Pairing from Repulsive Interactions in Quantum Hall Physics

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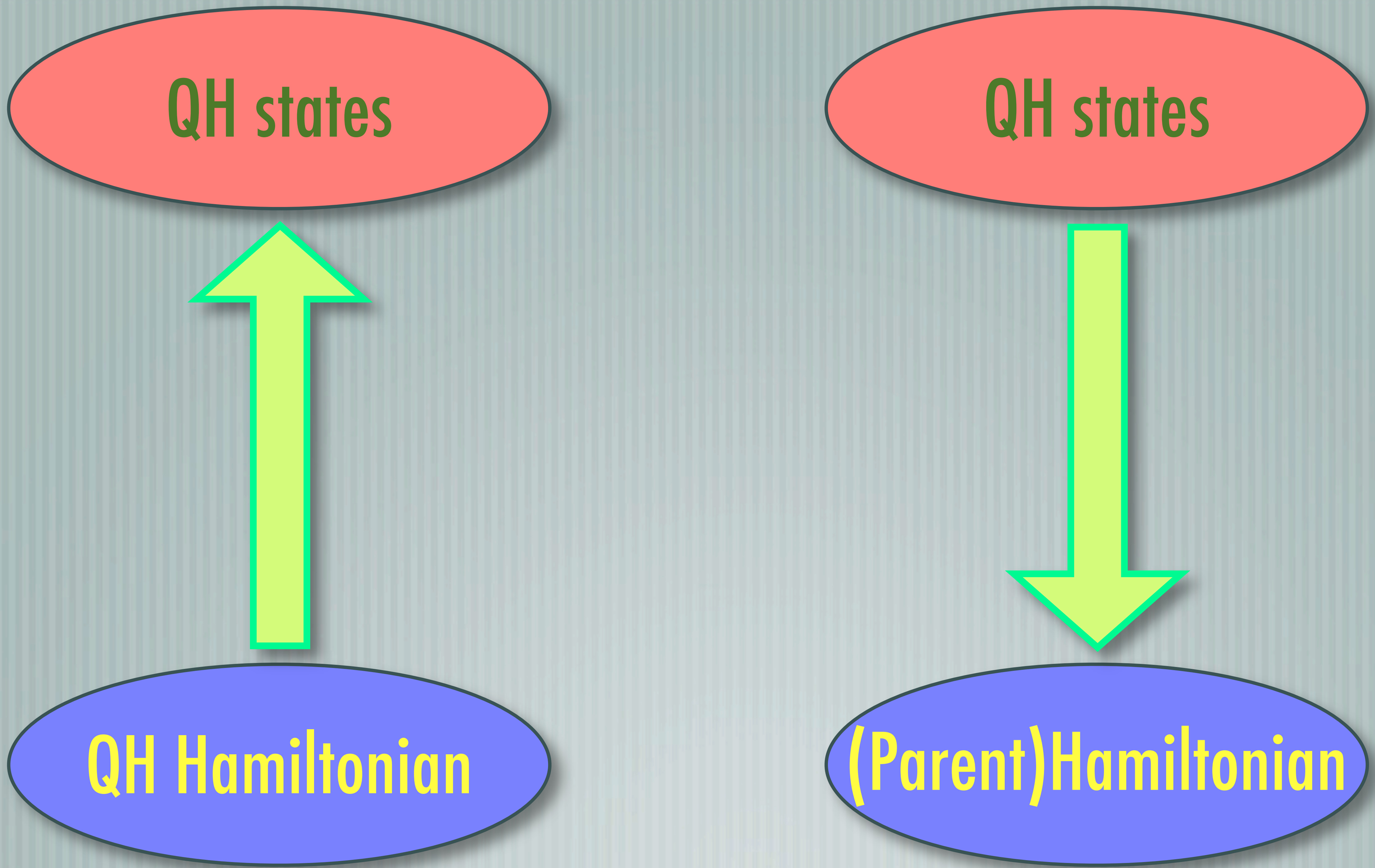
Zohar Nussinov: Washington University - St. Louis

Jorge Dukelsky: CSIC - Madrid

arXiv:1306.3268

Bottom-Up Approach to QH



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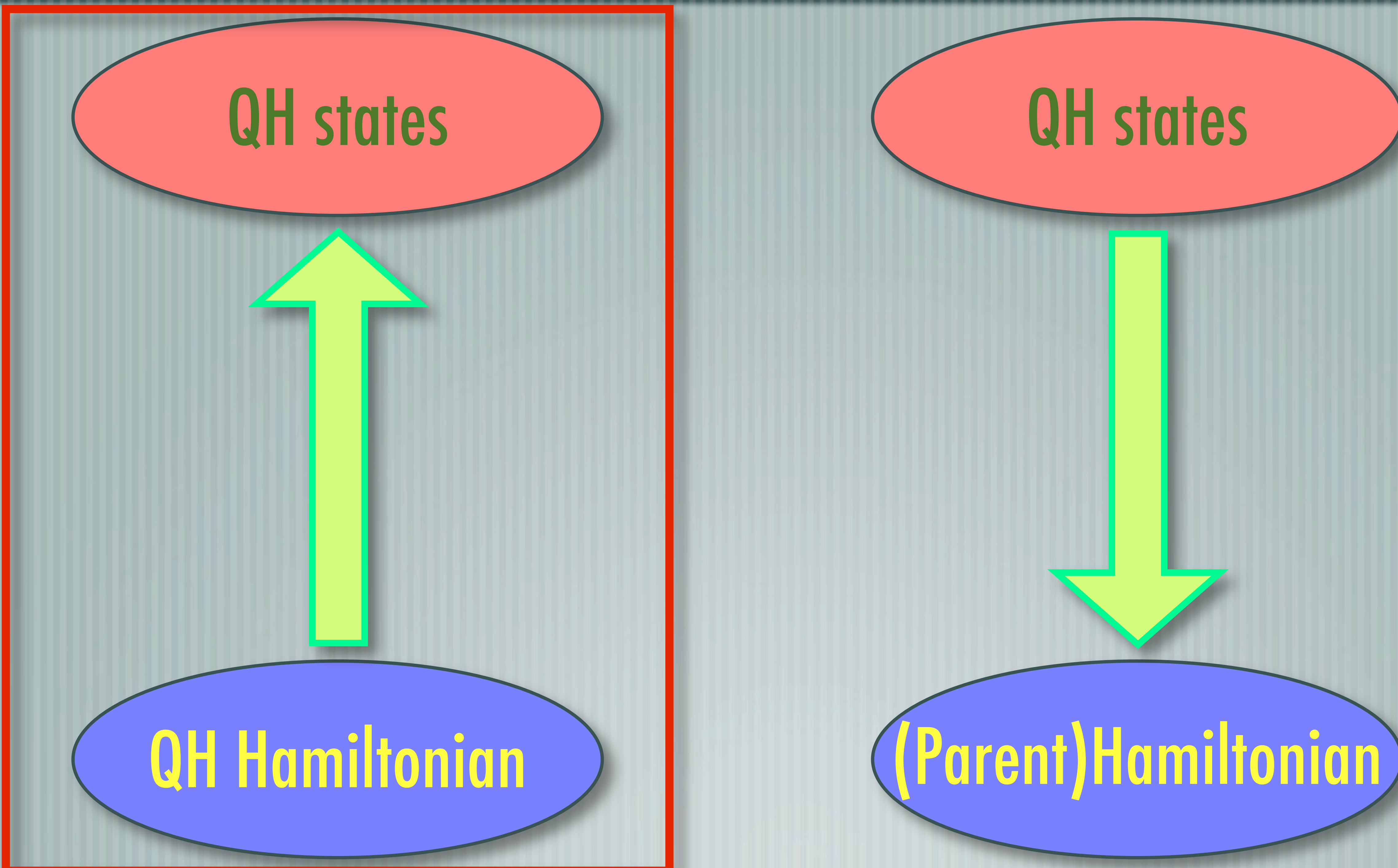


Motivation for this work

- FQH fluids are archetypical examples of interacting systems displaying TQO

- We need to deeply understand its excitations if we want to use its supposedly non-Abelian features for fault tolerant topological information processing

- Derivation of states with filling fractions other than Laughlin's? Need some organizing principle (parent Hamiltonians?)

- How about edge modes?

Main Messages

- A deep connection between Pairing and Quantum Hall Physics
- The Quantum Hall Hamiltonian is the direct sum of exactly-solvable hyperbolic Richardson-Gaudin models (the most general interaction is a sum of separable potentials)
- Second quantization formulation of Quantum Hall which is a "guiding center" language
- The most general interaction is a sum of separable potentials

Why Topological Quantum Order?

- New states of matter where the traditional Landau paradigm is not applicable

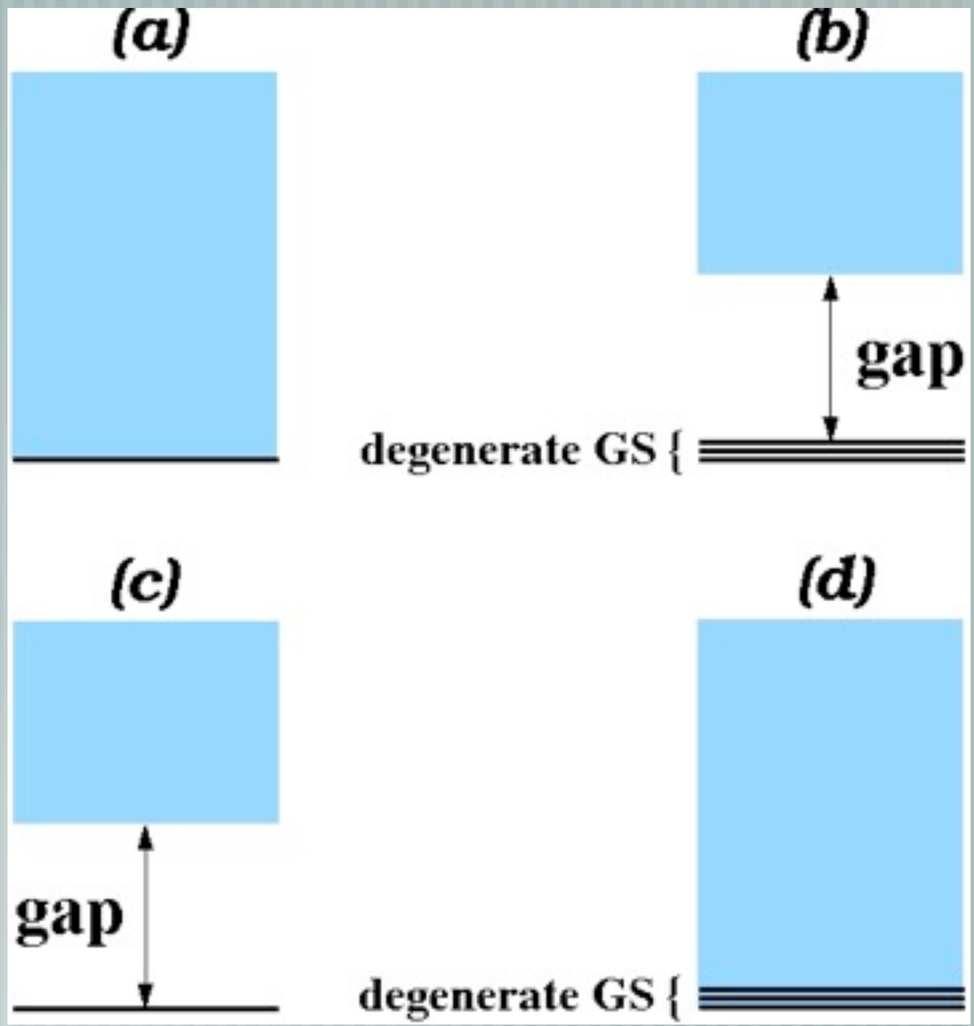
  A new quantum vacuum (TQM)
  (Different from Landau vacua)

  Can we engineer them?

- Topological Quantum Computation: Hardware Fault-tolerance

  Robustness against local perturbations

Spectra



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  Defeating Decoherence

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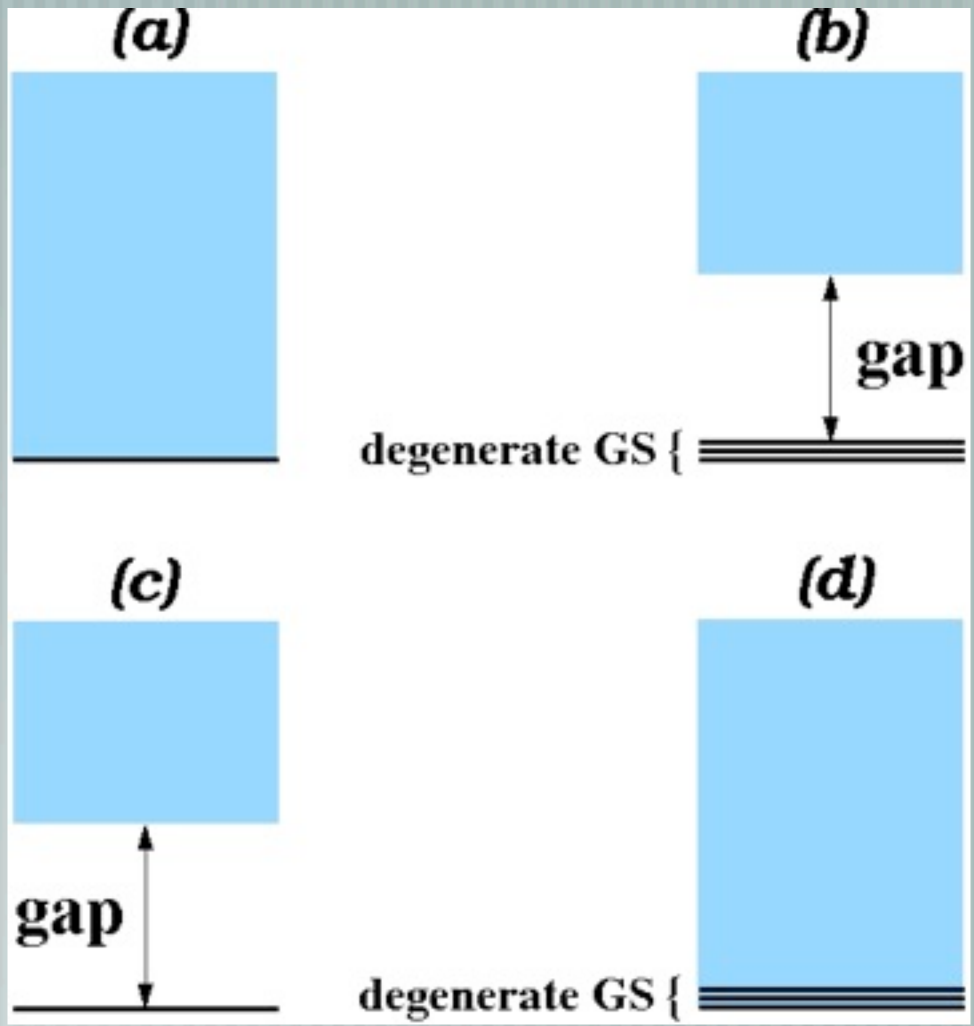

Why Topological Quantum Order?

