Vanishing spin stiffness in the spin-1/2 Heisenberg chain for any nonzero temperature

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1D ideal spin conductors are typically correlated integrable models whose real part of the spin conductivity shows a delta peak and thus a finite spin stiffness at T>0. The dc conductivity is infinite at T>0 and the system exhibits ballistic transport

As discussed in the following, whether at zero spin density and finite temperatures the spin stiffness of the spin-1/2 XXX chain is finite or vanishes remains an unsolved and controversial issue, as different approaches yield contradictory results

The main goal of this talk is to rigorously show that in the thermodynamic limit at finite temperatures the spin stiffness of the spin-1/2 XXX chain vanishes within the (micro)canonical ensemble for fixed total z-component spin projection, including at zero spin projection, i.e. it vanishes both at zero spin density and in the limit of zero spin density

The more general anisotropic spin-1/2 Heisenberg XXZ chain with anisotropy parameter larger than or equal to zero and exchange integral J is a paradigmatic example of an integrable strongly correlated system

In the limit of a large system, the thermodynamic Bethe ansatz becomes exact for the model Hamiltonian in the absence of a uniform vector potential

M. Takahashi and M. Suzuki, Prog. Theor. Phys. 48, 2187 (1972)

As discussed in the following, there persists doubts about the validity of calculating the spin stiffness from the thermodynamic Bethe ansatz eigenvalues of the Hamiltonian in a uniform vector potential without the knowledge of matrix elements

X. Zotos, Phys. Rev. Lett. 82, 1764 (1999)
J. Benz, T. Fukui, A. Klümper, and C. Scheeren, J. Phys. Soc. Jpn. Suppl. 74, 181 (2005)

Integrable models have a set of orthogonal commuting linearly extensive local and quasi local conserved quantities such that,

$$\langle \hat{Q}_j \hat{Q}_{j'} \rangle = \delta_{j,j'} \langle \hat{Q}_j^2 \rangle$$

They provide a rigorous lower bound for the stiffness encoded in an inequality due to Mazur,

$$D \geq \frac{1}{2L} \sum_j \langle \hat{J} \hat{Q}_j \rangle^2 / \langle \hat{Q}_j^2 \rangle$$

P. Mazur, Physica 43, 533 (1969)

(j summation runs over all linearly extensive conserved quantities)

Quasilocal conserved operators are nonlocal operators for which,

$$\langle \hat{Q}^\dagger \hat{Q} \rangle \propto L$$

T. Prosen, arXiv:1406.2258

R. G. Pereira, V. Pasquier, J. Sirker, I. Affleck, arXiv:1406.2306

For the anisotropic spin-1/2 Heisenberg XXZ chain there are recent rigorous high-temperature results relying on the model's deformed symmetries and corresponding quasilocal conserved operators. These symmetries can be related to a dense set of commensurate easy-plane anisotropies,

$\lambda = \pi l/l'$ where l, l' are integers and $\Delta = \cos\lambda$ for $0 \leq \Delta \leq 1$

The lower bound reached by accounting for such quasi local conservation laws leads to a rigorous high-temperature Mazur's inequality,

$$D(T) \geq \left(\frac{J}{4}\right)^2 \left(1 - (l')\frac{\sin\left(\frac{2\pi}{l'}\right)}{2\pi}\right)\left(\frac{\sin\left(\frac{\pi l}{l'}\right)}{\sin\left(\frac{\pi}{l'}\right)}\right)^2 \frac{1}{T}$$

T. Prosen, and E. Ilievski, Phys. Rev. Lett. 111, 057203 (2013)

For $l = 1$ and $\Delta = \cos(\pi/l')$ such a lower bound equals the spin-stiffness expression derived by Zotos by use of the thermodynamic Bethe ansatz in a uniform vector potential

X. Zotos, Phys. Rev. Lett. 82, 1764 (1999)

The isotropic point of the spin-1/2 XXZ chain (the spin-1/2 XXX model) is the most experimentally relevant for the spin-lattice relaxation rate and other physical quantities

J. Sirker, R. G. Pereira, and I. Affleck, Phys. Rev. B 83, 035115 (2011)
N. Motoyama, H. Eisaki, and S. Uchida, Phys. Rev. Lett. 76, 3212 (1996)
K. R. Thurber, A. W. Hunt,T. Imai, and F. C. Chou, Phys. Rev. Lett. 87, 247202 (2001)

