Thermodynamic limit of the six-vertex model with reflecting end

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Introduction

We would like to compute the free-energy and entropy of the six-vertex model with boundaries different from periodic or domain wall.

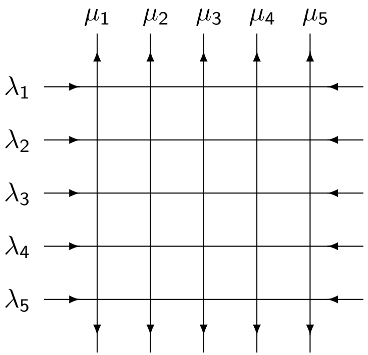
DWBC

In the computation of scalar product os Bethe states,

$$|\psi\rangle_N = B(\lambda_N)\cdots B(\lambda_2)B(\lambda_1)\,|\Uparrow\rangle\,,$$

appears the

$$Z_N^{DWBC}(\{\lambda\},\{\mu\}) = \langle\Downarrow|\,B(\lambda_N)\cdots B(\lambda_2)B(\lambda_1)\,|\Uparrow\rangle\,.$$



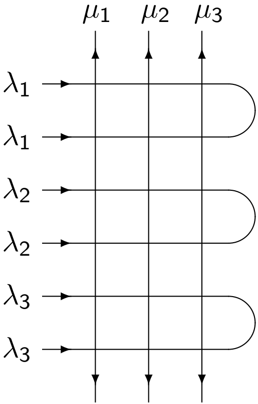
Tsuchiya partition function

In the case of open spin chains, the scalar product

$$|\phi\rangle_N = \mathcal{B}(\lambda_N)\cdots\mathcal{B}(\lambda_2)\mathcal{B}(\lambda_1)\,|\Uparrow\rangle\,.$$

leads to another partition function for the six-vertex model

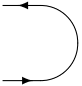
$$Z_N(\{\lambda\},\{\mu\}) = \langle\Downarrow|\,\mathcal{B}(\lambda_N)\cdots\mathcal{B}(\lambda_2)\mathcal{B}(\lambda_1)\,|\Uparrow\rangle\,.$$



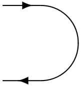
Reflecting ends

The diagonal K-matrix plays the role of the reflecting end,

$$K(\lambda) = \begin{pmatrix} k_{11}(\lambda) & 0 \\ 0 & k_{22}(\lambda) \end{pmatrix}.$$



k_{11}



k_{22}

Boltzmann weights

One can also redefine the Boltzmann weights to the case $-1 < \Delta < 1$. In this case, we have

$$a(\lambda) = \sin(\gamma - \lambda), \qquad b(\lambda) = \sin(\gamma + \lambda), \qquad c(\lambda) = \sin(2\gamma),$$

where $0 < \gamma < \pi/2$ and $\Delta = -\cos(2\gamma)$.

$$k_{11}(\lambda) = \frac{\sin(\xi + \lambda + \gamma)}{\sin(\xi)}, \qquad k_{22}(\lambda) = \frac{\sin(\xi - \lambda - \gamma)}{\sin(\xi)},$$

where ξ is the boundary parameter.

Tsuchiya determinant formula

$$
\begin{aligned}
Z_N &= (\sin(2\gamma))^N \prod_{i=1}^{N} \sin(2(\lambda_i+\gamma)) \frac{\sin(\xi-\mu_i)}{\sin(\xi)} \\
&\times \frac{\prod_{i,j=1}^{N} \sin(\gamma-(\lambda_i-\mu_j))\sin(\gamma+\lambda_i-\mu_j)\sin(\gamma-(\lambda_i+\mu_j))\sin(\gamma+\lambda_i+\mu_j)}{\prod_{\substack{i,j=1\\ i<j}}^{N} -\sin(\lambda_j-\lambda_i)\sin(\mu_i-\mu_j)\sin(\lambda_j+\lambda_i)\sin(\mu_i+\mu_j)} \\
&\times \det M,
\end{aligned}
$$

where M is a $N \times N$ matrix, whose matrix elements are $M_{ij} = \phi(\lambda_i, \mu_j)$ with

$$
\phi(\lambda,\mu) = \frac{1}{\sin(\gamma-(\lambda-\mu))\sin(\gamma+\lambda-\mu)\sin(\gamma-(\lambda+\mu))\sin(\gamma+\lambda+\mu)}.
$$