- Functionalities other than computer hardware:

Quantum Memories

Precision measurements (quantum metrology)?

Background independent “emergent” space?
(Toy Models of Quantum Gravity)

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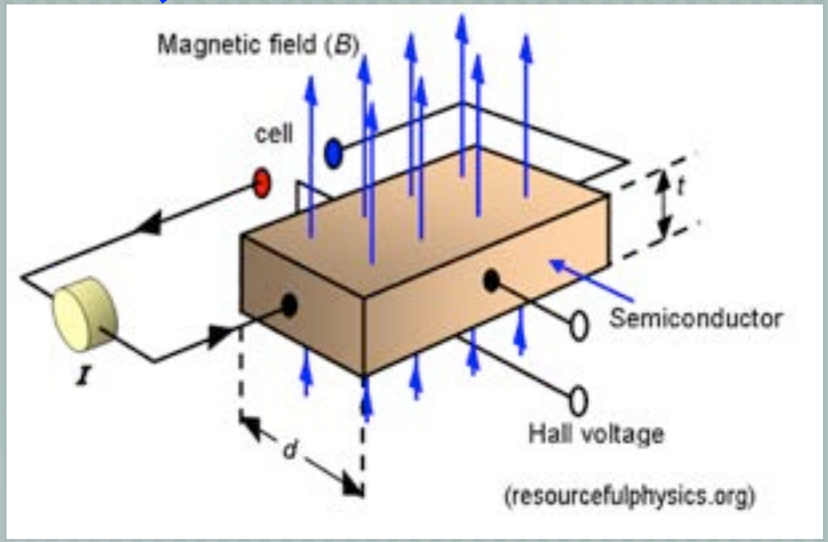
Given current interests in topological insulators (superconductors) and in building a Quantum computer

Is there a unifying theory (such as Landau) for TQM?

Old Examples of TQM



- Fractional Quantum Hall Liquids
- Kitaev's Toric code model

$$H = -\sum_s A_s - \sum_p B_p$$

$$A_s = \prod_{j \in \mathrm{star}(s)} \sigma_j^x \qquad B_p = \prod_{j \in \mathrm{boundary}(p)} \sigma_j^z$$

- Some spin liquids

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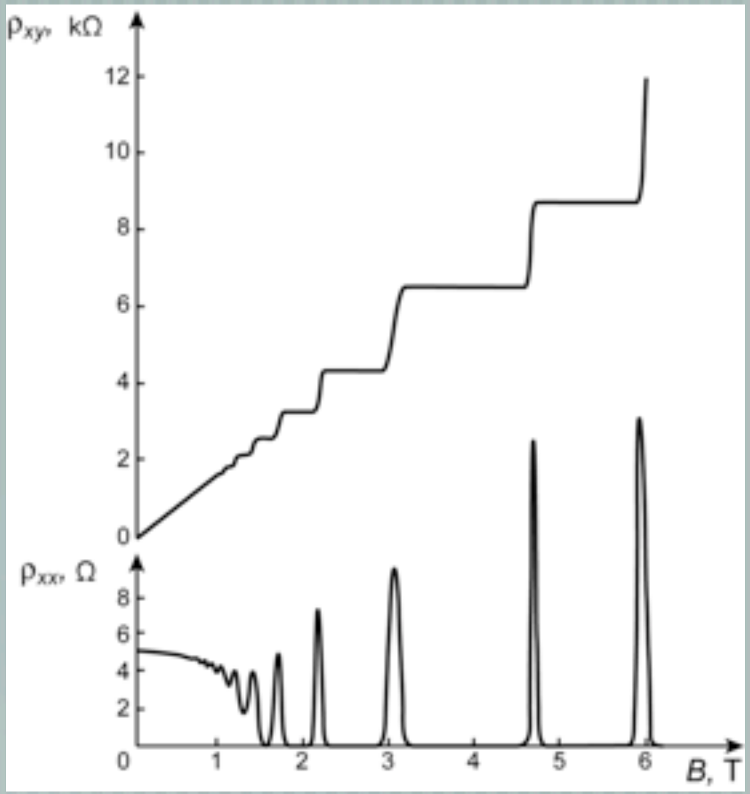

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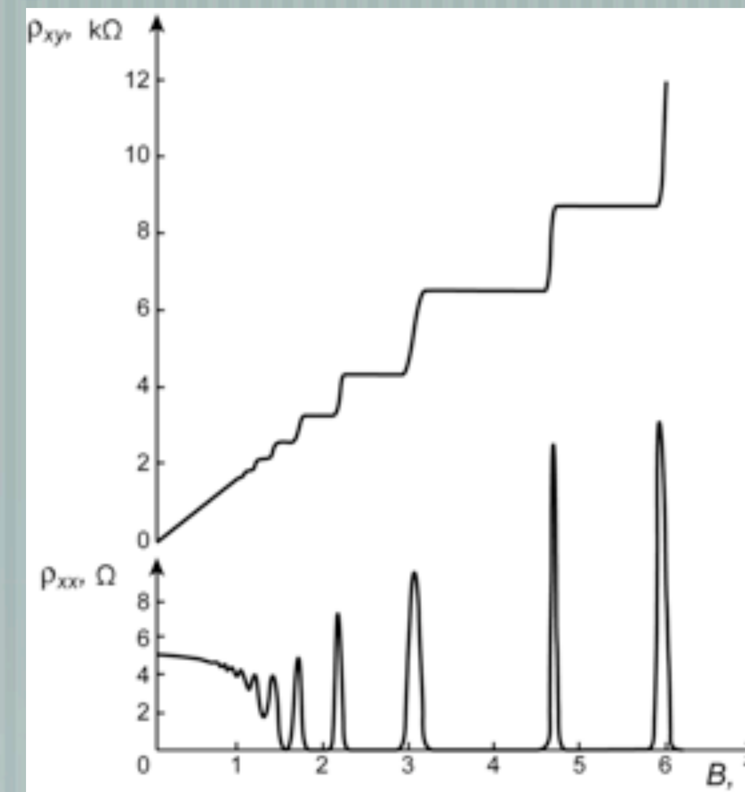

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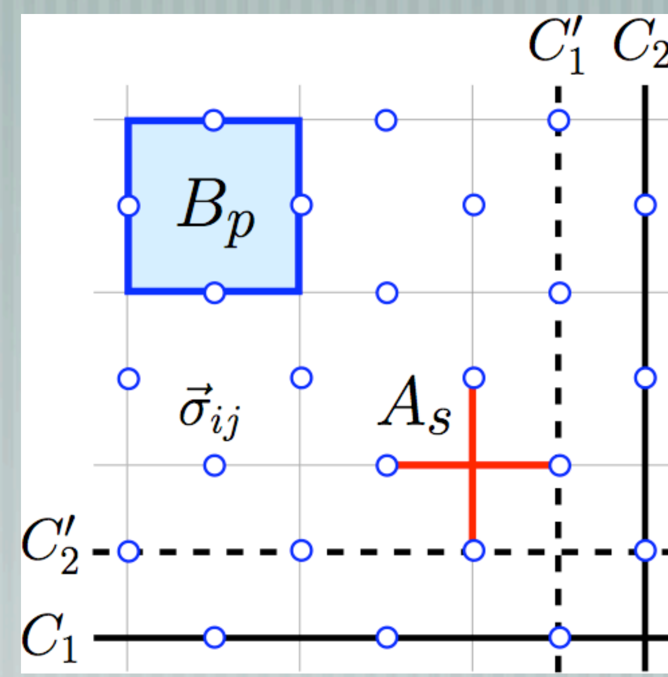

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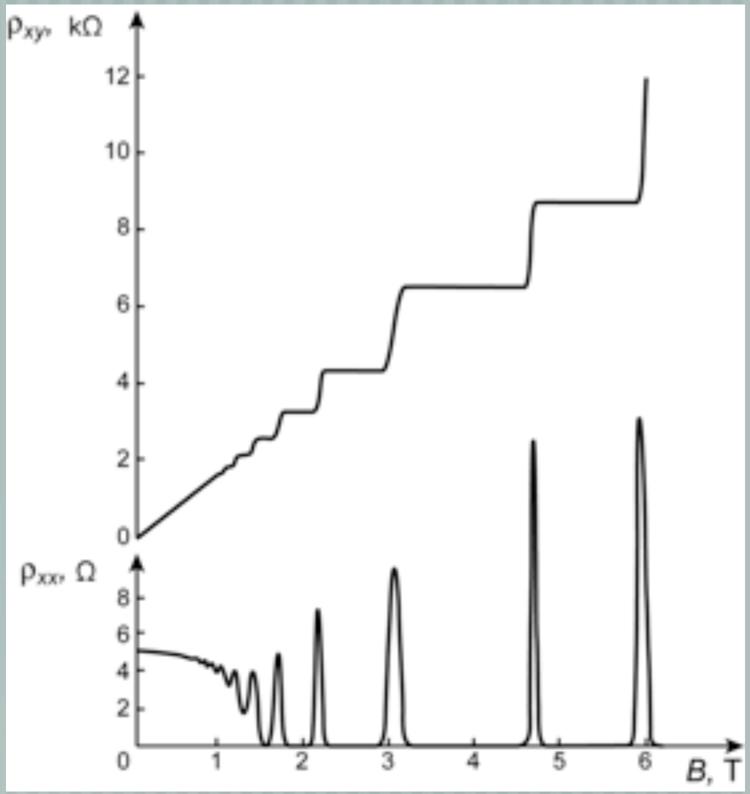

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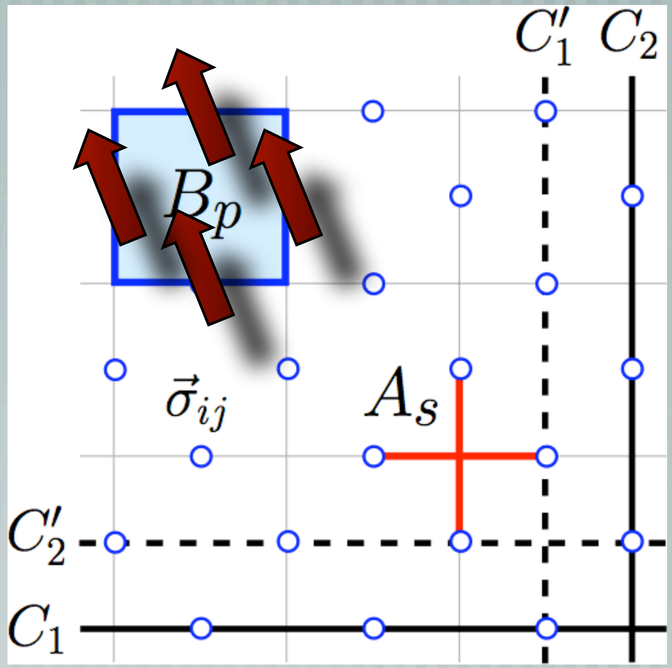

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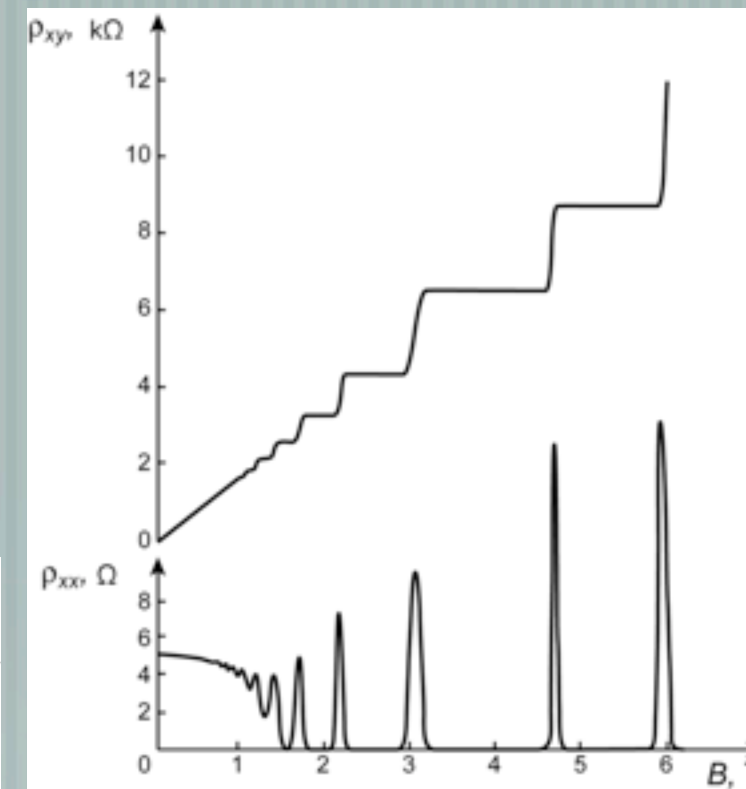

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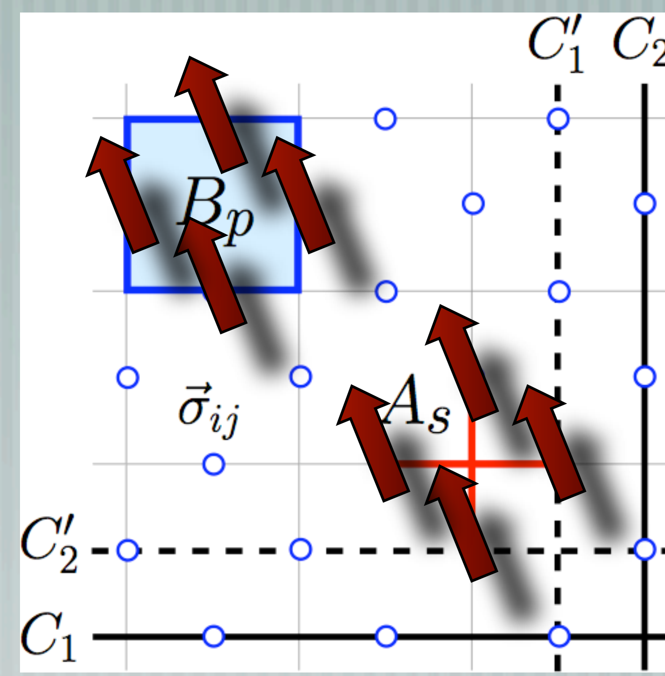