It is also the case that poses most challenging technical problems for theory. At the isotropic point the above Mazur's inequality is inconclusive, as it vanishes at zero spin density

Moreover, close to the isotropic point the numerical investigation of the spin stiffness expressions within the thermodynamic Bethe ansatz calculating it from the eigenvalues of the Hamiltonian in a uniform vector potential without the knowledge of matrix elements is difficult since the number of equations to solve diverges

X. Zotos, Phys. Rev. Lett. 82, 1764 (1999)
J. Benz, T. Fukui, A. Klümper, and C. Scheeren, J. Phys. Soc. Jpn. Suppl. 74, 181 (2005)

Our discussions focus mainly on the spin 1/2 isotropic Heisenberg model (XXX chain) at zero spin density. The Hamiltonian reads,

$$\hat{H}_{Hei} = J \sum_{j=1}^{L} \hat{\vec{S}}_j \cdot \hat{\vec{S}}_{j+1}$$

The real part of the spin conductivity can be written as,

$$\sigma(\omega, T) = 2\pi \, \underset{\uparrow}{D(T)} \, \delta(\omega) + \sigma_{reg}(\omega, T)$$

spin stiffness

$$D(T) = \frac{1}{2TL} \sum_{\nu} p_{\nu} \sum_{\nu' (\epsilon_{\nu} = \epsilon_{\nu'})} |\langle \nu | \hat{J} | \nu' \rangle|^2$$

If the T>0 stiffness is finite there is ballistic spin transport

X. Zotos and P. Prelovsek, Phys. Rev. B 53, 983 (1996)

The z-component spin current in the stiffness expression reads,

$$\hat{J} = -i\,J\sum_{j=1}^{L}(\hat{S}_j^+\hat{S}_{j+1}^- - \hat{S}_{j+1}^+\hat{S}_j^-)$$

The spin stiffness is directly related to the long-time asymptotic current-current correlation function as,

$$\begin{aligned} D(T) &= \frac{1}{2LT}\lim_{t\to\infty}\langle\hat{J}(t)\hat{J}(0)\rangle \\ &= \frac{1}{2TL}\sum_{\nu}p_\nu\sum_{\nu'(\epsilon_\nu=\epsilon_{\nu'})}|\langle\nu|\hat{J}|\nu'\rangle|^2 \end{aligned}$$

Ballistic spin transport then means that the correlation functions do not completely decay in time

Different approaches yield contradictory results on whether at zero spin density and finite temperatures the spin stiffness of the spin-1/2 XXX chain is finite or vanishes

Several (mainly) numerical approaches lead to a finite value for the spin stiffness of the XXX chain at finite temperature, as for instance those used in the following papers:

C. Karrasch, J. Hauschild, S. Langer, and F. Heidrich-Meisner, Phys. Rev. B 87, 245128 (2013)
C. Karrasch, J. H. Bardarson, and J. E. Moore, Phys. Rev. Lett. 108, 227206 (2012)
J. Benz, T. Fukui, A. Klümper, and C. Scheeren, J. Phys. Soc. Jpn. Suppl. 74, 181 (2005)
S. Fujimoto and N. Kawakami, Phys. Rev. Lett. 90, 197202 (2003)
F. Heidrich-Meisner, A. Honecker, D. C. Cabra, and W. Brenig, Phys. Rev. B 68, 134436 (2003)
J. V. Alvarez and C. Gros, Phys. Rev. Lett. 88, 077203 (2002)

