frac{M_b^2}{M_a^2 - M_b^2}\right)[B] \qquad 1^+ \to 1^- + \gamma$$

$$\alpha_\mu = U\partial_\mu U^\dagger \qquad \beta_\mu = U^\dagger\partial_\mu U$$

$$f_\pi^2 Tr(\alpha_\mu\alpha^\mu) \qquad \kappa Tr([\alpha_{\mu,}\alpha_\upsilon][\alpha^\mu_,\alpha^\nu])$$

$$f_\pi^2 Tr(\alpha_\mu\alpha^\mu) + \kappa Tr([\alpha_{\mu,}\alpha_\upsilon][\alpha^\mu_,\alpha^\nu])$$

$$D_A = \partial_A - iA_A \ , \qquad F_{AB} = i[D_A, D_B] \ , \qquad L_0 = -\frac{1}{4\tilde{e}^2}F_{AB}F^{AB}$$

Yang-Mills D=5 Chern-Simons term

$$\mathcal{L}_{CS} = c\epsilon^{ABCDE}\,\mathrm{Tr}\Big(A_A\partial_B A_C\partial_D A_E - \frac{3i}{2}A_A A_B A_C \partial_D A_E - \frac{3}{5}A_A A_B A_C A_D A_E\Big)$$

The instanton is an ("the") important topological object in Yang-Mills field theories

$\Pi_3(SU(2))$: instanton naturally lives on a D=4 Euclidean space, mapping SU(2) onto S_3

In D=5, the instanton becomes a stable, static, soliton

Compactify to D=4, with an A_5 zero-mode, the instanton becomes the Skyrmion; exact correspondence with chiral lagrangians.

QED Chern-Simons term

D=3: Knot Theory $\Longleftrightarrow$ "Gauss' Linking Theorem"

$$L_{CS} = \epsilon_{ijk} A^i \partial^j A^k$$

Bulk Physics: Photon Mass Term

Deser, Jackiw, Templeton, Schonfeld,
Siegel; Niemi, Semenoff, Y.S. Wu

D=5:

$$L_{CS} = \epsilon_{ABCDE} A^A \partial^B A^C \partial^D A^E$$

Bulk Physics: New interactions amongst KK-modes

Topological object: "instantonic soliton"

Deser's Theorem
Ramond and CTH

Associated Conserved Topological Currents:

Singlet: $$J_A = \epsilon_{ABCDE}\,\mathrm{Tr}(G^{BC}G^{DE})$$

Adjoint: $$J^a_A = \epsilon_{ABCDE}\,\mathrm{Tr}(\frac{\lambda^a}{2}\{G^{BC},G^{DE}\})$$

These currents come from a "completion" of the Lagrangian

Adjoint current - 2nd Chern character:

$$c\epsilon^{ABCDE}\,\mathrm{Tr}(A_A\partial_B A_C\partial_D A_E - \frac{3i}{2}A_A A_B A_C\partial_D A_E - \frac{3}{5}A_A A_B A_C A_D A_E)$$

Singlet currents - auxiliary characters:

$$c'\epsilon_{ABCDE}V^A\,\mathrm{Tr}(G^{BC}G^{DE})$$

The topology of the D=5 pure Yang Mills theory can be directly matched to the D=4

Chiral Lagrangian theory obtainable via deconstruction

Bianchi ID's, etc.: CTH, CTH & Zachos

Mathematically exact matchings:

Instantonic Soliton $\Longleftrightarrow$ Skyrmion

Gauge currents $\Longleftrightarrow$ Chiral currents

Chern-Simons term + boundary term $\Longleftrightarrow$ WZW term

ANOMALIES, CHERN-SIMONS TERMS AND
CHIRAL DELOCALIZATION IN EXTRA DIMENSIONS.
FERMILAB-PUB-06-010-T (Jan 2006) 46p. [HEP-TH 0601154]

(to appear in Phys. Rev. D)

LECTURE NOTES FOR
MASSLESS SPINOR AND MASSIVE SPINOR
TRIANGLE DIAGRAMS.