Homegeneous limit

- Taking $\lambda_i \to \lambda$ and $\mu_j \to \mu$.

$$
\begin{aligned}
Z_N(\lambda, \mu) &= \left[\sin(2\gamma)\sin(2(\lambda+\gamma)) \frac{\sin(\xi-\mu)}{\sin(\xi)} \right]^N \\
&\times \frac{[\sin(\gamma-(\lambda-\mu))\sin(\gamma+\lambda-\mu)\sin(\gamma-(\lambda+\mu))\sin(\gamma+\lambda+\mu)]^{N^2}}{C_N\,[-\sin(2\lambda)\sin(2\mu)]^{\frac{N(N-1)}{2}}} \\
&\times \tau_N(\lambda, \mu),
\end{aligned}
$$

where $C_N = \left[\prod_{k=1}^{N-1} k!\right]^2$. The determinant is given by

$$\tau_N(\lambda, \mu) = \det(H),$$

where the H-matrix elements are $H_{i,j} = (-\partial_\mu)^{j-1}\partial_\lambda^{i-1}\phi(\lambda, \mu)$.

Bidimensional Toda equation

$$-\tau_N \partial^2_{\mu\lambda} \tau_N + (\partial_\mu \tau_N)(\partial_\lambda \tau_N) = \tau_{N+1}\tau_{N-1},$$

and can be conveniently written as

$$-\partial^2_{\mu\lambda} [\log(\tau_N)] = \frac{\tau_{N+1}\tau_{N-1}}{\tau_N^2}, \qquad N \geq 1,$$

which is supplemented by the initial data $\tau_0 = 1$ and $\tau_1 = \phi(\lambda, \mu)$.

Special solutions

The partition function can be cast directly in simple expressions for some special points.

$$Z_N(\lambda,\mu;\gamma=\frac{\pi}{4}) = \left(\frac{\sin(\xi\mp\mu)}{\sin(\xi)}\right)^N (\cos(2\lambda))^{\frac{N(N+1)}{2}} (\sin(2\mu))^{\frac{N(N-1)}{2}}.$$

For the cases where $\mu=\pm(\lambda+\gamma)$ and $\mu=\pm(\lambda-\gamma)$,

$$Z_N(\lambda,\pm\lambda\pm\gamma)) = \left(\frac{\sin(\xi\mp(\lambda+\gamma))}{\sin(\xi)}\right)^N (\sin(2\gamma))^{N^2} (-\sin(2\lambda))^{\frac{N(N-1)}{2}} (\sin(2(\lambda+\gamma)))^{\frac{N(N+1)}{2}},$$

$$Z_N(\lambda,\pm\lambda\mp\gamma) = \left(\frac{\sin(\xi\mp(\lambda-\gamma))\sin(2(\gamma+\lambda))}{\sin(\xi)}\right)^N (\sin(2\gamma))^{N^2} (\sin(2\lambda)\sin(2(\gamma-\lambda))^{\frac{N(N-1)}{2}}.$$

The thermodynamic limit is trivial in these cases. The free energy $F=-\lim_{N\to\infty}\frac{\log(Z_N)}{2N^2}$ (we set temperature to 1) is given respectively by

$$\begin{aligned} e^{-2F(\lambda,\mu;\gamma=\pi/4)} &= \sqrt{\cos(2\lambda)\cos(2\mu)}, \\ e^{-2F(\lambda,\pm(\lambda+\gamma))} &= \sin(2\gamma)\sqrt{-\sin(2\lambda)\sin(2(\lambda+\gamma))}, \\ e^{-2F(\lambda,\pm(\lambda-\gamma))} &= \sin(2\gamma)\sqrt{\sin(2\lambda)\sinh(2(\gamma-\lambda))}. \end{aligned}$$

