Junctions of Tonks-Girardeau gases

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Outline

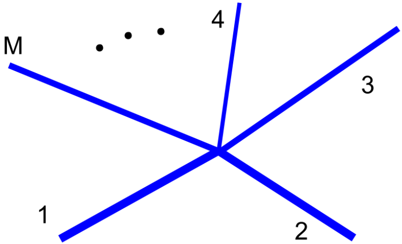
The problem

The Lieb-Lininger Hamiltonian...

$$H_{LL} = \int_0^{\mathcal{L}} dx \left[\frac{\hbar^2}{2m} \partial_x \Psi^\dagger(x) \partial_x \Psi(x) + \frac{c}{2} \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x) \right]$$

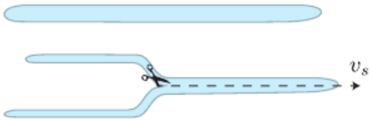
$$\left[\Psi(x), \Psi^\dagger(y) \right] = \delta(x - y)$$

... on a star junction



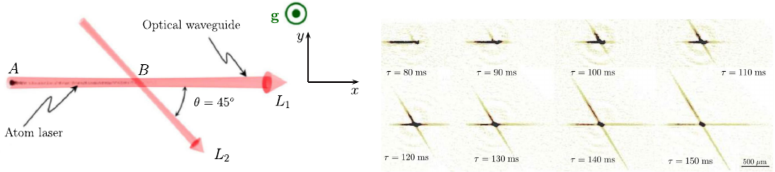
Experimental motivation

- "unzipping" of a bosonic gas, atom chips:



(Langen et al. Nat. Phys. 9, 640–643, Agarwal et al. 1402.6716, Folman et al. Phys. Rev. Lett. 84, 4749)

- Optically guided beam splitter for propagating matter waves:

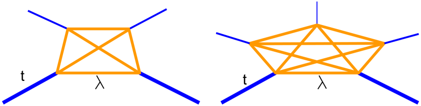


(Gattobigio et al. Phys. Rev. Lett. 109, 030403;
Houde et al. Phys. Rev. Lett. 85, 5543)

The star junction

Model the junction $\rightarrow$ lattice description

$$H = \sum_{\alpha=1}^{M} H_U^{(\alpha)} + H_J$$



On each leg: Bose-Hubbard (Jaksch et al. PRL 81, 3108)

$$\begin{aligned} H_U^{(\alpha)} &= -t \sum_{j=1}^{L-1} \left(b_\alpha^\dagger(j) b_\alpha(j+1) + b_\alpha^\dagger(j+1) b_\alpha(j) \right) \\ &\quad + \frac{U}{2} \sum_{j=1}^{L} b_\alpha^\dagger(j) b_\alpha^\dagger(j) b_\alpha(j) b_\alpha(j) \end{aligned}$$

tunnelling between different legs: "junction" Hamiltonian

$$H_J = -\lambda \sum_{\alpha<\beta}^{M} \left(b_\alpha(1)^\dagger b_\beta(1) + b_\beta(1)^\dagger b_\alpha(1) \right)$$

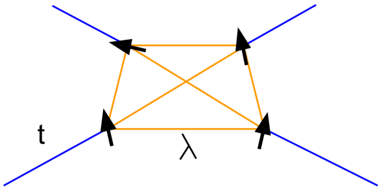
Only the *total* number of bosons N is conserved.

The Tonks-Girardeau gas

Simplification: infinite repulsion $U \to \infty$, low density

$$b_\alpha(j) \to \sigma_\alpha^-(j) \qquad b_\alpha^\dagger(j) \to \sigma_\alpha^+(j)$$

$$2b_\alpha^\dagger(j)b_\alpha(j) - 1 \to \sigma_\alpha^z(j)$$



$$H_\infty^{(\alpha)} = -t\sum_{j=1}^{L-1}\left(\sigma_\alpha^+(j)\sigma_\alpha^-(j+1) + \sigma_\alpha^+(j+1)\sigma_\alpha^-(j)\right)$$

$$H_J = -\lambda\sum_{\alpha<\beta}^{M}\Big(\sigma_\alpha^+(1)\sigma_\beta^-(1) + \sigma_\beta^+(1)\sigma_\alpha^-(1)\Big)$$

