The SM, the Higgs and beyond

Outline

- Lecture 2 – Symmetries and QFT

Fundamental principles of particle physics

Our description of the fundamental interactions and particles rests on two fundamental structures :

- Quantum Mechanics
- Symmetries

Symmetries

Central to our description of the fundamental forces :

Relativity - Lorentz transformations $SO(3,1)$

Lie symmetries – Gauge transformations $SU(3) \otimes SU(2) \otimes U(1)$

Copernican principle : “Your system of co-ordinates and units is nothing special”

⇒ Physics independent of system choice

Symmetries

Classification of symmetries in Standard Model:

Local vs global

SO(3,1) $SU(3)\otimes SU(2)\otimes U(1)$

$U(1)_{Baryon}$ $SU(2)_{Isospin}$

Continuous vs *Discrete*

All of the above

C,P,T – Charge conjugation, Parity, Time inversion…

We also talk about
Abelian syms – *generators* commute
Vs
Non-Abelian - *generators* do not commute

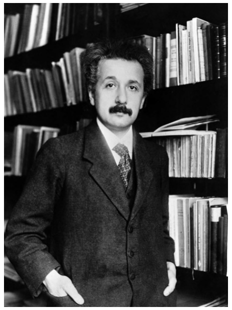
$$[J_i, J_j] = i\hbar \sum_{k=1}^{3} \varepsilon_{ijk} J_k$$

The J_i **are the "generators" of the group. SO(3) (SU(2))**

$$R(\alpha) = e^{-i\alpha.\mathbf{J}}$$

Their commutation relations define a "Lie algebra"†.

Special relativity



- Space time point $a^{\mu} = (ct, x, y, z)$ not invariant under translations

- Space-time vector $(a + \Delta a)^{\mu} - a^{\mu} = \Delta a^{\mu} = (c\Delta t, \Delta x, \Delta y, \Delta z)$

  Invariant under translations …but not invariant under rotations or boosts

- Einstein postulate : the real invariant distance is

$$\left(\Delta a^{0}\right)^{2} - \left(\Delta a^{1}\right)^{2} - \left(\Delta a^{2}\right)^{2} - \left(\Delta a^{3}\right)^{2} = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} \Delta a^{\mu} \Delta a^{\nu} = \Delta a^{\mu} \Delta a_{\mu} = \left(\Delta a\right)^{2}$$

$$g_{\mu\nu} = diag(+1, -1, -1, -1)$$

- Physics invariant under all transformations that leave all such distances invariant :

  Translations and The SO(3,1) Lorentz transformations

Lorentz transformations :

$$x^{\mu} \rightarrow \Lambda^{\mu}_{\nu} x^{\nu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} x^{\nu} = x'^{\mu} \quad \Rightarrow \quad g_{\mu\nu} x'^{\mu} x'^{\nu} = g_{\mu\nu} x^{\mu} x^{\nu} \quad \Rightarrow \quad g_{\mu\nu} \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} = g_{\alpha\beta}$$

(Summation assumed)

Solutions :

3 rotations R

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- 3 boosts B

$$\begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Space reflection – parity P

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Time reflection, time reversal T

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Fundamental principles of particle physics