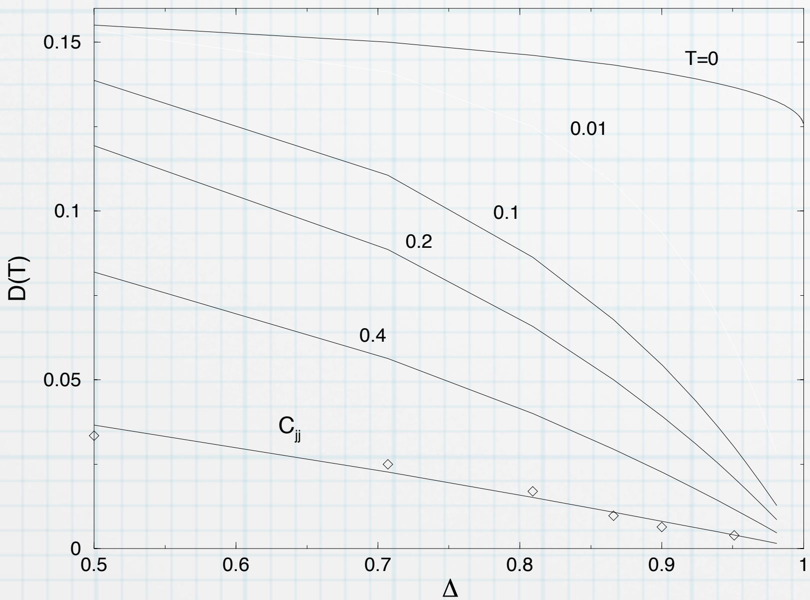
Other methods lead to results consistent with a very small or zero value for it, as those reported for instance in,

T. Prosen, and E. Ilievski, Phys. Rev. Lett. 111, 057203 (2013)
M. Znidaric, Phys. Rev. Lett. 106, 220601 (2011)
J. Herbrych, P. Prelovsek, and X. Zotos, Phys. Rev. B 84, 155125 (2011)
J. Sirker, R. G. Pereira, and I. Affleck, Phys. Rev. B 83, 035115 (2011)
J. Sirker, R. G. Pereira, and I. Affleck, Phys. Rev. Lett. 103, 216602 (2009)
N.M.R. Peres, P.D. Sacramento, D.K. Campbell, and J.M.P.C., Phys. Rev. B 59, 7382 (1999)
X. Zotos, Phys. Rev. Lett. 82, 1764 (1999)

Here we are mainly interested in methods using the thermodynamic Bethe anstaz

Different methods to derive the stiffness from the thermodynamic Bethe ansatz

1- The spin stiffness of the spin-1/2 XXZ chain has been calculated numerically within the thermodynamic Bethe ansatz from the eigenvalues of the Hamiltonian in a uniform vector potential without the knowledge of matrix elements



X. Zotos, Phys. Rev. Lett. 82, 1764 (1999)

The method of that study was first used for the charge stiffness of the 1D Hubbard model

S. Fujimoto and N. Kawakami, J. Phys. A 31, 465 (1998)

The doubts about the validity of that method stem from the divergences emerging in the integrands of Eqs. (24) and (25) of the following reference,

J. Benz, T. Fukui, A. Klümper, and C. Scheeren, J. Phys. Soc. Jpn. Suppl. 74, 181 (2005)

2- Alternatively, the authors of the latter reference used a phenomenological method that relies on a spinon and anti-spinon basis for the thermodynamic Bethe ansatz. They reached a larger value for the finite-T spin stiffness at zero spin density, which remains finite at the isotropic point

More recent very careful studies excluded the large spin stiffness found by that phenomenological method. They indicate that spin transport at finite temperatures is dominated by a diffusive contribution, the spin stiffness of the spin-1/2 XXX chain being very small or zero at zero spin density, as also found by the first above mentioned method

J. Sirker, R. G. Pereira, and I. Affleck, Phys. Rev. B 83, 035115 (2011)

3- Here we calculate the spin stiffness of the spin-1/2 XXX chain by a third method that also uses the thermodynamic Bethe ansatz. Our method is rigorous

Rather than calculating the spin stiffness from the eigenvalues of the Hamiltonian in a uniform vector potential without the knowledge of matrix elements, our method relies on the evaluation of matrix elements in the absence of a vector potential

J.M.P.C., T. Prosen, and D.K. Campbell, arXiv:1407.0732

First, we derive the spin stiffness at zero spin density and find that it vanishes in the thermodynamic limit

However, since vanishing spin density may in the thermodynamic limit also be approached by any finite fixed z-component spin projection value, or fixed window of spin projection values, and then letting the system length going to infinity, we carefully estimate the spin stiffness for finite spin projection values. We do that by calculating a suitable stiffness upper bound