Observation

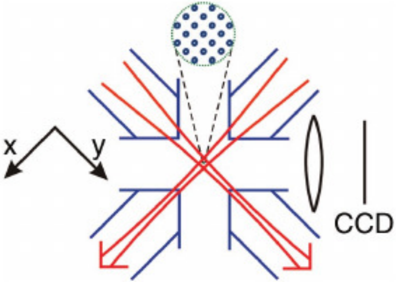


Fig. 2. Scheme illustrating the experiment. The large, blue-detuned crossed beam pairs form the 2D optical lattice that strongly confines atoms in 1D tubes (out of the page). The arrangement of the tubes is illustrated in the magnified circle. The smaller, red-detuned crossed traveling waves trap the atoms axially (out of the page, which corresponds to up in the experiment). The collection of tubes is imaged transversely, without resolving individual tubes.

(Kinoshita et al. Science 305, 1125; Paredes et al. Nature 429, 277-281)

Junctions of 3 legs

1. Introduce an additional spin degree of freedom (Crampé et al. NPB 871:526):

$$\eta^a = \sigma^a(0) \prod_{\beta \neq a} \prod_{j=1}^{L} \sigma^z_\beta(j) \qquad a = 1, 2, 3$$

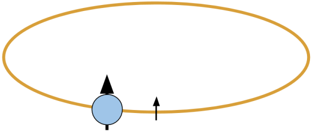
2. Perform a Jordan-Wigner tranform:

$$\sigma^-_\alpha(j) = \eta^\alpha \prod_{l<j} e^{i\pi c^\dagger_\alpha(l) c_\alpha(l)} c^\dagger_\alpha(j) \qquad \left\{ c_\alpha(j), c^\dagger_\beta(k) \right\} = \delta_{\alpha,\beta}\delta_{j,k}$$

3. Continuum limit:

$$H_{bulk} = -\frac{1}{2} \int dx \Psi^{\alpha\dagger}(x) \frac{d^2}{dx^2} \Psi_\alpha(x) - \lambda \vec{\eta} \cdot \Psi^{\alpha\dagger}(0) \left(\vec{S}\right)_{\alpha\beta} \Psi^\beta(0)$$

Three-component fermions interacting with a spin-$\frac{1}{2}$ localized magnetic impurity
$\rightarrow$ four-channel Kondo model, topological



Junction of more legs

- Hardcore bosons (spins) on M branches
  Jordan-Wigner (Altland et al. PRL 113:076401):

$$\sigma_\alpha^-(j) = \gamma_\alpha \prod_{l<j} e^{i\pi c_\alpha^\dagger(l) c_\alpha(l)} c_\alpha(j) \qquad \sigma_\alpha^z(j) = 2c_\alpha^\dagger(j)c_\alpha(j) - 1$$

  - $\left\{c_\alpha(j), c_\beta^\dagger(k)\right\} = \delta_{\alpha,\beta}\delta_{j,k} \quad \{c_\alpha(j), c_\beta(k)\} = 0$
  - Klein factors $\gamma_\alpha = \gamma_\alpha^\dagger \qquad \{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha,\beta}$

  the impurity "spin" η is made out of a pair of Majorana fermions:

$$\eta^a = \frac{1}{2i}\epsilon^{abc}\gamma_b\gamma_c$$

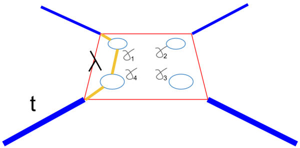
- Fermions on the branches: there would be no γ

$$H = -i\sum_{\alpha=1}^{M}\int dx \psi_\alpha^\dagger(x)\partial_x\psi_\alpha(x) - \lambda\sum_{\alpha\neq\beta}^{M}\psi_\alpha^\dagger(0)\psi_\beta(0)$$

The topological Kondo model

Continuum limit at low temperature (ψ_α are the continuum version of the lattice fermions):

$$H = -i \sum_{\alpha=1}^{M} \int dx \psi_\alpha^\dagger(x) \partial_x \psi_\alpha(x) - \lambda \sum_{\alpha \neq \beta} \gamma_\alpha \gamma_\beta \psi_\alpha^\dagger(0) \psi_\beta(0)$$



- At very low temperatures one enters into a non-Fermi liquid regime, whose physics is analogous to that of the Kondo fixed point
- $SO(M)_2$ critical point
- Unstable fixed point at $\lambda = 0$
- Anisotropy is irrelevant

Some exact results:

- At $T \to 0$, the residual entropy of the junction is:

$$S_{imp} = \begin{cases} \log \sqrt{M} & odd \\ \log \sqrt{\frac{M}{2}} & even \end{cases}$$

$\to$ even-odd effect.