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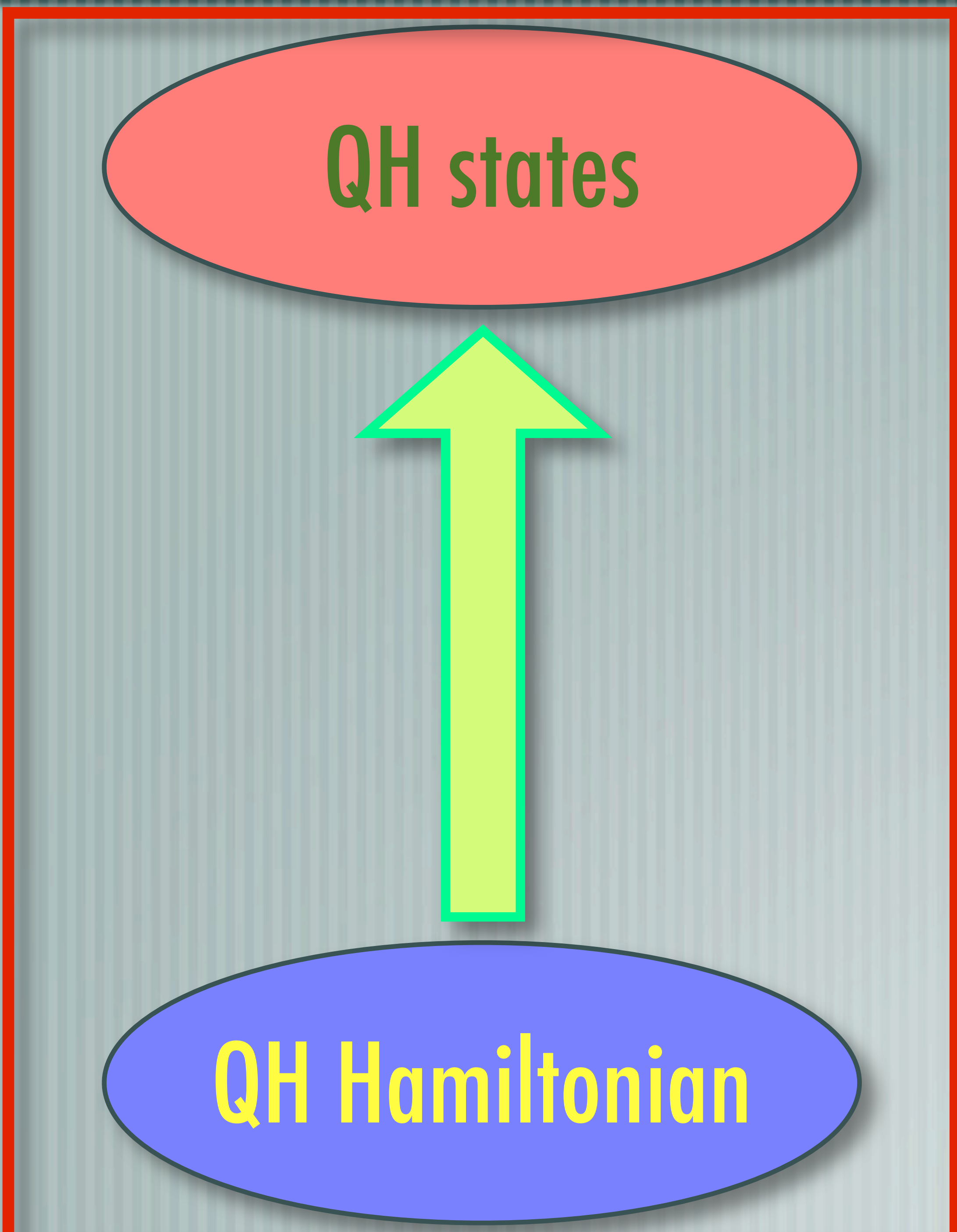
Outline



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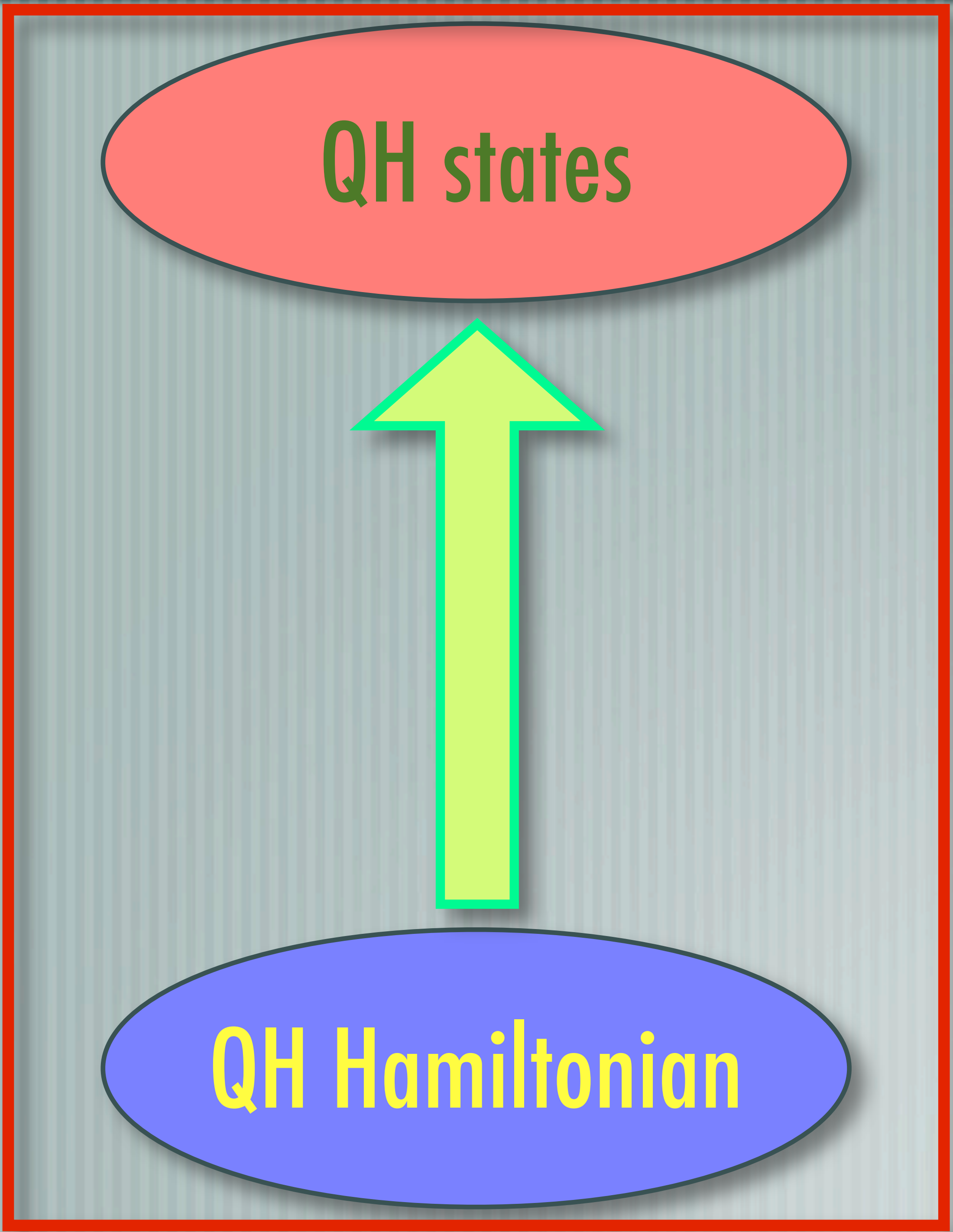

- Setup the QH Hamiltonian in second quantization

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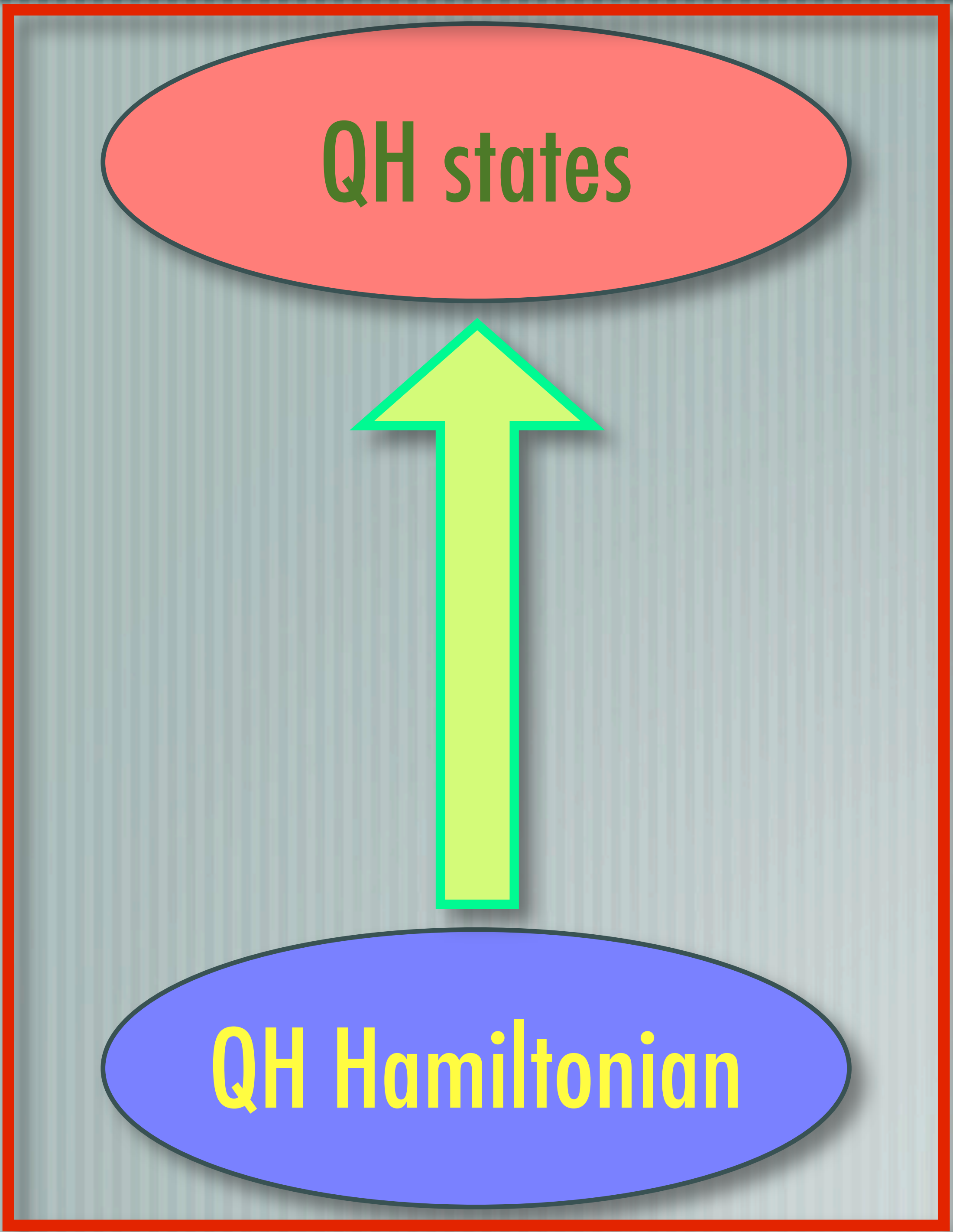

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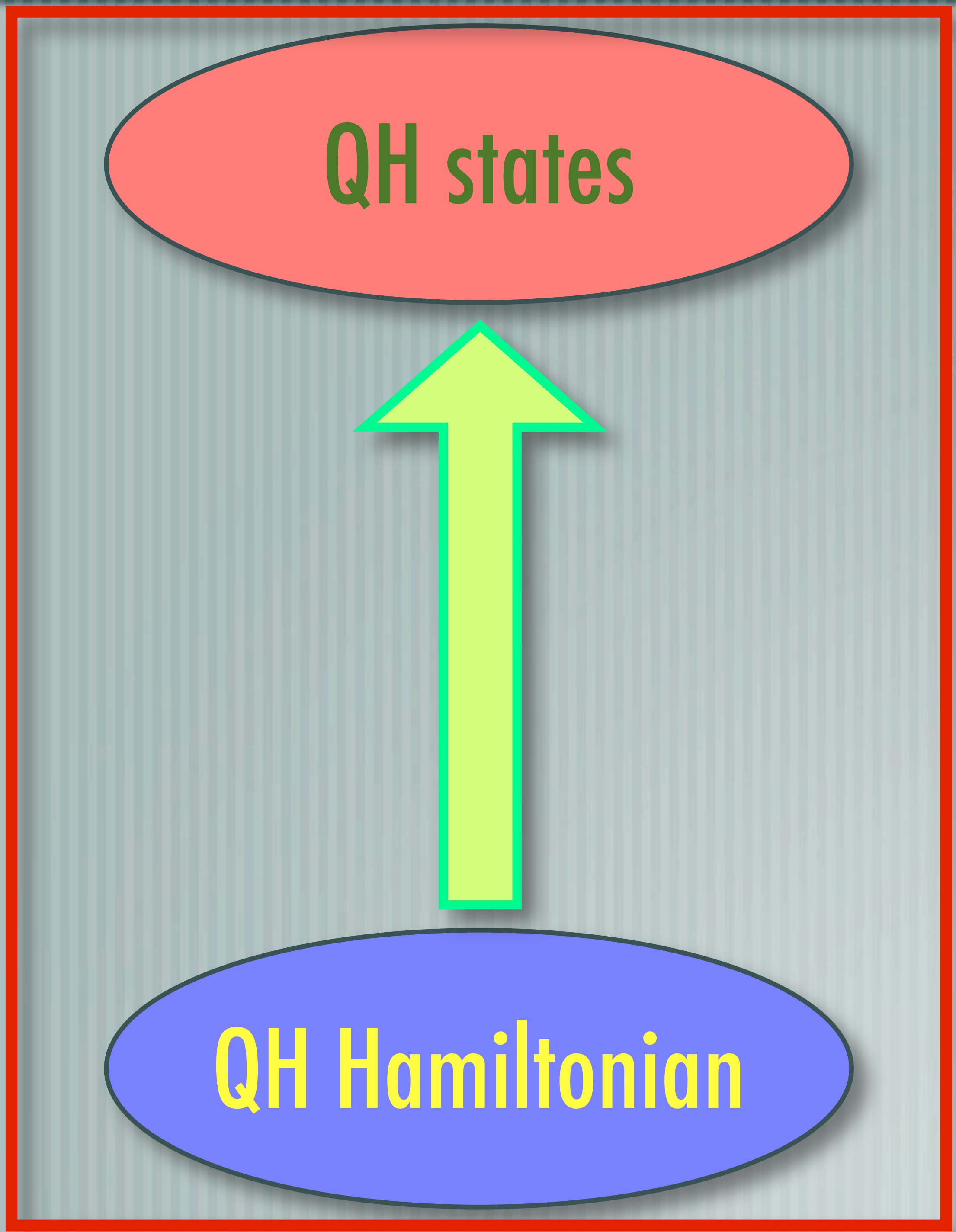