Our matrix-element derivation of the spin-1/2 XXX chain spin stiffness at zero spin density shows technicality similarities with the method recently used to calculate the charge stiffness of the 1D Hubbard model at half filling

J.M.P.C., S.-J. Gu, and M.J. Sampaio, J. Phys. A 47, 255004 (2014)

J.M.P.C., S.-J. Gu, and P. D. Sacramento, Ann. Phys. 339, 484 (2013)

That charge stiffness vanishes for on-site repulsion U>0 in the thermodynamic limit for all temperatures, in contrast to the predictions of,

S. Fujimoto and N. Kawakami, J. Phys. A 31, 465 (1998)

Why the method introduced by these authors seems to lead to a correct spin stiffness value for the spin-1/2 XXZ chain and a wrong value for the charge stiffness of the half-filled 1D Hubbard model is an issue that is shortly discussed in the end of this talk

Our derivation of matrix elements uses the following commutation relations between the current operator and the generators of the spin SU(2) symmetry,

$$\left[\hat{J},\hat{S}^{\pm}\right] = \left[\hat{S}^{z},\hat{J}^{\pm}\right] = \pm\hat{J}^{\pm}\,; \quad \left[\hat{J}^{\pm},\hat{S}^{\mp}\right] = \pm 2\hat{J}$$

$$\left[\hat{J},\hat{S}^{z}\right] = 0\,; \quad \left[\hat{J},(\hat{\vec{S}})^{2}\right] = \hat{J}^{+}\hat{S}^{-} - \hat{S}^{+}\hat{J}^{-}$$

Here the two SU(2) symmetry operator components of the current operator (other than that of the z-component) read,

$$\hat{J}^{+} = (\hat{J}^{-})^{\dagger} = 2i\,J\sum_{j=1}^{L}(\hat{S}^{+}_{j}\hat{S}^{z}_{j+1} - \hat{S}^{+}_{j+1}\hat{S}^{z}_{j})$$

At zero spin density the Bethe states are both lowest-weight states (LWSs) and highest-weight states of the spin SU(2) symmetry algebra, so that,

$$\hat{S}^{-}|l_{\mathrm{r}},S,0\rangle = 0\,; \qquad \hat{S}^{+}|l_{\mathrm{r}},0,0\rangle = \hat{S}^{-}|l_{\mathrm{r}},0,0\rangle = 0$$

Our thermodynamic Bethe ansatz refers to the energy eigenstates of the spin-1/2 XXX chain that are LWSs of the spin SU(2) algebra, which here we call Bethe states

M. Takahashi, Prog. Theor. Phys. 46, 401 (1971)

The energy eigenstates that are not LWSs are generated from the corresponding Bethe states as follows,

$$|l_{\rm r}, S, S^z\rangle = \frac{1}{\sqrt{\mathcal{C}}}(\hat{S}^{+})^{n_s}|l_{\rm r}, S, -S\rangle$$

where,

$$\mathcal{C} = (n_s!)\prod_{j=1}^{n_s}(\,2S+1-j\,) \qquad n_s = 1, ..., 2S$$

$l_{\rm r}$ → refers to all quantum numbers other than the spin and spin projection needed to uniquely specify an energy eigenstate

Within our representation for the energy eigenstates the current matrix elements involved in the spin stiffness expression can be written as,

$$\langle l_{\rm r}, S, S^z|\hat{J}|l'_{\rm r}, S', S^z\rangle = \frac{\langle l_{\rm r}, S, -S|(\hat{S}^-)^{n_s}\hat{J}(\hat{S}^+)^{n'_s}|l'_{\rm r}, S', -S'\rangle}{\sqrt{\mathcal{C}\mathcal{C}'}}$$

From the use of both the above commutators involving the spin current operator and state transformation laws one finds the following relation involving matrix elements between energy eigenstates with the same finite spin value,

$$\langle l_{\rm r}, S, S^z|\hat{J}|l'_{\rm r}, S, S^z\rangle = -\frac{S^z}{S}\langle l_{\rm r}, S, -S|\hat{J}|l'_{\rm r}, S, -S\rangle\,; \qquad S \geq 1/2$$

where

$$S^z = -S + n_s \qquad\qquad n_s = 1, ..., 2S$$

In the thermodynamic limit, only matrix elements between such energy eigenstates contribute to the spin stiffness at finite temperature, as discussed in the following