The energy of the system is written in terms of the solutions of the Bethe equations = quantization of momenta (Altland et al. JPA 47:265001).

- Solution for arbitrary λ at fixed number of bosons.
- Different for even and odd number of legs

The structure of Bethe equations (for odd $M = 2K + 1$)

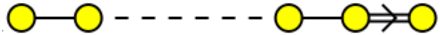
$$E = \frac{1}{2i} \sum_{j=1}^{M_1} \log e_2 \left(x_j^{(1)} \right)$$

each root configuration corresponds to a state and is a solution of:

$$\left[e_2 \left(x_j^{(1)} \right) \right]^N \prod_{k=1}^{M_2} e_1 \left(x_j^{(1)} - x_k^{(2)} \right) = \prod_{k=1}^{M_1} e_2 \left(x_j^{(1)} - x_k^{(1)} \right)$$

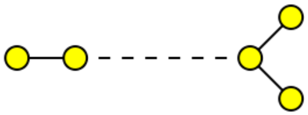
$$\prod_{k=1}^{M_{l-1}} e_1 \left(x_j^{(l)} - x_k^{(l-1)} \right) \prod_{k=1}^{M_{l+1}} e_1 \left(x_j^{(l)} - x_k^{(l+1)} \right) = \prod_{k=1}^{M_l} e_2 \left(x_j^{(l)} - x_k^{(l)} \right)$$

$$e_{1/2} \left(x_j^{(K)} - \frac{1}{\lambda} \right) \prod_{k=1}^{M_{K-1}} e_1 \left(x_j^{(K)} - x_k^{(K-1)} \right) = \prod_{k=1}^{M_K} e_1 \left(x_j^{(K)} - x_k^{(K)} \right)$$



$$e_n(x) = \frac{x - in/2}{x + in/2}$$

The structure of Bethe equations (for even $M = 2K$)



$$\left[e_2\left(x_j^{(1)}\right)\right]^N = \frac{\prod_{k=1}^{M_1} e_2\left(x_j^{(1)} - x_k^{(1)}\right)}{\prod_{k=1}^{M_2} e_1\left(x_j^{(1)} - x_k^{(2)}\right)}$$

$$1 = \frac{\prod_{k=1}^{M_2} e_2\left(x_j^{(2)} - x_k^{(2)}\right)}{\prod_{k=1}^{M_1} e_1\left(x_j^{(2)} - x_k^{(1)}\right)\prod_{k=1}^{M_3} e_1\left(x_j^{(2)} - x_k^{(3)}\right)}$$

$$1 = \frac{\prod_{k=1}^{M_l} e_2\left(x_j^{(l)} - x_k^{(l)}\right)}{\prod_{k=1}^{M_{l-1}} e_1\left(x_j^{(l)} - x_k^{(l-1)}\right)\prod_{k=1}^{M_{l+1}} e_1\left(x_j^{(l)} - x_k^{(l+1)}\right)} \qquad 2 < l < K-2$$

$$1 = \frac{\prod_{k=1}^{M_{K-2}} e_2\left(x_j^{(K-2)} - x_k^{(K-2)}\right)\prod_{k=1}^{M_{K-3}} e_{-1}\left(x_j^{(K-2)} - x_k^{(K-3)}\right)}{\prod_{k=1}^{M_K} e_1\left(x_j^{(K-2)} - x_k^{(K)}\right)\prod_{k=1}^{M_{K-1}} e_1\left(x_j^{(K-2)} - x_k^{(K-1)}\right)}$$

$$e_1\left(x_j^{(K-1)} - \frac{1}{\lambda}\right) = \frac{\prod_{k=1}^{M_{K-1}} e_2\left(x_j^{(K-1)} - x_k^{(K-1)}\right)}{\prod_{k=1}^{M_{K-2}} e_1\left(x_j^{(K-1)} - x_k^{(K-2)}\right)}$$