Quantum Mechanics $q(t)$

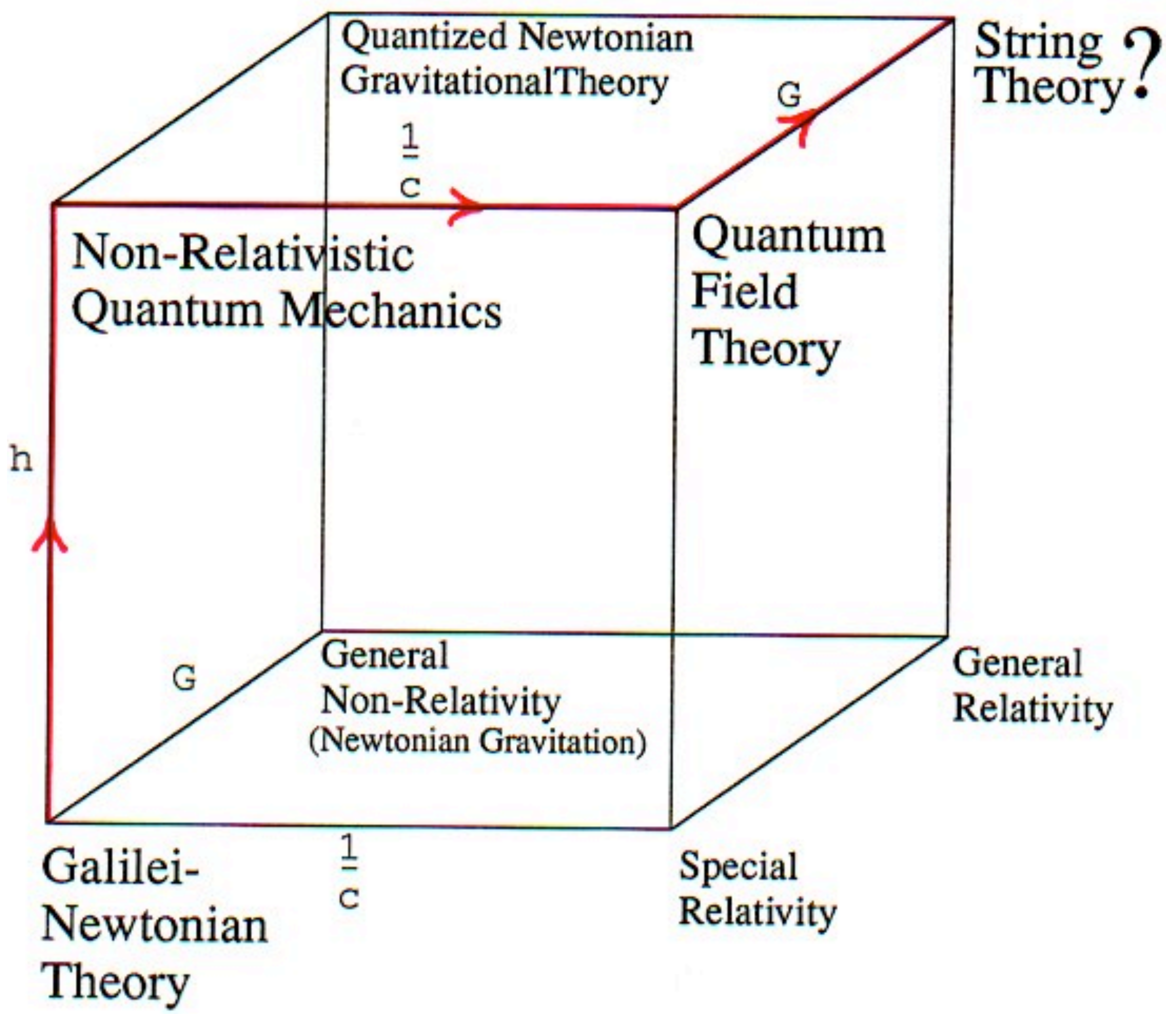
+

Relativity

} Quantum Field theory

$$q(t) \qquad \rightarrow \qquad q_x(t) = q(t, x)$$

Bronshtein's 'cube of theories'



Action principle

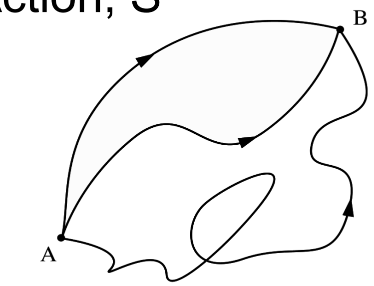
Action $$S = \int_{t_1}^{t_2} L\, dt$$

Lagrangian $$L = T - V$$

(Nonrelativistic mechanics)

- Classical path … minimises action

Action, S



$$S = \int_{t_A}^{t_B} (K.E. - P.E.)dt$$

- Quantum mechanics … sum over all paths with amplitude $\propto e^{iS/\hbar}$

"Principle of Least Action"
Feynman Lectures in Physics
Vol II Chapter 19

(Lagrangian invariant under all the symmetries of nature

-makes it easy to construct viable theories)

Compare with Hamiltonian formulation:

$$H = T + V$$

Why Quantum field theory?

Quantum Mechanics : Quantization of dynamical system of particles

Quantum Field Theory : Application of QM to dynamical system of fields

- Not all relativistic processes can be explained by single particle since $E=mc^2$ allows pair creation – happens all the time at LHC

- (Relativistic) QM has physical problems. For example it violates causality

  See slides of G. Ross on school homepage

Relativistic (quantum) field theory

$$L = \int \mathcal{L}\, d^3x, \quad \mathcal{L} \quad \text{lagrangian density}$$

Klein Gordon field $\phi(x)$

$$\mathcal{L} = \underbrace{\left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x)}_{T} - \underbrace{m^2 \phi(x)^\dagger \phi(x)}_{V}$$