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- Relate to a Pairing problem
- Study the Ker of the QH problem in terms of the Kers of the Pairing problems

Quantum Hall Physics

An Exercise in Second Quantization

Dimensional Reduction/Holography

The correlation function inequalities are general and not specific to any model. In general they lead to:

- Effective dimensional reduction

- Exact dimensional reduction:

Inequalities become equalities

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Duality connecting the two theories

TQO is a property of States not of the Spectrum

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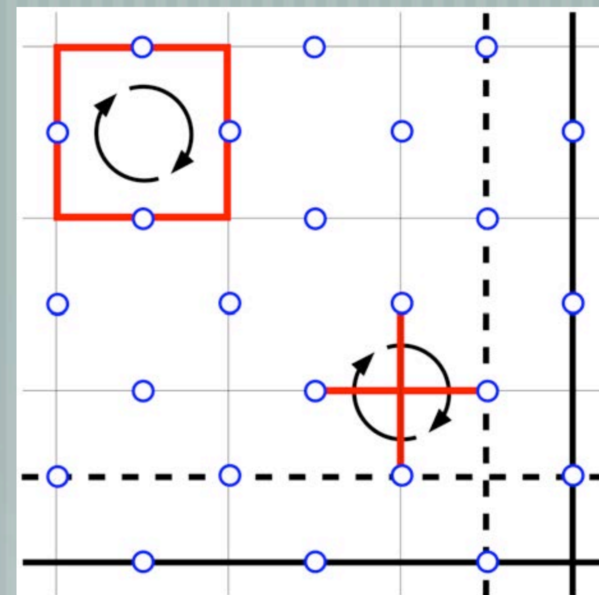
Kitaev's toric code model:



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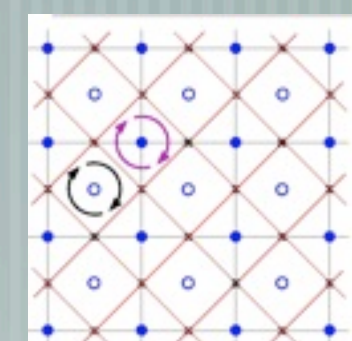
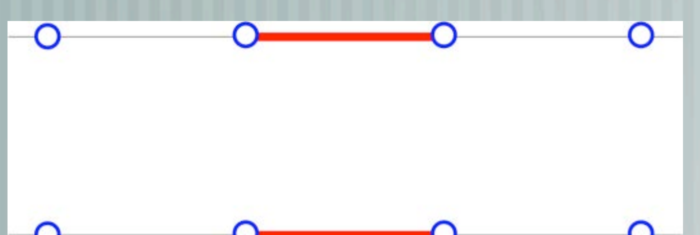


Duality mappings: Non-local
(Identical spectra)

(Nussinov-Ortiz 2006)

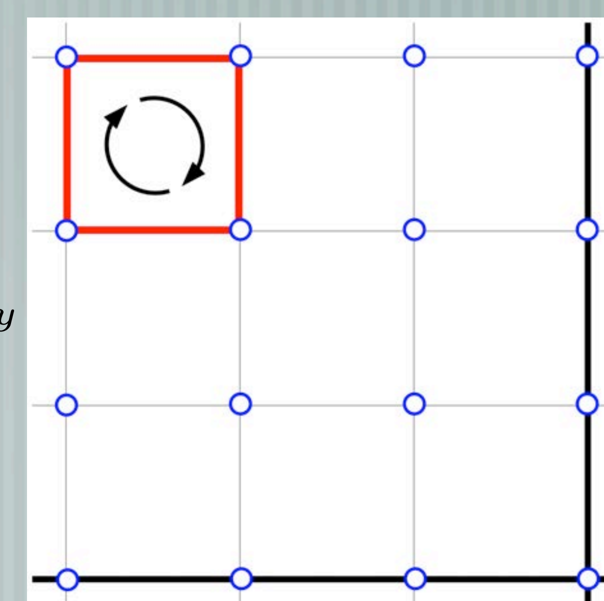
2 Ising chains:



$$H_I = -\sum_s \sigma^z_s \sigma^z_{s+1} - \sum_p \sigma^z_p \sigma^z_{p+1}$$

Wen's plaquette model:

$$H_W = -\sum_i \sigma^x_i \sigma^y_{i+\hat{e}_x} \sigma^x_{i+\hat{e}_x+\hat{e}_y} \sigma^y_{i+\hat{e}_y}$$



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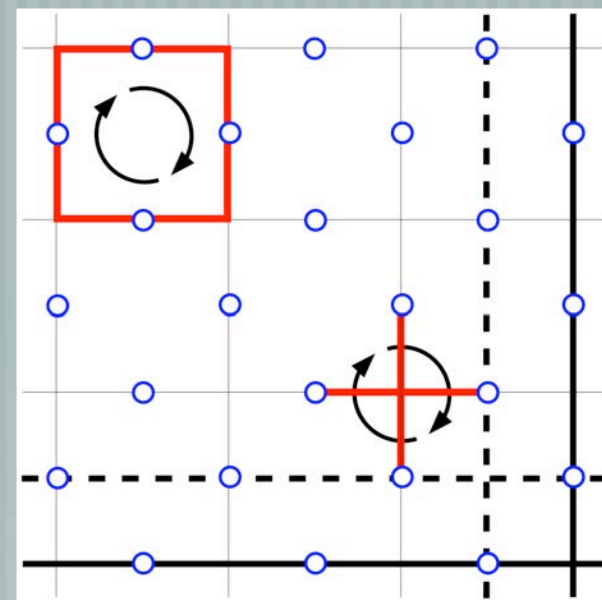

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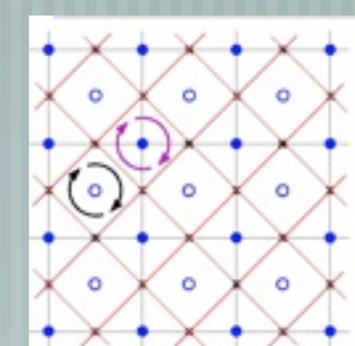

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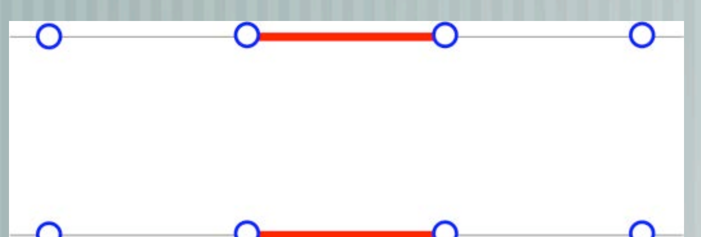

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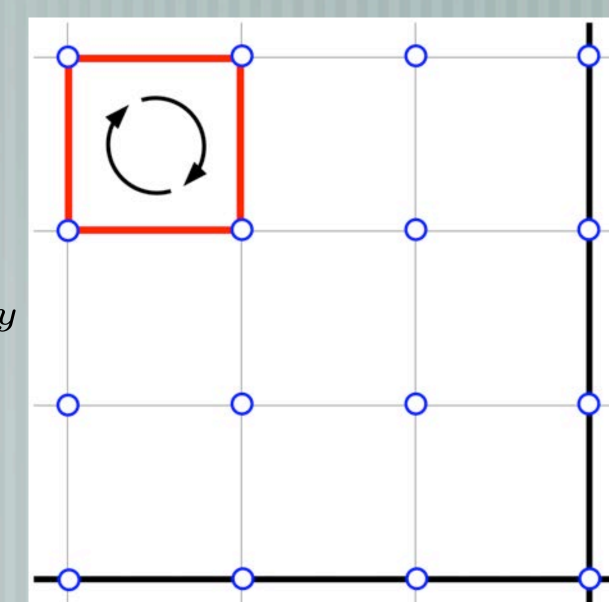
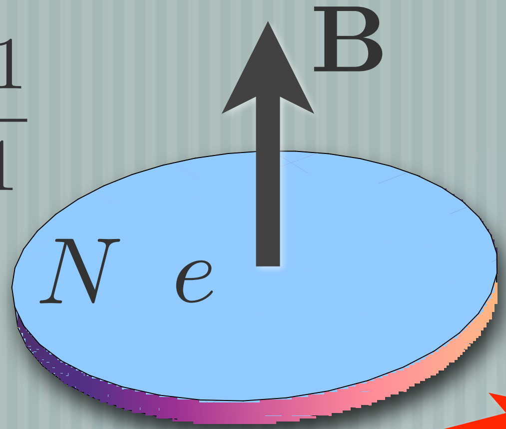


Entanglement is non-local with respect to the local language

Dimensional Reduction - QH Physics

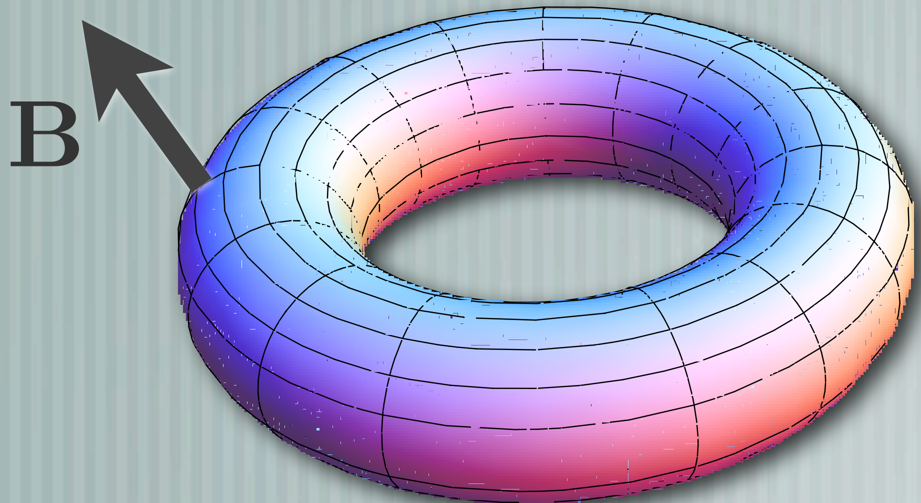
First Quantization

$$\nu = \frac{N-1}{L-1}$$



dynamical momenta

$$H_{\mathsf{QH}} = \sum_{\mathsf{i}=1}^{N} \frac{\mathbf{\Pi}_{\mathsf{i}}^2}{2m} + \sum_{\mathsf{i}<\mathsf{j}} V(\mathbf{x}_{\mathsf{i}} - \mathbf{x}_{\mathsf{j}})$$



2D continuous geometries

Dimensional Reduction - QH Physics

First Quantization | **Second Quantization**

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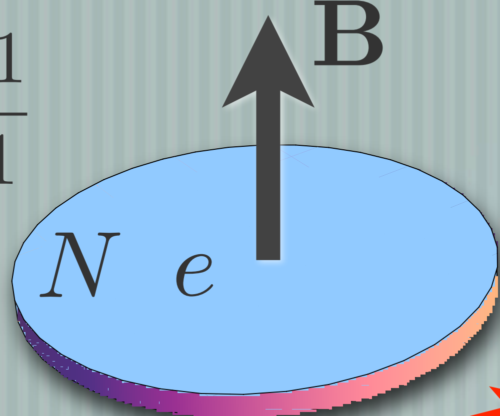

$$\hat{P}_{\mathsf{LLL}} H_{\mathsf{QH}} \hat{P}_{\mathsf{LLL}}$$

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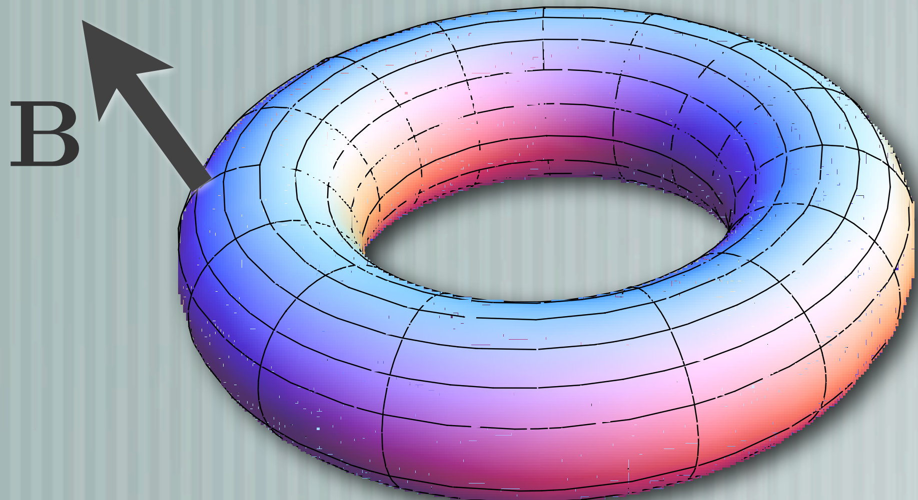
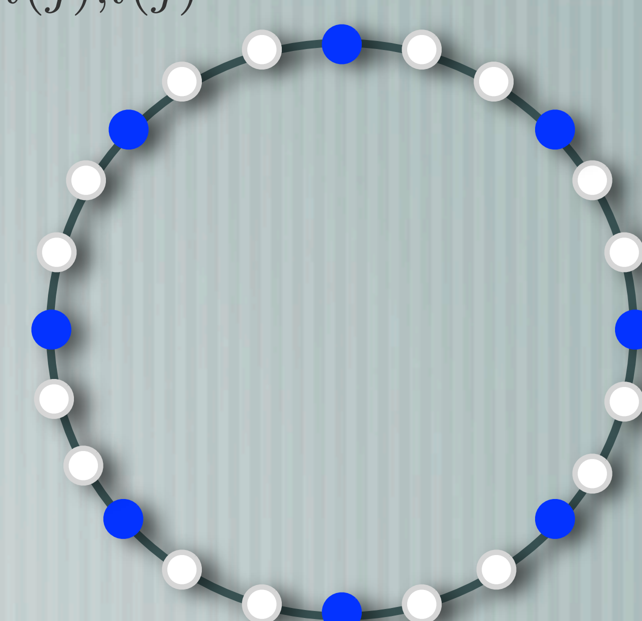
$$\widehat{H}_{\mathsf{QH}} = \sum_{0<j<L-1} \sum_{k(j),l(j)} V_{j;kl}\; c^{\dagger}_{j+k} c^{\dagger}_{j-k} c_{j-l} c_{j+l}$$





2D continuous geometries | 1D orbital lattices

Second Quantization Counting:

- j takes the $2L-3$ values:

$$j_{\mathsf{min}} = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, j_{\mathsf{m}} = \frac{L-1}{2}, \ldots, j_{\mathsf{max}} = L - \frac{3}{2}$$