VSASM

We can also fix both spectral parameters and anisotropy parameter γ, such as

$$Z_N(0,0;\frac{\pi}{3}) = A_1^{VSASM} = \prod_{k=0}^{N-1}(3k+2)\frac{(6k+3)!(2k+1)!}{(4k+2)!(4k+3)!} = 1, 3, 26, 646, \ldots$$

which is a combinatorial point connected to the number of vertically symmetric alternating sign matrices (VSASM) due to (Kuperberg 2002)
Othe special cases are

$$Z_N(0,0;\frac{\pi}{4}) = 2^N A_2^{VSASM} = 2^{N^2},$$

and

$$Z_N(0,0;\frac{\pi}{6})/3^N = A_3^{VSASM} = \frac{3^{N(N-3)/2}}{2^N}\prod_{k=1}^{N}\frac{(k-1)!(3k)!}{k((2k-1)!)^2} = 1, 5, 126, \ldots,$$

where A_x^{VSASM} are the x-enumeration of the vertically symmetric alternating sign matrices (Kuperberg 2002).

Thermodynamic limit

$$Z_N(\lambda,\mu) = e^{-2N^2 F(\lambda,\mu)+O(N)},$$

where $F(\lambda,\mu)$ is the bulk free energy and unit temperature.
We suppose the following ansatz for the large size behaviour of the determinant $\tau_N(\lambda,\mu)$,

$$\tau_N(\lambda,\mu) = C_N e^{2N^2 f(\lambda,\mu)+O(N)},$$

where

$$e^{-2F(\lambda,\mu)} = \frac{\sin(\gamma-(\lambda-\mu))\sin(\gamma+\lambda-\mu)\sin(\gamma-(\lambda+\mu))\sin(\gamma+\lambda+\mu)}{\sqrt{-\sin(2\lambda)\sin(2\mu)}}e^{2f(\lambda,\mu)},$$

Liouville equation

Substituting the ansatz in the Toda equation (1), we obtain

$$-2\partial^2_{\mu\lambda} f(\lambda, \mu) = e^{4f(\lambda,\mu)},$$

which is the Liouville equation, whose general solution has the form of

$$e^{2f(\lambda,\mu)} = \frac{\sqrt{-u'(\lambda)v'(\mu)}}{(u(\lambda) + v(\mu))},$$

for arbitrary C^2 functions $u(\lambda)$, $v(\mu)$.

Solution

Our strategy is to chose $e^{2f(\lambda,\mu)}$ to match with the solution at $\gamma = \pi/4$. This leave us a γ dependent parameter to be determined. However the λ, μ dependence was already determined.

$$e^{2f(\lambda,\mu)} = \frac{\alpha\sqrt{-\sin(\alpha\lambda)\sin(\alpha\mu)}}{\cos(\alpha\lambda)+\cos(\alpha\mu)} = \frac{\alpha\sqrt{-\sin(\alpha\lambda)\sin(\alpha\mu)}}{2\cos(\frac{\alpha}{2}(\lambda-\mu))\cos(\frac{\alpha}{2}(\lambda+\mu))} \tag{1}$$

where the parameter $\alpha = \alpha(\gamma)$ and $\alpha(\pi/4) = 4$.