FERMILAB-TM-2341-T (Jan 2006) 17p. [HEP-TH 0601155]

EXACT EQUIVALENCE OF THE D=4 GAUGED
WESS-ZUMINO-WITTEN TERM AND THE D=5
YANG-MILLS CHERN-SIMONS TERM.

FERMILAB-PUB-06-046-T (Mar 2006) 25p. [HEP-TH 0603060]

Orbifold Boundary Conditions:

(a) Horava-Witten

(b) Magnetic Josephson Junction

Spectrum: (a) A_μ zero mode and KK tower

(b) No A_5 zero mode

(c) All A_5 modes eaten -> longitudinal dof's

Flipped Orbifold Boundary Conditions:

(a) parity reversed Horava-Witten

(b) Josephson Junction

Spectrum: (a) A_5 zero mode

(b) No A_{mu} zero mode

(c) All other A_5 modes eaten -> longitudinal dof's

Only zero-mode gauge invariance is manifest:

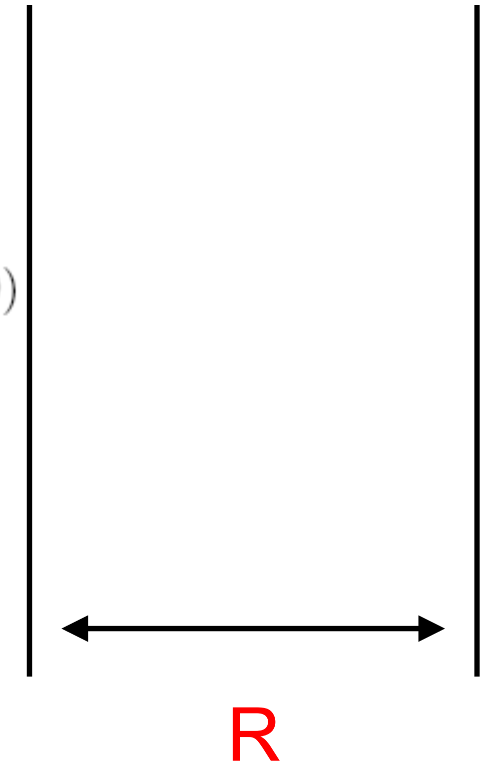
$$
\begin{aligned}
\delta B_\mu(x,y) &= \partial_\mu \theta(x,y) - \partial_\mu \int_{y_0=0}^{y} \partial'_y \theta(x,y')\, dy' \\
&= \partial_\mu \theta(x,y) - \partial_\mu(\theta(x,y) - \theta(x,0)) \\
&= \partial_\mu \theta(x) \qquad\qquad \theta(x) \equiv \theta(x,0)
\end{aligned}
$$

$B_\mu(x,y)$ is a "Stueckelberg field" with respect to y dependent gauge transformations.

Zero mode on branes:

$$\int_I d^4x\, \overline{\psi}_L i \not{D}_L \psi_L$$

$$D_{L\mu} = \partial_\mu - iA_\mu(x_\mu, 0)$$



$$\int_{II} d^4x\, \overline{\psi}_R i \not{D}_R \psi_R$$

$$D_{R\mu} = \partial_\mu - iA_\mu(x_\mu, R)$$

Dirac mass:

$$m\overline{\psi}_L(x_\mu, 0) W \psi_R(x_\mu, R) + h.c.$$

Wilson Line:

$$W = \exp(i \int_0^R A_5(x_\mu, x_5) dx^5)$$

Gauge transformation in D=5:

$$A_A(x_\mu, y) \to A_A(x_\mu, y) + \partial_A \theta(x_\mu, y)$$

$$\psi_L(x_\mu) \rightarrow \exp(i\theta(0, x_\mu))\psi_L(x_\mu)$$

$$\psi_R(x_\mu) \rightarrow \exp(i\theta(R, x_\mu))\psi_R(x_\mu)$$

The theory is anomalous:

$$\to S_{branes} - \int_I d^4x\, \theta(x_\mu, 0)\partial_\mu J_L^\mu - \int_{II} d^4x\, \theta(x_\mu, R)\partial_\mu J_R^\mu$$

$$\partial_\mu J_L^\mu = -\frac{1}{48\pi^2} F^{\mu\nu}(0)\widetilde{F}_{\mu\nu}(0)$$

$$\partial_\mu J_R^\mu = \frac{1}{48\pi^2} F^{\mu\nu}(R)\widetilde{F}_{\mu\nu}(R)$$