- Sums over $k(j)$ involve $\mathcal{C}(j)$ active orbital levels:

$$\sum_{k(j)} = \sum_{0<k\leq\min(j,L-1-j)} \quad \text{with} \quad \mathcal{C}(j) = \min([j+\frac{1}{2}],[L-\frac{1}{2}-j])$$

This is a guiding center formulation with the geometrical information (dynamical momenta) encoded in the matrix elements $V_{j;kl}$

Separability of Pseudopotentials

Given an arbitrary spherically symmetric interaction:

$$V(\mathbf{x}_{\mathsf{i}} - \mathbf{x}_{\mathsf{j}}) = \sum_{m \geq 0} g_m V_m = \sum_{m \geq 0} g_m \sum_{\mathsf{i}<\mathsf{j}} P_m(\mathsf{ij})$$

with $g_m \geq 0$ and $P_m(\mathsf{ij})$ a projector onto the subspace of relative angular momentum m of the pair (ij)

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We have shown that in second quantization: $\hat{H}_{\mathsf{QH}} = \sum_{m \geq 0} g_m \; \hat{H}_{V_m}$

with $$\hat{H}_{V_m} = \sum_{0<j<L-1} \sum_{k(j),l(j)} \eta_k \eta_l \; c^{\dagger}_{j+k} c^{\dagger}_{j-k} c_{j-l} c_{j+l}$$

For the 1st Haldane pseudopotential or Trugman-Kivelson model:

geometry	L (Laughlin)	N_Φ	η_k	$\phi_r(z)$
disk	$mN-m+1$	L	$k\,2^{-j}\sqrt{\frac{1}{2\pi j}\binom{2j}{j+k}}$	$\frac{1}{\sqrt{2\pi 2^r r!}}\,z^r e^{-\frac{1}{4}\|z\|^2}$
cylinder	$mN-m+1$	L	$\kappa^{3/2}\,k\,e^{-\kappa^2 k^2}$	$\sqrt{\kappa}\,e^{-\frac{1}{2}(x-r\kappa)^2+ir\kappa y}$
sphere	$mN-m+1$	$L-1$	$k\frac{N_\Phi+1}{4\sqrt{2\pi j}}\sqrt{\binom{2N_\Phi}{2j}^{-1}\frac{(6N_\Phi-5)N_\Phi}{(2N_\Phi-1)(2N_\Phi-2j)}\binom{N_\Phi}{j+k}\binom{N_\Phi}{j-k}}$	$\sqrt{\frac{N_\Phi+1}{4\pi}\binom{N_\Phi}{r}}[e^{-i\frac{\varphi}{2}}\sin(\frac{\theta}{2})]^r[e^{i\frac{\varphi}{2}}\cos(\frac{\theta}{2})]^{N_\Phi-r}$
torus	mN	L	$\kappa^{3/2}\sum_{s\in\mathbb{Z}}(k+sL)\,e^{-\kappa^2(k+sL)^2}$	$\sum_{s\in\mathbb{Z}}\phi^{\text{cylinder}}_{r+sL}$

- In the case of the cylinder for arbitrary m

$$\eta_k = \frac{e^{-\kappa^2 k^2}}{2^{\frac{m}{2}}\sqrt{m!}}\,H_m[\sqrt{2}\,\kappa k] \longrightarrow \quad \text{Hermite poly}$$

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We have shown that geometries with the same genus number can be related through similarity transformations

Strongly-Coupled States of Matter

Generalized Gaudin Problems

Generalized Gaudin Algebra

- **GGA:** $(\kappa = x, y, z, \text{ and } W = X, Y, Z)$

$$m \neq \ell \begin{cases} [\mathsf{S}^\kappa_m, \mathsf{S}^\kappa_\ell] = 0 \,, \\ [\mathsf{S}^x_m, \mathsf{S}^y_\ell] = i(Y_{m\ell}\, \mathsf{S}^z_m - X_{m\ell}\, \mathsf{S}^z_\ell) \,, \\ [\mathsf{S}^y_m, \mathsf{S}^z_\ell] = i(Z_{m\ell}\, \mathsf{S}^x_m - Y_{m\ell}\, \mathsf{S}^x_\ell) \,, \\ [\mathsf{S}^z_m, \mathsf{S}^x_\ell] = i(X_{m\ell}\, \mathsf{S}^y_m - Z_{m\ell}\, \mathsf{S}^y_\ell) \,, \end{cases} \qquad m \to \ell \begin{cases} [\mathsf{S}^\kappa_m, \mathsf{S}^\kappa_m] = 0 \,, \\ [\mathsf{S}^x_m, \mathsf{S}^y_m] = -i\ \mathsf{f}(\eta_m) \frac{\partial \mathsf{S}^z_m}{\partial \eta_m} \,, \\ [\mathsf{S}^y_m, \mathsf{S}^z_m] = -i\ \mathsf{f}(\eta_m) \frac{\partial \mathsf{S}^x_m}{\partial \eta_m} \,, \\ [\mathsf{S}^z_m, \mathsf{S}^x_m] = -i\ \mathsf{f}(\eta_m) \frac{\partial \mathsf{S}^y_m}{\partial \eta_m} \,, \end{cases}$$

$$W_{m\ell} = W(\eta_m, \eta_\ell) \in \text{ antisymmetric} \qquad \lim_{\varepsilon \to 0} \varepsilon W(x, x+\varepsilon) = \mathsf{f}(x)$$

From Jacobi identities: $\begin{cases} Z_{m\ell} X_{\ell n} + Z_{nm} Y_{\ell n} + X_{nm} Y_{m\ell} = 0 \\ X^2_{m\ell} - Z^2_{m\ell} = \Gamma_1 \,,\ X^2_{m\ell} - Y^2_{m\ell} = \Gamma_2 \end{cases}$ **Gaudin eqns.**

Generalized Gaudin Algebra (GGA) + quantum invariants allow derivation of several families of exactly-solvable models including the BCS reduced Hamiltonian

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$$
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From Jacobi identities:
$$
\begin{cases} Z_{m\ell} X_{\ell n} + Z_{nm} Y_{\ell n} + X_{nm} Y_{m\ell} = 0 \\ X^2_{m\ell} - Z^2_{m\ell} = \Gamma_1 \,, \ X^2_{m\ell} - Y^2_{m\ell} = \Gamma_2 \end{cases}
$$
Gaudin eqns.

Quantum Invariants: $[H_m, H_\ell] = 0$

$$H(\eta_m) \equiv H_m = \mathsf{S}^x_m \mathsf{S}^x_m + \mathsf{S}^y_m \mathsf{S}^y_m + \mathsf{S}^z_m \mathsf{S}^z_m = \vec{\mathsf{S}}_m \cdot \vec{\mathsf{S}}_m$$

Diagonalizing XXZ invariants: $H_m|\Phi\rangle = \omega(\eta_m)|\Phi\rangle$,

$$\mathsf{S}^-_m|0\rangle = 0 \quad , \quad \mathsf{S}^z_m|0\rangle = F(\eta_m)|0\rangle \quad \forall \eta_m \, , \quad |0\rangle \text{ lowest-weight vector}$$

- **Bethe ansatz:** $$|\Phi\rangle = \prod_{\ell=1}^{M} \mathsf{S}^+_\ell |0\rangle = \prod_{\ell=1}^{M} (\mathsf{S}^x_\ell + i\mathsf{S}^y_\ell)|0\rangle \ ,$$
- **Eigenvalue:**

$$\omega(\eta_m) = F^2(\eta_m) - \mathsf{f}(\eta_m) \frac{\partial}{\partial \eta_m} F(\eta_m) + \sum_{\ell=1}^{M} \left(\Gamma - 2Z_{m\ell} \, F(\eta_m) + \sum_{n(\neq \ell)=1}^{M} Z_{m\ell} Z_{mn} \right)$$

- **Bethe equation:**

$$F(\eta_\ell) + \sum_{n(\neq\ell)=1}^{M} Z_{n\ell} = 0 \ , \qquad \qquad \ell = 1, \cdots, M$$

Monday, August 5, 2013

Solutions of the XXZ Gaudin equation:

$$Z_{m\ell}X_{\ell n} + Z_{nm}X_{\ell n} + X_{nm}X_{m\ell} = 0$$

$$X_{\ell n} = g\frac{\sqrt{1+st_\ell^2}\sqrt{1+st_n^2}}{t_\ell - t_n},\ Z_{\ell n} = g\frac{1+st_\ell t_n}{t_\ell - t_n}\ ,\quad \Gamma = sg^2,\ t_i = -g/Z_{ri},\ |s| = 0, 1$$

1. *Rational:* $\Gamma = 0, s = 0,$

$$X(\eta_\ell, \eta_n) \quad = \quad Z(\eta_\ell, \eta_n) = g\frac{1}{\eta_\ell - \eta_n},$$

   with $t_i = \eta_i$,

2. *Trigonometric:* $\Gamma > 0, s = +1,$

$$X(\eta_\ell, \eta_n) \quad = \quad g\frac{1}{\sin(\eta_\ell - \eta_n)}\ ,\ Z(\eta_\ell, \eta_n) = g\cot(\eta_\ell - \eta_n)\ ,$$

   with $t_i = \tan(\eta_i)$,

3. *Hyperbolic:* $\Gamma < 0, s = -1,$

$$X(\eta_\ell, \eta_n) \quad = \quad g\frac{1}{\sinh(\eta_\ell - \eta_n)}\ ,\ Z(\eta_\ell, \eta_n) = g\coth(\eta_\ell - \eta_n)\ ,$$

   with $t_i = \tanh(\eta_i)$.

Exactly-solvable models derived from the GGA:

(I) Find realizations of the algebra: e.g. $\bigoplus_{\mathbf{j}} su(2) \quad \{S^+_{\mathbf{j}}, S^-_{\mathbf{j}}, S^z_{\mathbf{j}}\}$

$$\mathsf{S}^{\pm}_m = \sum_{\mathbf{j}\in\mathcal{T}} X_{m\mathbf{j}} S^{\pm}_{\mathbf{j}} \ , \ \mathsf{S}^z_m = -\frac{1}{2}\mathbb{1} - \sum_{\mathbf{j}\in\mathcal{T}} Z_{m\mathbf{j}} S^z_{\mathbf{j}}$$

(II) Rewrite H_m : $\quad H_m[\vec{\mathsf{S}}_m] \rightarrow H_m[\vec{S}_{\mathbf{j}}]$

(III) Use analytic properties of X **and** Z : $\quad R_{\mathbf{i}} = \dfrac{1}{\mathsf{f}(\eta_{\mathbf{i}})} \oint_{\Gamma_{\mathbf{i}}} \dfrac{d\eta_m}{2\pi i} H_m$

Constants of motion: $\quad [R_{\mathbf{i}}, R_{\mathbf{j}}] = 0$

$$\boxed{R_{\mathbf{i}} = S^z_{\mathbf{i}} + 2 \sum_{\mathbf{j}\in\mathcal{T}(\neq \mathbf{i})} \left(\frac{X_{\mathbf{ij}}}{2} (S^+_{\mathbf{i}} S^-_{\mathbf{j}} + S^-_{\mathbf{i}} S^+_{\mathbf{j}}) + Z_{\mathbf{ij}}\, S^z_{\mathbf{i}} S^z_{\mathbf{j}} \right)}$$

(IV) BCS example: $\quad H = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} R_{\mathbf{k}}(X, X) \quad \begin{cases} S^+_{\mathbf{k}} = c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} = (S^-_{\mathbf{k}})^\dagger \\ S^z_{\mathbf{k}} = \frac{1}{2}(n_{\mathbf{k}\uparrow} + n_{-\mathbf{k}\downarrow} - 1) \end{cases}$

Some Examples of exactly-solvable Gaudin models

Gaudin Algebra	Representation	l	Model
XXX	$\bigoplus_l su(2)$-F-P		BCS Richardson
		N	Nuclear Pairing
			BCS $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$
	$\bigoplus_l su(2)$-F-S	N	Particle-hole-like
	$\bigoplus_l su(1,1)$-B	N	B BCS
	$\bigoplus_l su(2) \oplus su(2)$	N	Central Spin
	$\bigoplus_l su(1,1) \oplus su(1,1)$	N	B Central Spin
XXZ	$\bigoplus_l su(2)$-F-P	2	Suhl-Matthias-Walker
	$\bigoplus_l su(1,1)$-B		Lipkin-Meshkov-Glick
		2	Interacting Boson (IBM1)
			Two-Josephson-coupled BECs
	$\bigoplus_l su(2) \oplus h_4$	N	Generalized Dicke, F-atom-molecule
	$\bigoplus_l su(1,1) \oplus h_4$	N	B-atom-molecule
	$\bigoplus_l su(2)$-F-S $\oplus su(2)$	N	Kondo-like impurity
	$\bigoplus_l h_4 \oplus su(2)$	N	Special Spin-Boson
XYZ	$\bigoplus_l su(2)$	N	Generalized XYZ Gaudin

Some Examples of exactly-solvable Gaudin models

$$\boxed{H = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}}\, n_{\mathbf{k}} + \frac{G}{V} \sum_{\mathbf{k},\mathbf{k}'} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}\,,}$$

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$p + ip$ **superconductor**

$p_x + ip_y$ Fermionic Superfluid

Hyperbolic Gaudin Hamiltonian

A particular realization of the hyperbolic Gaudin model is:

$$H_h = \sum_k \eta_k S_k^z - G \sum_{k,k'} \sqrt{\eta_k \eta_{k'}} S_k^+ S_{k'}^-$$

with Eigenspectrum:

$$|\Phi_M\rangle = \prod_{\alpha=1}^{M} \left(\sum_k \frac{\sqrt{\eta_k}}{\eta_k - E_\alpha} S_k^+ \right) |\nu\rangle \qquad E(\Phi_M) = \langle \nu | H_h | \nu \rangle + \sum_{\alpha=1}^{M} E_\alpha$$

and Gaudin (Bethe) equations: $\left(Q = \frac{1}{2G} - \frac{L}{2} + M - 1\right)$

$$\sum_i \frac{s_i}{\eta_i - E_\alpha} - \sum_{\alpha', \alpha' \neq \alpha} \frac{1}{E_{\alpha'} - E_\alpha} - \frac{Q}{E_\alpha} = 0$$