For large L there are two temperature T regimes:

(i) T smaller than the energy eigenvalues level spacing

(ii) T larger than the energy eigenvalues level spacing

In the present thermodynamic limit, the T regime (i) shrinks to T=0, while the T region (ii) includes all of T>0

X. Zotos, F. Naef, and P. Prelovsek, Phys. Rev. B. 55, 11029 (1997)

Moreover, in that limit the persistent currents vanish

N. Yu and M. Fowler, Phys. Rev. B 45, 11795 (1992)

One then finds by summing over momentum k and -k subspaces, the result that within the T>0 regime (ii) the expression of the stiffness involves only current expectation values

X. Zotos, F. Naef, and P. Prelovsek, Phys. Rev. B. 55, 11029 (1997)
P. Ginsparg, in Fields, Strings and Critical Phenomena (North-Holland, Amsterdam,1990)

$$D(T) = \frac{1}{2TL} \sum_{\nu} p_\nu |\langle \nu | \hat{J} | \nu \rangle|^2 \, ; \qquad T > 0 \, ; \qquad L \to \infty$$

The above relations are valid only for energy eigenstates with spin S>1/2. Concerning matrix elements between S=0 energy eigenstates, it follows trivially from the operator algebra relations that the spin current operator can be written as,

$$\hat{J} = \frac{1}{2}[\hat{J}^+, \hat{S}^-]$$

It is then straightforward to show that matrix elements between S=0 energy eigenstates vanish,

$$\begin{aligned}\langle l_\mathrm{r}, 0, 0|\hat{J}|l'_\mathrm{r}, 0, 0\rangle &= \frac{1}{2}(\langle l_\mathrm{r}, 0, 0|\hat{J}^+\hat{S}^-|l'_\mathrm{r}, 0, 0\rangle \\ &- \langle l_\mathrm{r}, 0, 0|\hat{S}^-\hat{J}^+|l'_\mathrm{r}, 0, 0\rangle) = 0\end{aligned}$$

The spin stiffness of the spin-1/2 XXX chain,

$$D(T) = \frac{1}{2TL} \sum_{l_{\rm r}} \sum_{S=|S^z|}^{L/2} p_{l_{\rm r},S,S^z} |\langle l_{\rm r}, S, S^z | \hat{J} | l_{\rm r}, S, S^z \rangle|^2$$

then vanishes at zero z-component spin projection for finite temperatures T>0,

$$D_{S^z=0}(T) = 0, \quad T > 0$$

This follows from the above matrix elements relations implying the validity of the following current expectation values results,

$$\begin{aligned} \langle l_{\rm r}, 0, 0 | \hat{J} | l_{\rm r}, 0, 0 \rangle &= 0 \\ \langle l_{\rm r}, S, S^z | \hat{J} | l_{\rm r}, S, S^z \rangle &= -\frac{S^z}{S} \langle l_{\rm r}, S, -S | \hat{J} | l_{\rm r}, S, -S \rangle \end{aligned}$$

Hence the spin current of all zero z-component spin projection energy eigenstates vanishes

However, it is important not to restrict ourselves only to spin density m=0, which may be sensitive to certain pathologies

A rigorous analysis requires that one considers the spin stiffness for m>0 and confirms that it vanishes in the limit of the density tending to 0

The use of the above matrix-elements relations allows writing the finite-temperature spin stiffness for arbitrary value of the z-component spin projection different from zero as,

$$D_{S^z}(T) = \frac{(2S^z)^2}{2LT} \sum_{S=|S^z|}^{L/2} \sum_{l_{\rm r}} p_{l_{\rm r},S,S^z} \frac{|\langle l_{\rm r}, S, -S|\hat{J}|l_{\rm r}, S, -S\rangle|^2}{(2S)^2}$$

In the following we express the energy eigenstate spin currents in terms of thermodynamic Bethe ansatz quantum number occupancy configurations and clarify the relation of such configurations to those of the model L spins 1/2