Solution

We must use the boundary condition given by $\mu = \pm(\lambda + \gamma)$ to determine α parameter. In doing so we see the only possible choice for the parameter is $\alpha(\gamma) = \pi/\gamma$.

$$e^{-2F(\lambda,\mu)} = \frac{\pi \sin(\gamma - \lambda + \mu) \sin(\gamma + \lambda - \mu) \sin(\gamma - \lambda - \mu) \sin(\gamma + \lambda + \mu)}{2\gamma \sqrt{-\sin(2\lambda) \sin(2\mu)}} \frac{\sqrt{-\sin(\frac{\pi\lambda}{\gamma}) \sin(\frac{\pi\mu}{\gamma})}}{\cos(\frac{\pi(\lambda-\mu)}{2\gamma}) \cos(\frac{\pi(\lambda+\mu)}{2\gamma})}. \tag{2}$$

- The other points $\mu = \pm(\lambda - \gamma)$ are naturally fulfilled.
- As an independent check, the solution obtained also reproduces the special points $\gamma = \pi/3, \pi/4, \pi/6$.

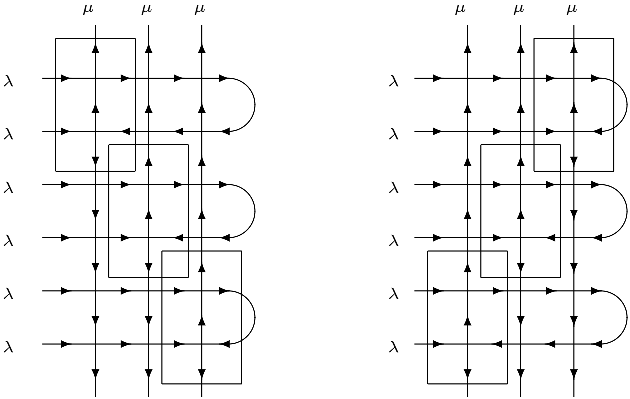
Ferrolectric phase: $\Delta > 1$

In the case $\Delta > 1$, one can obtain the expression for the free energy looking at the leading order state. The expression for the free energy can be written as

$$e^{-2F(\lambda,\mu)} = \sinh(\lambda - |\mu| + |\gamma|)\sqrt{\sinh(\lambda + |\mu| - \gamma)\sinh(\lambda + |\mu| + \gamma)}.$$

$\gamma > 0$ $\gamma < 0$

However due to the lack of additional boundary condition, we are unable to fix the suitable solution of Liouville equation.



Entropy

The number of alternating sign matrix (ASM) is given by

$$Z_N^{DWBC}(\lambda - \mu = \frac{\pi}{3}; \gamma = \frac{\pi}{3}) = A_N^{ASM} = \prod_{k=0}^{N-1} \frac{(3k+1)!}{(N+k)!} = 1, 2, 7, 42, 429, \cdots . \quad (3)$$

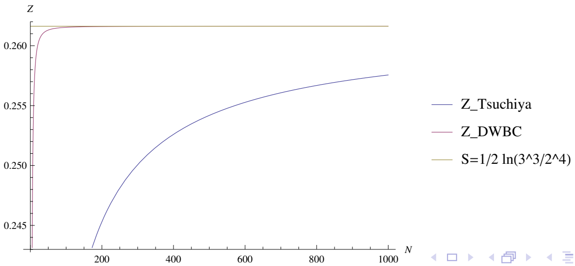
Taking the large limit we obtain the entropy of the six-vertex model with domain-wall boundary

$$S_{DWBC} = \frac{1}{2} \ln \left(\frac{3^3}{2^4} \right) . \quad (4)$$

The six-vertex model with reflecting end (Tsuchiya partition function) is related to the number of vertically symmetric alternating sign matrices (VSASM)

$$Z_N(0, 0; \frac{\pi}{3}) = A_1^{VSASM} = \prod_{k=0}^{N-1} (3k+2) \frac{(6k+3)!(2k+1)!}{(4k+2)!(4k+3)!} = 1, 3, 26, 646, \ldots \quad (5)$$

- Taking the large limit ($N \to \infty$) we again obtain the same value for the entropy, which means $S_{TSUCHIYA} = S_{DWBC}$.



Concluding remarks

- We determined the free-energy in the disordered phase ($|\Delta| < 1$).
- The leading ferroelectric state was identified.
- The entropy of the six-vertex model with reflecting end was found to be the same as DWBC.

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