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Pairon

One can choose the SU(2) fermionic representation:

$$S_k^+ = \frac{k_x + ik_y}{|k|} c_k^\dagger c_{-k}^\dagger \qquad S_k^- = \frac{k_x - ik_y}{|k|} c_{-k} c_k$$

$$S_k^z = \frac{1}{2}\left(c_k^\dagger c_k + c_{-k}^\dagger c_{-k} - 1\right)$$

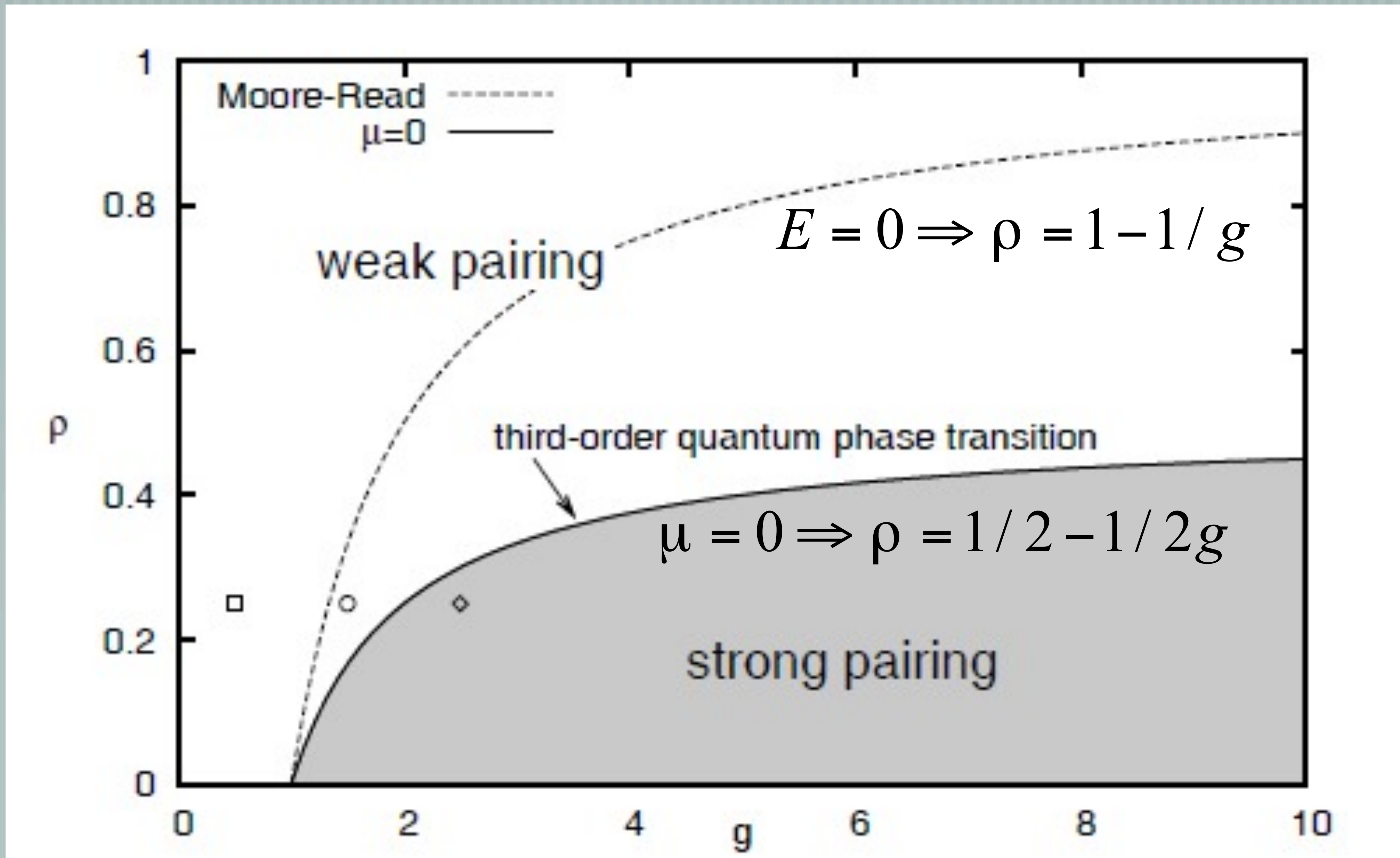
And by also choosing: $\eta_k = k^2$

One obtains the p+ip superconducting model:

$$H_{p_x+ip_y} = \sum_{k,k_x>0} \frac{k^2}{2}\left(c_k^\dagger c_k + c_{-k}^\dagger c_{-k}\right) - G \sum_{\substack{k,k_x>0,\\ k',k'_x>0}} (k_x + ik_y)(k'_x - ik'_y) c_k^\dagger c_{-k}^\dagger c_{-k'} c_{k'}$$

Quantum Phase Diagram

The phase diagram can be parametrized in terms of the density $\rho = M/L$ and the rescaled coupling $g = GL$



Quantum Phase Diagram

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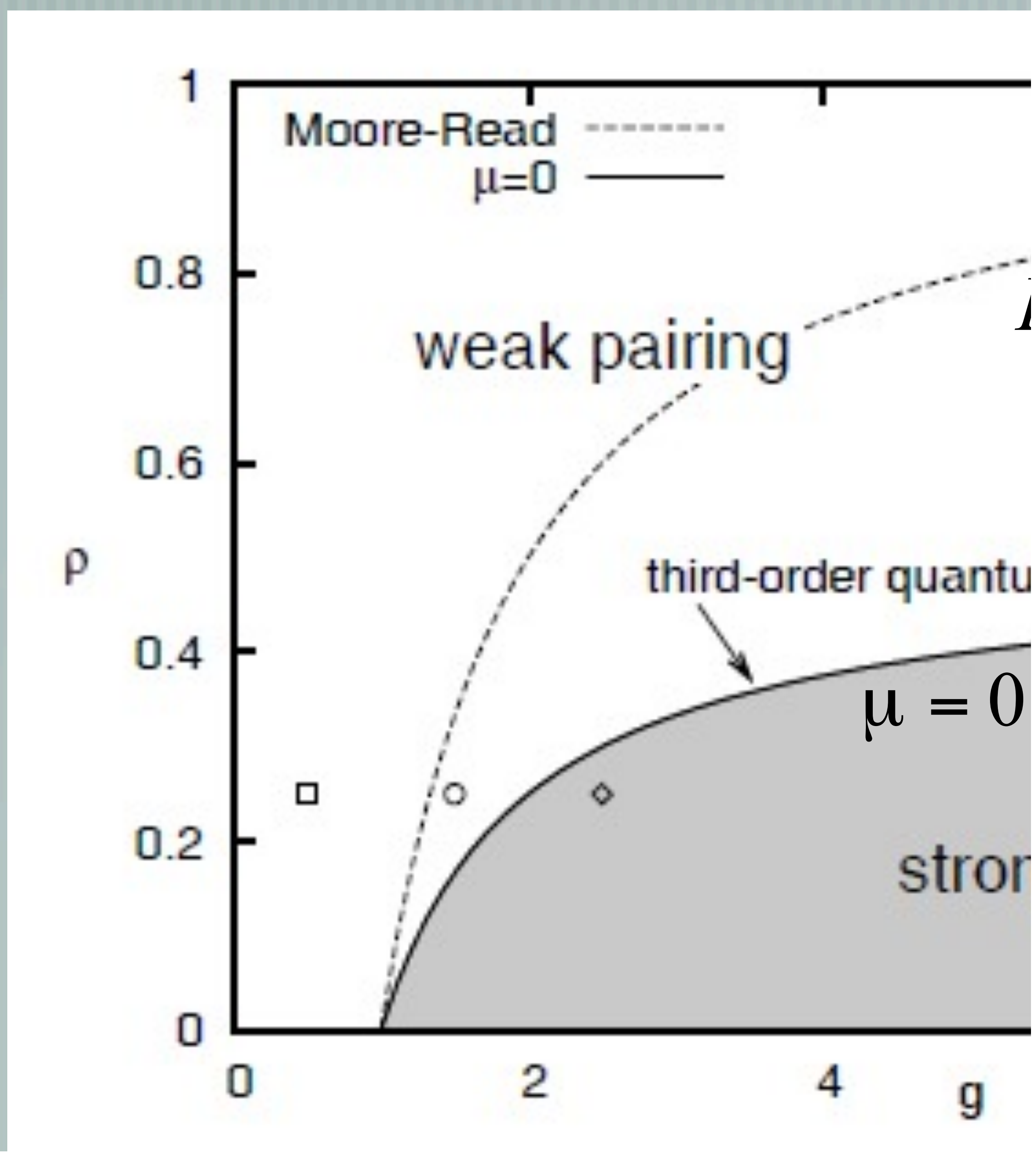

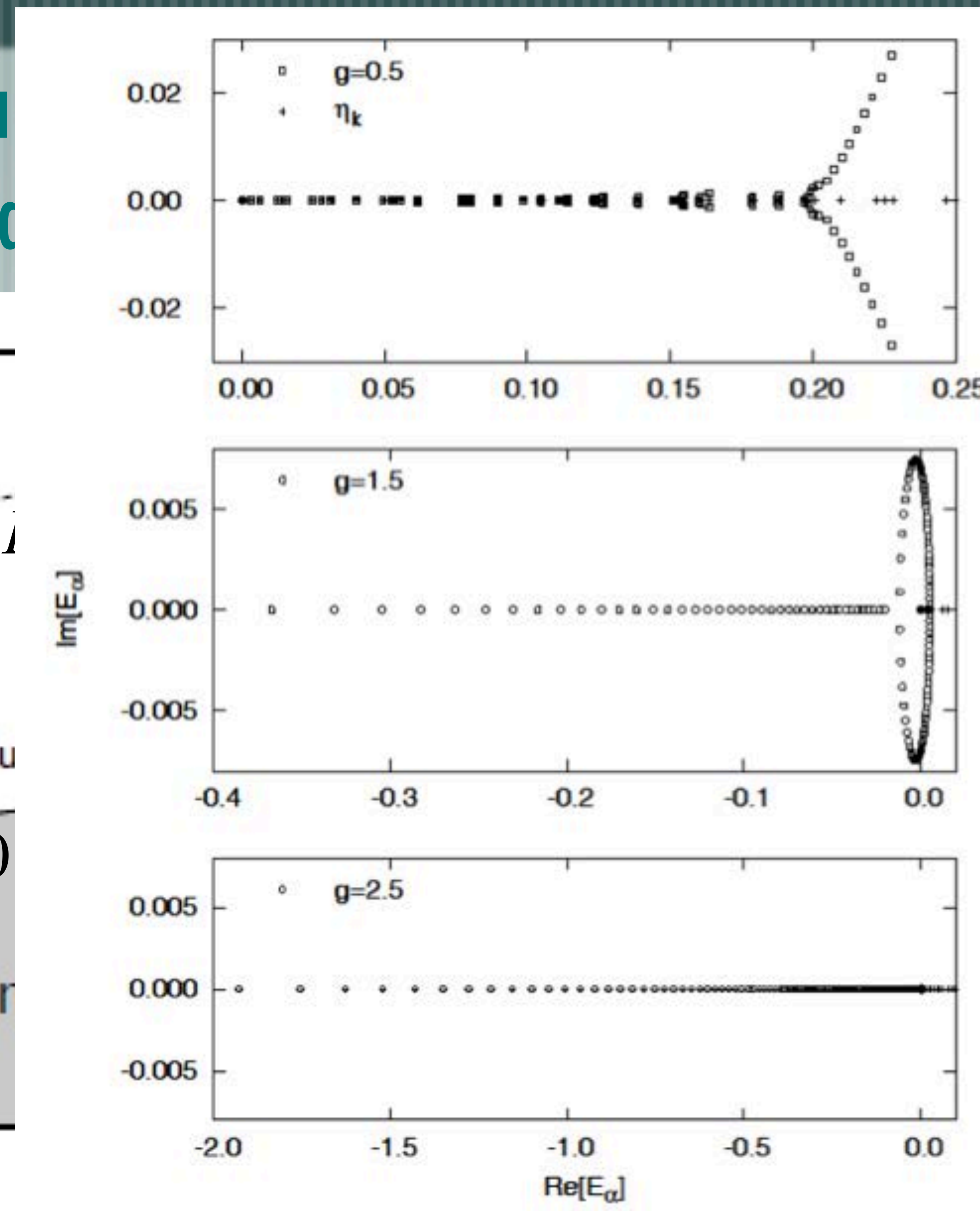

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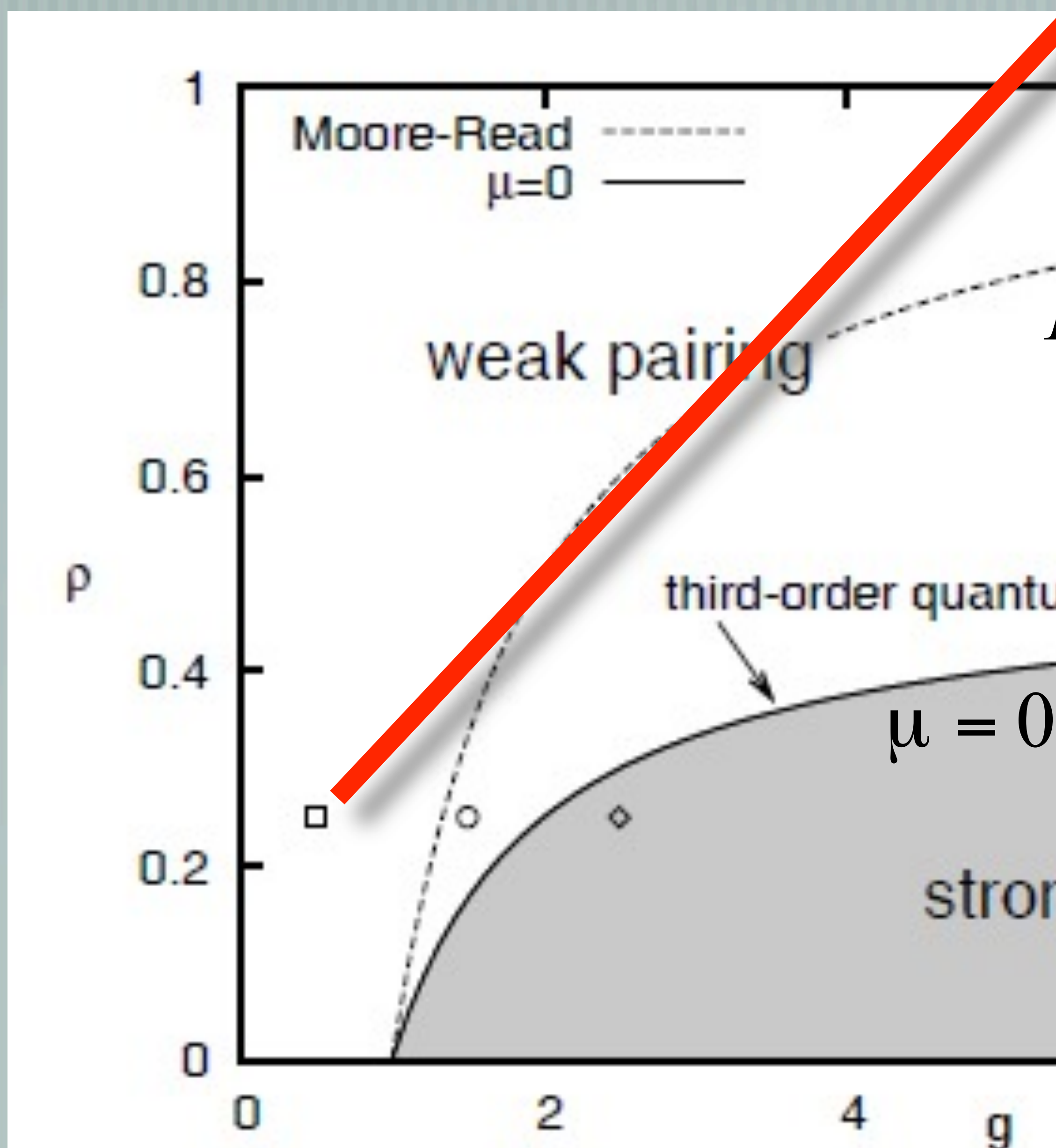