The number of independent occupancy configurations of the L spins 1/2 equals the Hilbert space dimension,

$$\sum_{2S=0\,(\text{integers})}^{L} \mathcal{N}(S) = 2^L$$

Here the is sum over the number of independent occupancy configurations of the L spins 1/2 in each of the sub-spaces of fixed spin S, whose dimension reads,

$$\mathcal{N}(S) = (2S+1)\,\mathcal{N}_{singlet}(S)$$

Such a dimension equals the product of the number of multiplet and singlet configurations, respectively. From the use of the spin SU(2) symmetry algebra one finds that the latter reads,

$$\mathcal{N}_{singlet}(S) = \binom{L}{L/2 - S} - \binom{L}{L/2 - S - 1}$$

The number of spins 1/2 that contribute to the multiplet and singlet configurations, respectively, are uniquely defined,

$$2S \;\; \text{spins} \;\; 1/2 \longrightarrow (2S+1)$$

$$L - 2S \;\; \text{spins} \;\; 1/2 \longrightarrow \mathcal{N}_{singlet}(S)$$

An important property for the present study is that only the first group of 2S spins 1/2 in the multiplet configurations contribute to the spin current $\langle \hat{J} \rangle = \langle [\hat{J}^{+}, \hat{S}^{-}] \rangle / 2$

In the following we relate the quantum numbers of the thermodynamic Bethe ansatz equations whose occupancy configurations generate the energy eigenstates with these two sets of spins 1/2

M. Takahashi, Prog. Theor. Phys. 46, 401 (1971)

$L - 2S$ spins $1/2$ in singlet configurations $\longrightarrow$ part of their degrees of freedom are contained in the n-pair configurations (often called strings of length n)

That each n-pair configuration contains degrees of freedom of n spins 1/2 singlet pairs is consistent with the sum rule obeyed by the n-pair configuration number values of the energy eigenstates,

$$\{M_n\} \longleftrightarrow \qquad L - 2S = \sum_{n=1}^{\infty} 2n M_n$$

Within the thermodynamic Bethe ansatz, there is for each type of n-pair configuration associated with a given fixed n value n=1,2,3,... a set of quantum numbers,

$$I_j^n \longrightarrow \begin{array}{l} 0, \pm 1, \pm 2, \ldots \\ \pm 1/2, \pm 3/2, \pm 5/2, \ldots \end{array}$$

Each type of n-pair configuration can be associated with a n-band whose momentum values have the usual spacing,

$$q_j = \frac{2\pi}{L} I_j^n \longleftrightarrow \qquad j = 1, \ldots, M_n^b$$

The number of a n-band occupied and unoccupied momentum values are well defined for each energy eigenstate,

$$M_n^b = M_n + M_n^h \longleftrightarrow M_n^h = 2S + \sum_{n'=n+1}^{\infty} 2(n'-n)M_{n'}$$

Actually, the energy eigenstates are generated by all possible occupancies of the n-band momentum values. Those have Pauli-like zero and one occupancies. Hence it is useful to introduce alternative n-band momentum distributions for n-particles and n-holes, respectively,

$$M_n(q_j) = 1 \longleftrightarrow M_n^h(q_j) = 0$$

$$M_n(q_j) = 0 \longleftrightarrow M_n^h(q_j) = 1$$

It is confirmed in the following that the degrees of freedom of the two sets of 2S and L-2S spins 1/2 are easiest identified within the n-hole representation of the energy eigenstates spin currents

Specifically, within the usual Bethe-ansatz n-particle representation the spin current of a LWS has the form,

$$\langle l_{\rm r}, S, -S|\hat{J}|l_{\rm r}, S, -S\rangle = \sum_{n=1}^{\infty}\sum_{j=1}^{M_n^b} M_n(q_j)\, j_n(q_j)$$

where the n-particle elementary current spectrum is defined by the relations,

$$j_n(q_j) = -2J\, f_n(k_n(q_j)) \qquad f_n(k) = g_n(k)\,\sin k$$

$$g_n(k) = \frac{2}{n}\,\frac{\cos^2(k/2)}{2\pi[\rho_n(\Lambda) + \rho_n^h(\Lambda)]} \qquad k \in [-\pi, \pi]$$

$$k_n(q_j) = 2\arctan(\Lambda_j^n)$$

Here Λ_j^n is the real part of the n-pair configuration rapidity of the energy eigenstate under consideration, which is uniquely defined by the coupled Bethe-anstaz equations