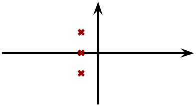
$$1 = \frac{\prod_{k=1}^{M_K} e_2\left(x_j^{(K)} - x_k^{(K)}\right)}{\prod_{k=1}^{M_{K-1}} e_1\left(x_j^{(K)} - x_k^{(K-1)}\right)}$$

Thermodynamics for arbitrary number of legs

$$\mathcal{L} \to \infty \qquad N \to \infty \qquad N/\mathcal{L} = const$$

- Take the logarithm of the Bethe equations
- "string" solutions:

$$\left\{ x^{(j)}_{n;\alpha} + \frac{i}{2}(n - 2l + 1) : x^{(j)}_{n;\alpha} \in \mathbb{R},\ l = 1, \ldots, n \quad j = 1, \ldots, K \right\}$$



- Densities $\rho^{(j)}_n$ count the number of $n-$string centers in the interval $[x, x + dx[$ at level (j)
- Densities $\tilde{\rho}^{(j)}_n$ count the number of $n-$string "holes" in the interval $[x, x + dx[$ at level (j)

Junction free energy

- Entropy S counts the number of states associated to a given density
- Consider the system in thermal equilibrium at temperature T and derive the saddle point equations for $F(T) = E - T\,S$.

The result is most conveniently written in terms of the functions:

$$\frac{\rho_n^{(j)}}{\tilde{\rho}_n^{(j)}} = e^{\phi_n^{(j)}}$$

Junction free energy at temperature T ($M \geq 3$):

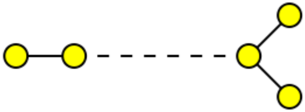
$$F_J(T) = -T \sum_{j=1}^{\lfloor M/2 \rfloor} \int_{\mathbb{R}} \frac{d\omega}{2\pi} \frac{e^{i\omega/\lambda} \cosh \frac{\omega}{2}}{\cosh \frac{(M-2)\omega}{4}} \frac{\sinh\left(\frac{j\omega}{2}\right)}{\sinh\left(\frac{\omega}{2}\right)} L_{-,1}^{(j)}(\omega) + F_{J,o}$$

with $L_{\pm,n}^{(j)}(x) = \log\left(1 + e^{\pm\phi_n^{(j)}(x)}\right)$. If the number of legs is odd, there is also

$$F_{J,o} = \int_{\mathbb{R}} \frac{dx}{2\pi} \frac{e^{i\omega/\lambda} \sinh \frac{(M-3)\omega}{4}}{2\cosh \frac{(M-2)\omega}{4} \sinh \frac{\omega}{2}} L_{+,1}^{\lfloor M/2 \rfloor}(\omega)$$

TBA equations

- even $M = 2K$:



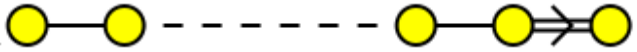
$$
\begin{aligned}
\phi_n^{(j)}(x) &= \frac{1}{T}\arctan\left(e^{\pi x}\right)\delta^{j,1}\delta_{n,2} - s * \left[L_{-,n-1}^{(j)} + L_{-,n+1}^{(j)}\right](x) \\
&\quad + s * \left[L_{+,n}^{(j-1)} + L_{+,n}^{(j+1)}\right] \qquad\qquad j < K-2 \\
\phi_n^{(K-2)}(x) &= -s * \left[L_{-,n-1}^{(K-2)} + L_{-,n+1}^{(K-2)}\right] + s * \left[L_{+,n}^{(K-3)} + L_{+,n}^{(K-1)} + L_{+,n}^{(K)}\right] \\
\phi_n^{(j)}(x) &= -s * \left[L_{-,n-1}^{(j)} + L_{-,n+1}^{(j)}\right] + s * L_{+,n}^{(K-2)} \qquad j = K-1, K
\end{aligned}
$$

$s(x) = \frac{\pi}{\sinh(\pi x)}$

- case $M = 6$ in Altland et al.