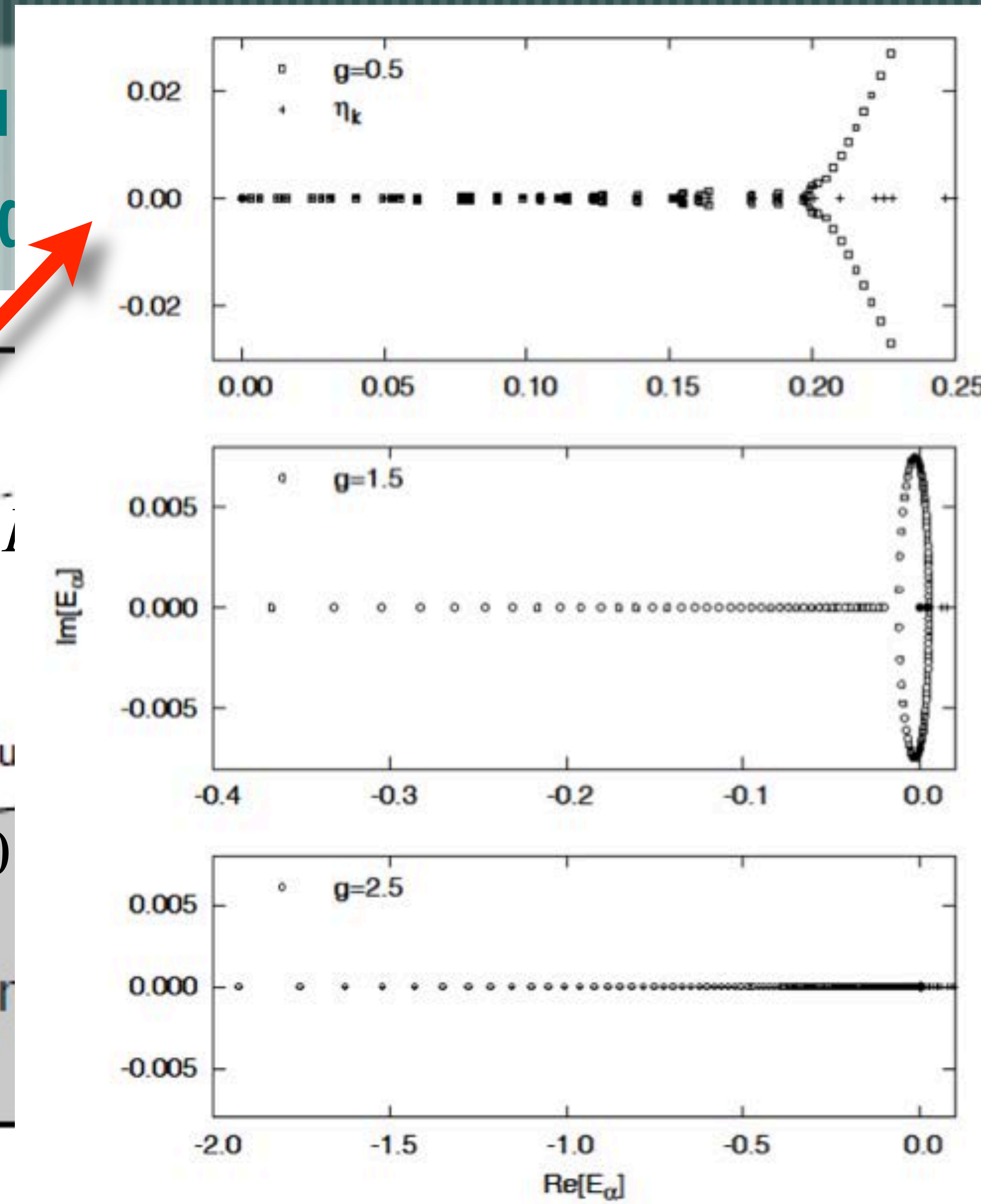

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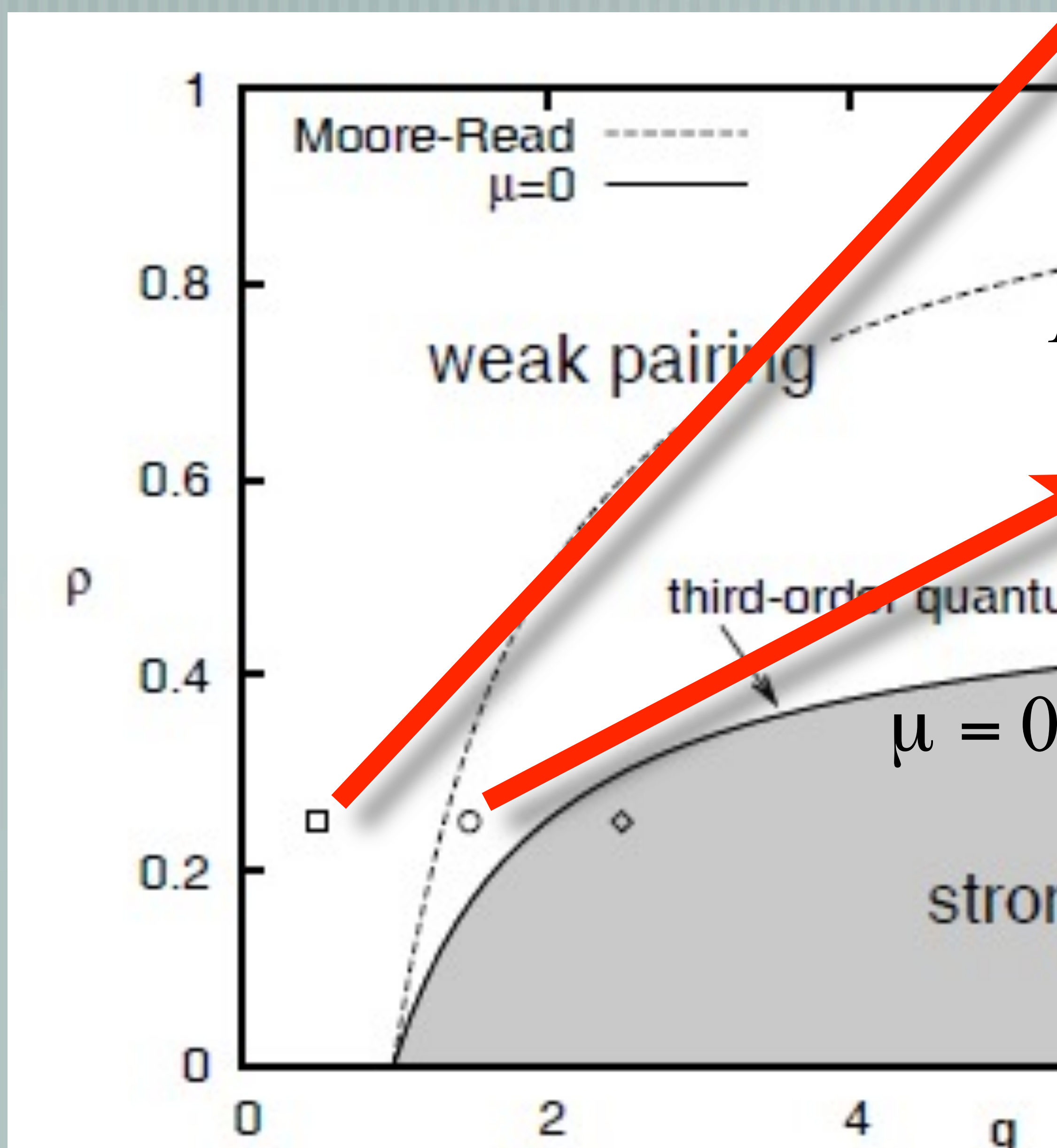

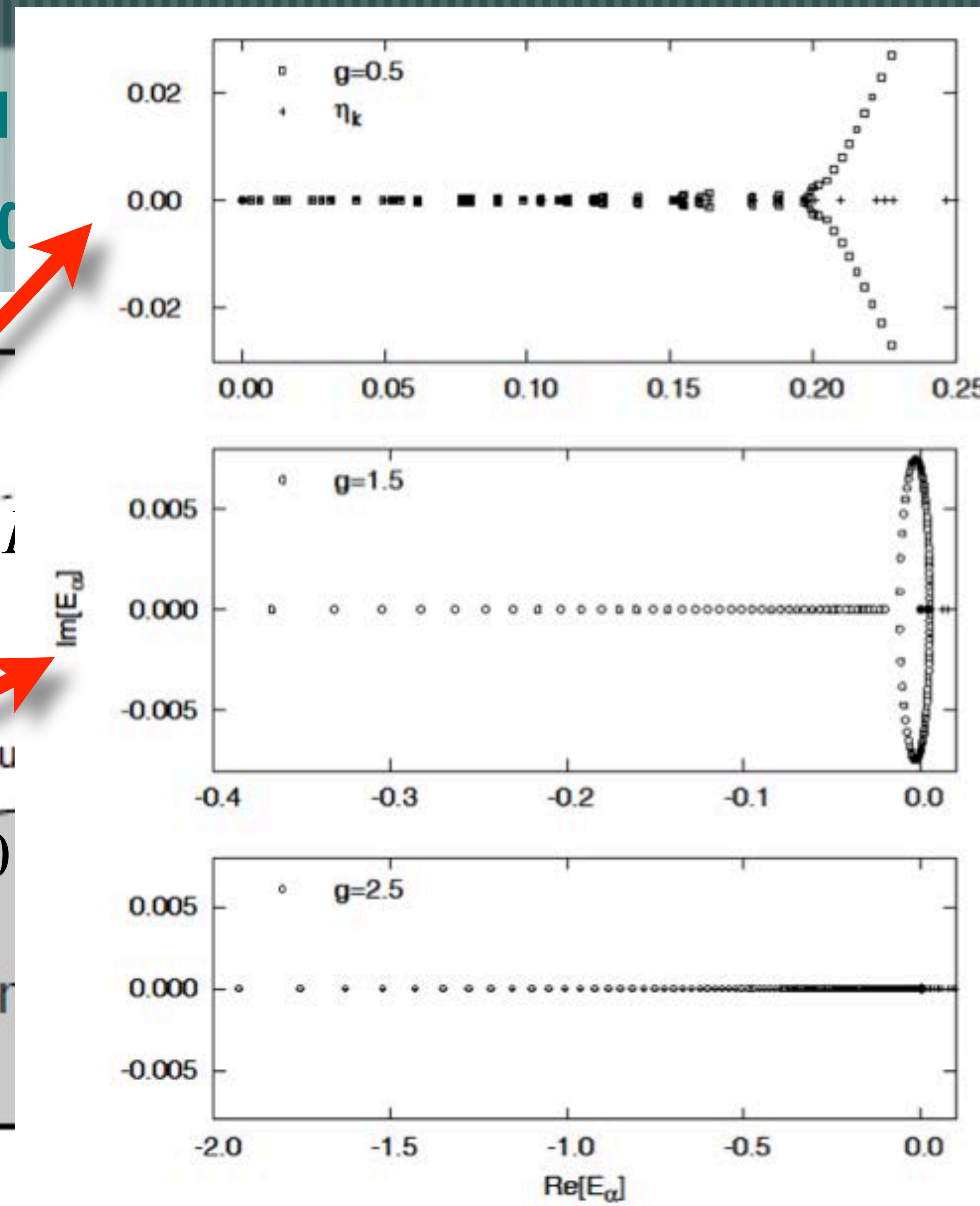

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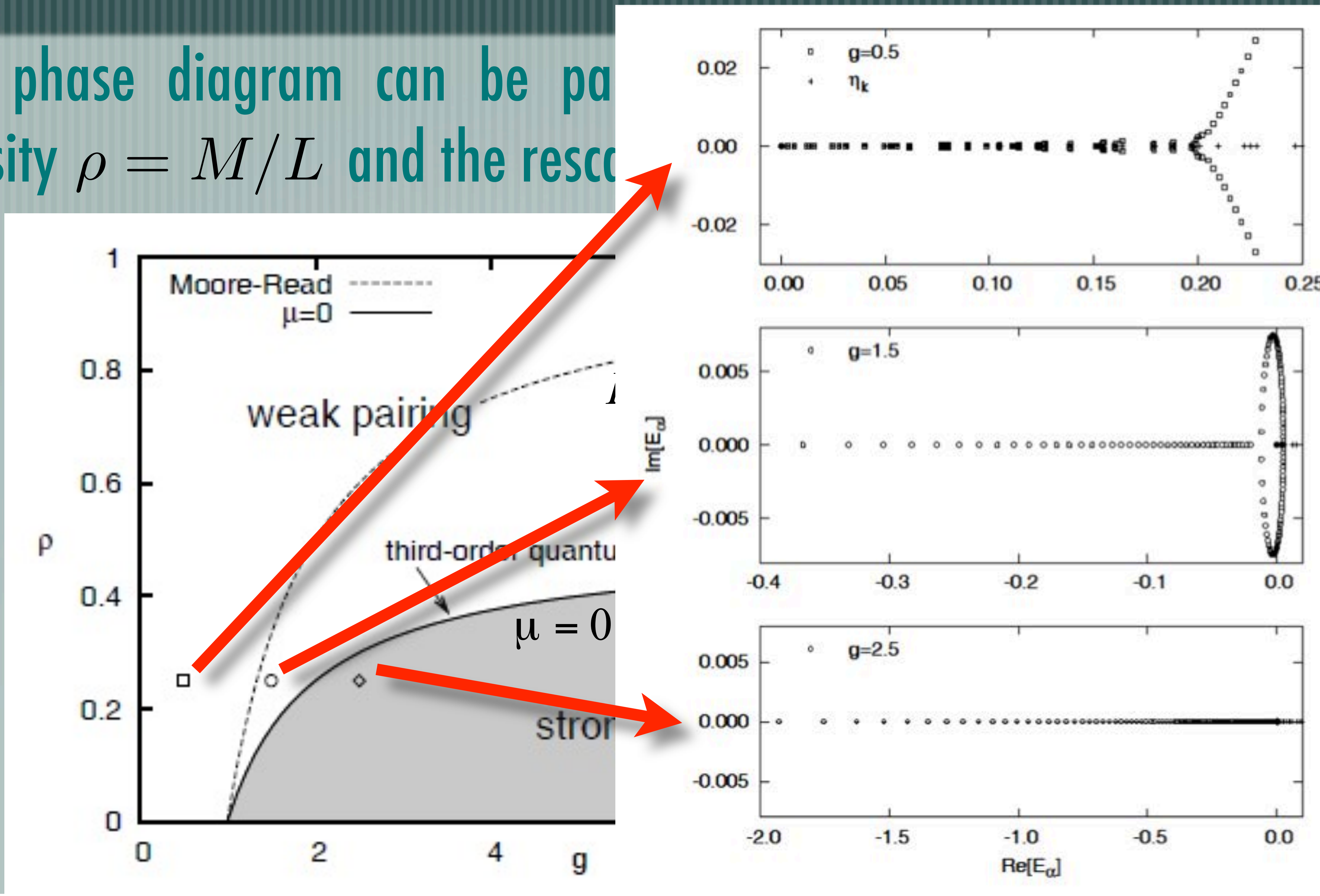