Consistent Anomalies

D=4 Effective Theory

$$S_{full} = \int d^4x \, [\, \overline{\psi}(i\partial\!\!\!/ + V\!\!\!/ + A\!\!\!/\gamma^5 - m)\psi + \frac{1}{12\pi^2}\sum_{nmk} c_{nmk} B^n_\mu B^m_\nu \widetilde{F}^{k\mu\nu}$$
$$- \frac{1}{4e^2} F^0_{\mu\nu} F^{0\mu\nu} - \frac{1}{4e'^2}\sum_{n\geq 1} F^n_{\mu\nu} F^{n\mu\nu} + \sum_{n\geq 1}\frac{1}{2e_n^2} M_n^2 B^n_\mu B^{n\mu}]$$

$$V_\mu = \sum_{n\ even} B^n_\mu, \qquad A_\mu = \sum_{n\ odd} B^n_\mu$$

if we truncate the theory on the zero mode B^0 and first KK-mode, B^1,

$$\frac{1}{12\pi^2} c_{100} B^1_\mu B^0_\nu \widetilde{F}^{0\mu\nu} = \frac{1}{6\pi^2}\epsilon^{\mu\nu\rho\sigma} A_\mu V_\nu \partial_\rho V_\sigma$$

D=4 Effective Theory Current Algebra

$$\tilde{J}^n_\mu = \frac{\delta S}{\delta B^{n\mu}} = \overline{\psi}\gamma_\mu\psi|_{n\ even} + \overline{\psi}\gamma_\mu\gamma^5\psi|_{n\ odd} + J^{n\ CS}_\mu$$

$$J^{n\ CS}_{mu} = \frac{\epsilon_{\mu\nu\rho\sigma}}{12\pi^2}\sum_{mk}\left[(c_{nmk} - c_{mnk} + c_{kmn} - c_{mkn})B^{m\nu}\partial^\rho B^{k\sigma}\right]$$

$$\partial^\mu J^n_\mu = \frac{1}{48\pi^2}\sum_{mk}\left(1-(-1)^{n+m+k}\right)F^m_{\mu\nu}\widetilde{F}^{k\mu\nu} \qquad \partial^\mu J^{n\ CS}_\mu = \frac{1}{48\pi^2}\sum_{m,k}(c_{nmk} - c_{mnk} + c_{nkm} - c_{knm})F^m_{\mu\nu}\widetilde{F}^{k\mu\nu}$$

KK-mode anomalies: $$\partial^\mu \tilde{J}^n_\mu = \frac{1}{24\pi^2}\sum_{m,k} d_{nmk}F^m_{\mu\nu}\widetilde{F}^{k\mu\nu}$$

$$\begin{aligned} d_{nmk} &= \frac{1}{2}\left[(1-(-1)^{n+m+k}) + (c_{nmk} - c_{mnk} + c_{nkm} - c_{knm})\right] \\ &= \frac{3}{2}[(-1)^{n+m+k} - 1]\frac{n^2(k^2+m^2-n^2)}{(k+m-n)(k+m+n)(k+n-m)(k-m-n)} \\ &= \frac{3}{2}c_{nmk} \end{aligned}$$

$$c_{0mk} = 0 \qquad \partial^\mu J^5_\mu = \frac{1}{16\pi^2}\left(c_{100}F_{\gamma\ \mu\nu}\widetilde{F}^{k\mu\nu}_\gamma + c_{111}F_{B\ \mu\nu}\widetilde{F}^{k\mu\nu}_B\right) = \frac{1}{8\pi^2}F_{\gamma\ \mu\nu}\widetilde{F}^{\mu\nu}_\gamma + \frac{1}{24\pi^2}F_{B\ \mu\nu}\widetilde{F}^{\mu\nu}_B$$