more TBA equations

- odd $M = 2K + 1$:



$$
\begin{aligned}
\phi_m^{(j)}(x) &= \delta^{j,1}\delta_{m,K}\frac{1}{T}\arctan\left(e^{\pi x}\right) - s * \left(L^{(j)}_{-,m-1} + L^{(j)}_{-,m+1}\right) \\
&\quad + s * \left(L^{(j-1)}_{+,m} + L^{(j+1)}_{-,m}\right) \qquad\qquad j < K-1 \\
\phi_m^{(K-1)}(x) &= -s * \left(L^{(K-1)}_{-,m-1} + L^{(K-1)}_{-,m+1}\right) + s * L^{(K-2)}_{+,m} \\
&\quad + \frac{s}{s_{1/2}} * L^{(K)}_{+,2m} + s * \left(L^{(K)}_{+,2m-1} + L^{(K)}_{+,2m+1}\right) \\
\phi_{2m}^{(K)}(x) &= -s_{1/2} * \left(L^{(K)}_{-,2m-1} + L^{(K)}_{-,2m+1}\right) + s_{1/2} * L^{(K-1)}_{+,m} \\
\phi_{2m-1}^{(K)}(x) &= -s_{1/2} * \left(L^{(K)}_{-,2m-2} + L^{(K)}_{-,2m}\right)
\end{aligned}
$$

$s_{1/2} = \frac{2\pi}{\cosh(2\pi x)}$

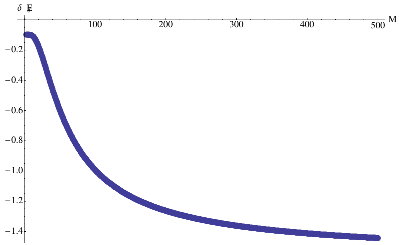
- cases $M = 3, 5$ in Altland et al.

The junction energy

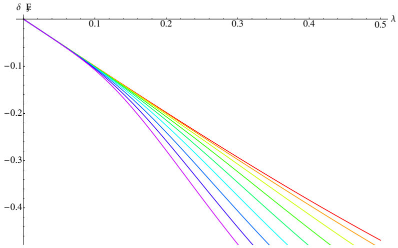
When different guides are connected through a junction term at low temperature, the energy of the system is shifted by the amount:

$$\delta E_J = i \log \frac{i\Gamma\left(\frac{M+2}{4(M-2)} + \frac{i}{(M-2)\lambda}\right)\Gamma\left(\frac{3M-2}{4(M-2)} - \frac{i}{(M-2)\lambda}\right)}{\Gamma\left(\frac{M+2}{4(M-2)} - \frac{i}{(M-2)\lambda}\right)\Gamma\left(\frac{3M-2}{4(M-2)} + \frac{i}{(M-2)\lambda}\right)}$$

This quantity is the same in system with even and odd number of legs



$\lambda = 0.1$



$M = 3, 4, \ldots, 11$

Extension: junctions of anyonic systems

Anyonic Lieb Lininger (Averin, Korepin, Patu)

$$H = \int_0^{\mathcal{L}} dx \left[\frac{\hbar^2}{2m} \partial_x \Psi^\dagger(x) \partial_x \Psi(x) + \frac{c}{2} \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x) \right]$$

$$\begin{aligned} \Psi_\alpha(j)\Psi_\beta(k) &= e^{i\pi\kappa\epsilon(j-k)\delta_{\alpha,\beta}} \Psi_\beta(k)\Psi_\alpha(j) \\ \Psi_\alpha(j)\Psi^\dagger_\beta(k) &= e^{-i\pi\kappa\epsilon(j-k)\delta_{\alpha,\beta}} \Psi^\dagger_\beta(k)\Psi_\alpha(j) + \delta_{j,k}\delta_{\alpha,\beta} \end{aligned}$$

Reshuffling of fields $\Psi \to \Phi$ and inclusion of the junction

$$\begin{aligned} H_b &= -i \sum_{\alpha=1}^{M} \int dx \Phi^\dagger_\alpha(x) \partial_x \Phi_\alpha(x) + H_K \\ H_J &= -\lambda \sum_{\alpha<\beta} \gamma_\alpha \gamma_\beta \Phi^\dagger_\alpha(0) \Phi_\beta(0) + h.c. \end{aligned}$$

Kind of topological Kondo.

Conclusions

- A junction of Tonks-Girardeau gases realizes a topological Kondo model
- Ground state energy for any coupling between legs and number of legs
- Free energy of the topological Kondo model as function of temperature and number of legs