Gaudin for Quantum Hall

Exactly-Solvable Model: Strong Coupling

Consider the general class of hyperbolic Gaudin models with:

$$\mathsf{S}^z(x) = -\frac{1}{2} - \sum_{k(j)} Z(x,\eta_k) S^z_{jk}, \quad \mathsf{S}^\pm(x) = \sum_{k(j)} X(x,\eta_k) S^\pm_{jk}$$

In this rep one can define $\mathcal{C}(j)$ constants of motion: (Fix j)

$$R_{jk} = S^z_{jk} - \sum_{l(j), l\neq k} X(\eta_k,\eta_l)\left(S^+_{jk}S^-_{jl} + S^-_{jk}S^+_{jl}\right) - 2\sum_{l(j), l\neq k} Z(\eta_k,\eta_l) S^z_{jk}S^z_{jl}$$

And from their linear combination obtain:

$$H_{\mathsf{G}j} = \sum_{k(j)} \epsilon_k S^z_{jk} - \sum_{k(j),l(j)} (\epsilon_k - \epsilon_l) X(\eta_k,\eta_l) S^+_{jk}S^-_{jl} - \sum_{k(j),l(j)} (\epsilon_k - \epsilon_l) Z(\eta_k,\eta_l) S^z_{jk}S^z_{jl}$$

The following parametrization (satisfying Jacobi's relation):

$$X(x,y) = -\bar{g}\frac{\sqrt{x}\sqrt{y}}{x-y}, \quad Z(x,y) = -\frac{\bar{g}}{2}\frac{x+y}{x-y}$$

and $\epsilon_k = \lambda_j \eta_k^2$ leads to the Hamiltonian:

$$H_{\mathsf{G}j} = \lambda_j (1 + \bar{g}(S_j^z - 1)) \sum_{k(j)} \eta_k^2 S_{jk}^z + \lambda_j \bar{g} \sum_{k(j),l(j)} \eta_k \eta_l S_{jk}^+ S_{jl}^-$$

where

$\mathbf{S}_{jk}^2$ and $\mathbf{S}_j = \sum_{k(j)} \mathbf{S}_{jk}$ are good quantum numbers

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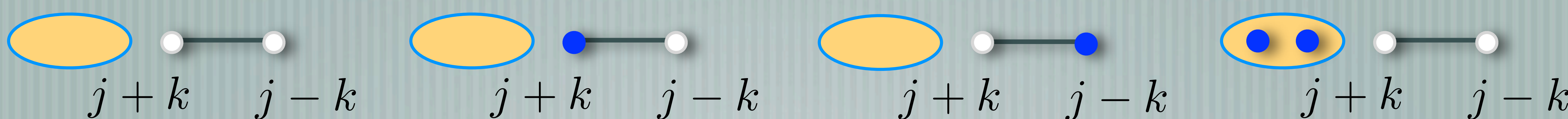
We want to consider the special case where vanishes

One can choose the SU(2) fermionic representation:

$$S^+_{jk} = c^\dagger_{j+k} c^\dagger_{j-k} \ , \ S^-_{jk} = c_{j-k} c_{j+k} \ , \ S^z_{jk} = \frac{1}{2}(n_{j+k} + n_{j-k} - 1)$$

such that acting on the vacuum $|\nu(j)\rangle$ containing only unpaired e-

$$S^-_{jk}|\nu(j)\rangle = 0 \qquad S^z_{jk}|\nu(j)\rangle = \frac{1}{2}(|\nu_{jk}| - 1)|\nu(j)\rangle \equiv -s_{jk}|\nu(j)\rangle$$



$\boxed{\nu_{jk} = 0}$ $\boxed{\nu_{jk} = +1}$ $\boxed{\nu_{jk} = -1}$ $\boxed{\nu_{jk} = 0}$

$$N = 2M + N_{\text{b}} + N_{\text{inactive}}$$

(L orbitals)

Paired

Unpaired= $N_{\text{b}} = \sum_{k(j)} |\nu_{jk}|$

Inactive levels

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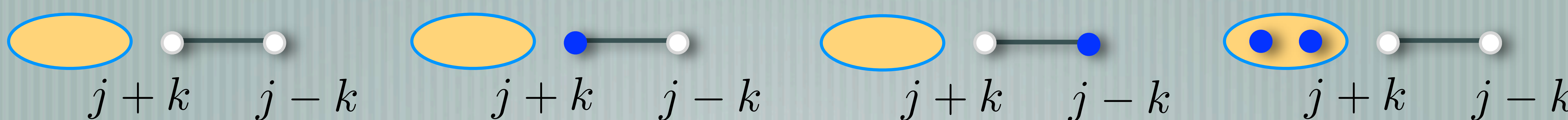

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(L orbitals)

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Unpaired= $N_{\text{b}} = \sum_{k(j)} |\nu_{jk}|$

Inactive levels

Additional symmetries become manifest in the fermionic language:

- Pauli blocking ($SU(2)$ gauge symmetry):

$$\tau^+_{jk} = c^\dagger_{j+k} c_{j-k} \ , \ \tau^-_{jk} = c^\dagger_{j-k} c_{j+k} \ , \ \tau^z_{jk} = \frac{1}{2}(n_{j+k} - n_{j-k})$$

- Total angular momentum (global symmetry):

$$\hat{J} = \sum_{r=0}^{L-1} r \ n_r \quad \Longrightarrow \quad [\hat{J}, S^\pm_{jk}] = \pm 2j S^\pm_{jk}$$

pair's angular momentum: $2j$

One classify eigenstates $|\Phi_{M\nu(j)}\rangle$ according to $\hat{J}$ and S^z_j

$$S^z_j |\Phi_{M\nu(j)}\rangle = (M - \sum_{k(j)} s_{jk}) \, |\Phi_{M\nu(j)}\rangle \qquad \hat{J}|\Phi_{M\nu(j)}\rangle = J|\Phi_{M\nu(j)}\rangle$$

By choosing: $\bar{g} = -1/(M - \sum_{k(j)} s_{jk} - 1)$

one obtains: $(g = \lambda_j \bar{g})$

$$\boxed{H_{\mathsf{G}j} = g \sum_{k(j),l(j)} \eta_k \eta_l \; c^\dagger_{j+k} c^\dagger_{j-k} c_{j-l} c_{j+l} = g \; T^+_{j1} T^-_{j1}}$$

Arbitrary Haldane pseudopotential

This model is exactly solvable for any η_k , the QH information is in part in their specific values

By choosing: $$\bar{g} = -1/(M - \sum_{k(j)} s_{jk} - 1)$$

one obtains: $(g = \lambda_j \bar{g})$

$$\boxed{H_{Gj} = g \sum_{k(j),l(j)} \eta_k \eta_l \; c^\dagger_{j+k} c^\dagger_{j-k} c_{j-l} c_{j+l} = g \; T^+_{j1} T^-_{j1}}$$

Arbitrary Haldane pseudopotential

geometry	L (Laughlin)	N_Φ	η_k	$\phi_r(z)$
disk	$mN - m + 1$	L	$k\, 2^{-j} \sqrt{\frac{1}{2\pi j} \binom{2j}{j+k}}$	$\frac{1}{\sqrt{2\pi 2^r r!}} \, z^r e^{-\frac{1}{4}\|z\|^2}$
cylinder	$mN - m + 1$	L	$\kappa^{3/2}\, k\, e^{-\kappa^2 k^2}$	$\sqrt{\kappa}\, e^{-\frac{1}{2}(x - r\kappa)^2 + ir\kappa y}$
sphere	$mN - m + 1$	$L - 1$	$k \frac{N_\Phi + 1}{4\sqrt{2\pi j}} \sqrt{\binom{2N_\Phi}{2j}^{-1} \frac{(6N_\Phi - 5)N_\Phi}{(2N_\Phi - 1)(2N_\Phi - 2j)} \binom{N_\Phi}{j+k} \binom{N_\Phi}{j-k}}$	$\sqrt{\frac{N_\Phi + 1}{4\pi} \binom{N_\Phi}{r}} [e^{-i\frac{\varphi}{2}} \sin(\frac{\theta}{2})]^r [e^{i\frac{\varphi}{2}} \cos(\frac{\theta}{2})]^{N_\Phi - r}$
torus	mN	L	$\kappa^{3/2} \sum_{s \in \mathbb{Z}} (k + sL)\, e^{-\kappa^2 (k+sL)^2}$	$\sum_{s \in \mathbb{Z}} \phi^{\text{cylinder}}_{r+sL}$

What can one learn from its eigenspectrum?

Given N electrons and L orbitals the filling fraction is: $\nu = \dfrac{N-1}{L-1}$

The dimension of the total Hilbert space:

$$\dim \mathcal{H}_L(N) = \binom{L}{N} = \sum_{J \in \mathcal{J}_L(N)} \dim \mathcal{H}_L(N, J)$$

where the set of allowed J

$$\mathcal{J}_L(N) = \left\{ \frac{N(N-1)}{2}, \frac{N(N-1)}{2}+1, \frac{N(N-1)}{2}+2, \cdots, N\left(L - \frac{(N+1)}{2}\right) \right\}$$

and $\dim \mathcal{H}_L(N, J)$ is determined from the generating function:

$$\mathcal{Z}(x, z) = \prod_{r=0}^{L-1} (1 + z x^r) = \sum_{\bar{J}=0}^{L(L-1)/2} \sum_{\bar{N}=0}^{L} \dim \mathcal{H}_L(\bar{N}, \bar{J})\, z^{\bar{N}} x^{\bar{J}}$$

Eigenvectors:

$$|\Phi_{M\nu(j)}\rangle = \prod_{\alpha=1}^{M} \mathsf{S}_j^+(E_\alpha)|\nu(j)\rangle \ , \ \mathsf{S}_j^+(E_\alpha) = \sum_{k(j)} \frac{\eta_k}{\eta_k^2 - E_\alpha} \ c_{j+k}^\dagger c_{j-k}^\dagger$$

There exists two classes of solutions:

All finite pairons: $\mathcal{E}_{M\nu(j)} = 0$

One infinite pairon: $\mathcal{E}_{M\nu(j)} = 2g \left(\sum_{k(j)} s_{jk}\, \eta_k^2 - \sum_{\alpha=1}^{M-1} E_\alpha \right)$

The Gaudin (Bethe) equation is:

$$\sum_{\beta(\neq\alpha)=1}^{M} \frac{E_\beta}{E_\beta - E_\alpha} - \sum_{k(j)} s_{jk} \ \frac{\eta_k^2}{\eta_k^2 - E_\alpha} = 0, \ \forall\alpha$$

Spectrum of Gaudin-Quantum Hall

Repulsive case $(g > 0)$:

$j_{\mathrm{m}} - 1$ $\quad$ $j_{\mathrm{m}} - 1/2$ $\quad$ $j_{\mathrm{m}} = \frac{L-1}{2}$ $\quad$ $j_{\mathrm{m}} + 1/2$ $\quad$ $j_{\mathrm{m}} + 1$

$H_{\mathsf{G}j}$

$\ldots$ $\quad$ $\ldots$

$\dim \mathcal{H}_L(N, J) - \dim \mathcal{H}_L(N-2, J-2j)$ (independent of η_k)

0 } # zeros: Large degeneracy (Null space)

$H_{\mathsf{G}j}$

0 }

Attractive case $(g < 0)$:

Strongly-coupled Superconductor

(unique ground state)

Ground States of the Full Pseudopotential Problem

Frustration-Free Properties

We have shown that in second quantization:

$$\hat{H}_{\mathsf{QH}} = \sum_{0<j<L-1} \sum_{m\geq 0} H^{m}_{\mathsf{G}j} = \sum_{m\geq 0} g_m \ \hat{H}_{V_m}$$

$\mathsf{Ker}(\hat{H}_{\mathsf{QH}})$ is the common null space of all the null spaces $\mathsf{Ker}(H^{m}_{\mathsf{G}j})$

Given N, L , the Hamiltonian $\hat{H}_{V_1}$ displays zero energy ground states $|\Psi^J_\nu\rangle$, whenever $\nu = \frac{p}{q} \leq \frac{1}{3}$. The zero energy state is unique when $\nu = \frac{1}{3}$, it is in the sector $J = J_{\mathsf{m}}$, and it is the Laughlin state

$\hat{H}_{V_1}$ is a frustration-free Hamiltonian for $\nu = \frac{p}{q} \leq \frac{1}{3}$

$$H_{\mathsf{G}j}|\Psi_\nu^J\rangle = 0, \quad \text{for all } j, \; j_{\mathsf{min}} \leq j \leq j_{\mathsf{max}} \Rightarrow T_{j1}^{-}|\Psi_\nu^J\rangle = 0$$

positive semi-definite

Corollary: All zero energy states have zero coefficients, in a Slater determinant expansion, for the basis states with:

$$(n_0 = 1, n_1 = 1) \;, \; (n_0 = 1, n_2 = 1) \;, \; (n_{L-3} = 1, n_{L-1} = 1), \; (n_{L-2} = 1, n_{L-1} = 1)$$

Conclusions

- Exact relation between QH Hamiltonians and Pairing models
- We determined the exact spectrum of the QH-Gaudin problem
- Proved separability of Haldane pseudopotentials (explicit construction)
- Topological equivalence of different geometries sharing the same genus number
- Quasi-hole generators in second quantization

Outlook

- String (long-range) order in Second Quantization
- Relation between Calogero-Sutherland and Gaudin
- How about $\nu = 1/2$?
- Electromagnetic response