Manifestly Lorentz invariant

Lagrangian formulation of the Klein Gordon equation

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Klein Gordon field $\phi(x)$

$$\mathcal{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$

Manifestly Lorentz invariant

Classical path : $\frac{\delta S}{\delta \phi} = 0 \qquad S = \int d^3x\, dt\, L$

$$\delta S = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \phi} - \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} = 0$$

Euler Lagrange equation (shown in exercises)

Lagrangian formulation of the Klein Gordon equation

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Euler Lagrange equation
(shown in exercises)

⇨ $$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

Klein Gordon equation

(You will derive a number of field equations from Lagrangians in the exercises)

New symmetries

$$L = \left(\partial_{\mu}\phi(x)\right)^{\dagger}\partial^{\mu}\phi(x) - m^{2}\phi(x)^{\dagger}\phi(x)$$

Is invariant under $\phi(x) \to e^{i\alpha}\phi(x)$

What is this symmetry in our classification scheme?

New symmetries

$$\mathcal{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$

Is invariant under $\phi(x) \to e^{i\alpha}\phi(x)$

…a *global, continuous Abelian* (U(1)) gauge symmetry

A symmetry implies a conserved current and charge.

e.g. Translation ⇨ Momentum conservation

Rotation ⇨ Angular momentum conservation

What conservation law does the U(1) invariance imply?

Noether current

$$\mathsf{L} = \left(\partial_{\mu}\phi(x)\right)^{\dagger}\partial^{\mu}\phi(x) - m^{2}\phi(x)^{\dagger}\phi(x)$$

Is invariant under $\phi(x) \rightarrow e^{i\alpha}\phi(x)$ …an Abelian (U(1)) gauge symmetry

(use inf. Transform δϕ and Euler lagrange eqs.)

$$\delta L = 0 \rightarrow 0 = i\partial^{\mu}\left(\frac{\partial L}{\partial(\partial^{\mu}\phi)}\phi\right) - (\phi \leftrightarrow \phi^{\dagger})$$

What is the physics of this equation?

Noether current

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(Inf. Transform δϕ and Euler lagrange eqs.)

$$\delta L = 0 \rightarrow 0 = i\partial^\mu \left(\frac{\partial L}{\partial(\partial^\mu \phi)} \phi \right) - (\phi \leftrightarrow \phi^\dagger)$$

Conserved current

$$\partial^\mu j_\mu = 0, \quad j_\mu = \frac{ie}{2}\left(\frac{\partial \mathsf{L}}{\partial(\partial^\mu \phi)}\phi - \frac{\partial \mathsf{L}}{\partial(\partial^\mu \phi^\dagger)}\phi^\dagger \right)$$

Noether current

The Klein Gordon current

$$\mathsf{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$

Is invariant under $\phi(x) \to e^{i\alpha}\phi(x)$ …an Abelian (U(1)) gauge symmetry

$$\partial^\mu j_\mu = 0, \quad j_\mu = \frac{ie}{2}\left(\frac{\partial \mathsf{L}}{\partial(\partial^\mu \phi)}\phi - \frac{\partial \mathsf{L}}{\partial(\partial^\mu \phi^\dagger)}\phi^\dagger\right)$$

$$j_\mu^{KG} = -ie\left(\phi^* \partial_\mu \phi - \phi\, \partial_\mu \phi^*\right)$$

This is of the form of the electromagnetic current for the KG field

The Klein Gordon current

$$\mathcal{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$

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$$\partial^\mu j_\mu = 0, \quad j_\mu = \frac{ie}{2}\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)}\phi - \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^\dagger)}\phi^\dagger\right)$$

$$j_\mu^{KG} = -ie\left(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*\right)$$

This is of the form of the electromagnetic current for the KG field:

$A_\mu \, j_\mu^{\mathrm{KG}}$ 'minimal coupling' to EM potential

$Q = \int d^3x \; j^0$ is the associated conserved charge

Aside - Additional terms

Terms allowed by U(1) symmetry

$$\mathcal{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x) + \lambda \left|\phi\right|^4 + \frac{\lambda'}{M^2}\left|\phi\right|^6 + \ldots$$

Renormalisable $D \leq 4$

If $M \gg 10^3 GeV$, "Effective" Field theory approximately renormalisable

Renormalizability is another principle taken for the construction of the SM –
In the above sense

U(1) *local gauge* invariance and QED

$$\phi(x) \to e^{i\alpha(x)Q}\phi(x)$$

Jargon: we are *gauging* the global U(1) symmetry

$$\mathrm{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$

Invariant?

U(1) local gauge invariance and QED

$$\phi(x) \to e^{i\alpha(x)Q}\phi(x)$$

$$\mathrm{L} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$ not invariant due to derivatives

$$\partial_\mu \phi \to \partial_\mu (e^{i\alpha(x)Q}\phi) = e^{i\alpha(x)Q}\partial_\mu \phi + iQe^{i\alpha(x)Q}\phi\, \partial_\mu \alpha(x)$$

To obtain invariant Lagrangian look for a modified derivative transforming covariantly

$$D_\mu \phi \to e^{i\alpha(x)Q} D_\mu \phi$$

U(1) local gauge invariance and QED

$$\phi(x) \rightarrow e^{i\alpha(x)Q}\phi(x)$$

$$L = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$ not invariant due to derivatives

To obtain invariant Lagrangian look for a modified derivative transforming covariantly

$$D_\mu \phi \rightarrow e^{i\alpha(x)Q} D_\mu \phi$$

Need to introduce a new *vector* field $A_\mu \rightarrow A_\mu + \partial_\mu \alpha$

$$D_\mu = \partial_\mu - iQA_\mu$$

$$\phi(x) \to e^{iQ\alpha(x)}\phi(x)$$

$$D_\mu \phi \to e^{i\alpha(x)Q} D_\mu \phi$$

$$A_\mu \to A_\mu + \partial_\mu \alpha$$

$$\mathcal{L} = \left(D_\mu \phi(x)\right)^\dagger D^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x)$$ is invariant under local U(1)

Note : $\partial_\mu \to D_\mu = \partial_\mu - iQA_\mu$ is equivalent to $p^\mu \to p^\mu + eA^\mu$

universal coupling of electromagnetism *follows* from local gauge invariance

i.e. $$\mathcal{L} = \mathcal{L}^{\mathrm{KG}} = \left(\partial_\mu \phi(x)\right)^\dagger \partial^\mu \phi(x) - m^2 \phi(x)^\dagger \phi(x) - j_\mu^{KG} A^\mu + O(e^2)$$

$$\phi(x) \rightarrow e^{iQ\alpha(x)}\phi(x)$$

$$D_\mu\phi \rightarrow e^{i\alpha(x)Q} D_\mu\phi$$

$$A_\mu \rightarrow A_\mu + \partial_\mu\alpha$$

$$\mathrm{L} = \left(D_\mu\phi(x)\right)^\dagger D^\mu\phi(x) - m^2\phi(x)^\dagger\phi(x)$$

is invariant under local U(1)

Note : $\partial_\mu \rightarrow D_\mu = \partial_\mu - iQA_\mu$ is equivalent to $p^\mu \rightarrow p^\mu + eA^\mu$

'Minimal coupling' of electromagnetism *follows* from local gauge invariance
Dynamics *follows* from ***symmetry***

The Euler lagrange equation give the KG equation:

$$(\partial_\mu\partial^\mu + m^2)\psi = -V\psi \quad where \quad V = -ie(\partial_\mu A^\mu + A^\mu\partial_\mu) - e^2A^2$$

The electromagnetic Lagrangian

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

$$F_{\mu\nu} \to F_{\mu\nu}, \qquad A_\mu \to A_\mu + \partial_\mu \alpha$$

$$\mathcal{L}^{EM} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$$

$M^2 A^\mu A_\mu$ Forbidden by gauge invariance

The Euler-Lagrange equations give Maxwell equations !

$$\frac{\partial \mathcal{L}}{\partial A^\nu} - \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} = 0$$

$$\partial_\mu F^{\mu\nu} = j^\nu$$

$$\left(\text{N.B. } \varepsilon_{\mu\nu\rho\sigma}\partial^\mu F^{\rho\sigma} = 0\right)$$

$$\equiv$$

$$\nabla.\mathbf{E} = \rho, \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla.\mathbf{B} = 0, \qquad \nabla \times \mathbf{B} - \frac{\partial E}{\partial t} = \mathbf{j}$$

EM dynamics follows from a local gauge symmetry!!

Suppose we have two fields with different U(1) charges :

$$\phi_{1,2}(x) \rightarrow e^{i\alpha Q_{1,2}} \phi_{1,2}(x)$$

$$\mathcal{L} = \left(\partial_\mu \phi_1(x)\right)^\dagger \partial^\mu \phi_1(x) - m^2 \phi_1(x)^\dagger \phi_1(x)$$
$$+ \left(\partial_\mu \phi_2(x)\right)^\dagger \partial^\mu \phi_2(x) - m^2 \phi_2(x)^\dagger \phi_2(x)$$

..no cross terms possible (corresponding to charge conservation)