To express the spin currents in the alternative n-hole representation, one must account for the exotic properties of the n-bands. In contrast to the usual solid-state bands, summing the elementary currents over all n-band momentum values gives not in general zero. It rather gives,

$$\sum_{j=1}^{M_n^b} j_n(q_j) = \frac{\sum_{n'=n+1}^{\infty} 2(n'-n)M_{n'}}{M_n^h} \sum_{j=1}^{M_n^b} M_n^h(q_j)\, j_n(q_j)$$

One then finds that within the Bethe-ansatz n-hole representation the spin current of a LWS has the form,

$$\langle l_{\mathrm{r}}, S, -S|\hat{J}|l_{\mathrm{r}}, S, -S\rangle = \sum_{n=1}^{\infty} \frac{2S}{M_n^h} \sum_{j=1}^{M_n^b} M_n^h(q_j)\, j_n^h(q_j)$$

where,

$$j_n^h(q_j) \equiv -j_n(q_j) = 2J\, f_n(k_n(q_j))$$

To dig deeper into the physical meaning of the n-band hole representation, we emphasize that the latter spin current expression can be rewritten as,

$$\langle l_{\mathrm{r}}, S, -S|\hat{J}|l_{\mathrm{r}}, S, -S\rangle = \sum_{n=1}^{\infty}\sum_{j=1}^{M_n^b} M_n^h(q_j)\, j_n^h(q_j) + \sum_{n=2}^{\infty} M_n\, j_n^p$$

Here the n-hole elementary currents have contributions both from the 2S spins 1/2 in multiplet configurations and the spins 1/2 in singlet n-pair configurations with n>1 pairs

The virtual elementary spin currents in the second term are carried by the n-pair configurations with n>1 pairs and read,

$$j_n^p = -\sum_{n'=1}^{n-1} \frac{2(n-n')}{M_{n'}^h} \sum_{j=1}^{M_{n'}^b} M_{n'}^h(q_j)\, j_{n'}^h(q_j)$$

Their role is to exactly cancelling the unwanted current contributions from such n-pair configurations to the n-hole elementary currents

As mentioned above, the use of the spin SU(2) symmetry algebra alone leads to a number fixed-S spin singlet configurations whose value is,

$$\mathcal{N}_{singlet}(S) = \binom{L}{L/2-S} - \binom{L}{L/2-S-1}$$

Consistently with the direct relation of the 2S-L spins 1/2 in such configurations to the Bethe-ansatz n-pair configurations, one finds that at fixed S the following dimension has exactly the same value,

$$\mathcal{N}_{singlet}(S) = \sum_{\{M_n\}} \prod_{n=1}^{\infty} \binom{M_n^b}{M_n}$$

Here the summation is over all sets of n-pair configuration numbers that obey the S-fixed sum rule,

$$\sum_{n=1}^{\infty} 2nM_n = L - 2S$$

For our spin stiffness computation, it is convenient to replace the former n-hole current representation,

$$\langle l_{\rm r}, S, -S|\hat{J}|l_{\rm r}, S, -S\rangle = \sum_{n=1}^{\infty} \frac{2S}{M_n^h} \sum_{j=1}^{M_n^b} M_n^h(q_j)\, j_n^h(q_j)$$

in the stiffness general T>0 expression,

$$D_{S^z}(T) = \frac{(2S^z)^2}{2LT} \sum_{S=|S^z|}^{L/2} \sum_{l_{\rm r}} p_{l_{\rm r},S,S^z} \frac{|\langle l_{\rm r}, S, -S|\hat{J}|l_{\rm r}, S, -S\rangle|^2}{(2S)^2}$$

Since the summations in this expression are very difficult to perform, our goal is to derive a rigorous upper bound for the stiffness valid for all z-component spin projection values. Specifically, we want to check whether such a upper bound vanishes in the limit of zero spin projection

Each spin-S subspace can be divided into a set of smaller reduced subspaces with the same S value and fixed values for the set of n-pair configuration numbers. We denote the maximum value of the current absolute value,

$$|\langle l_{\mathrm{r}}, S, -S|\hat{J}|l_{\mathrm{r}}, S, -S\rangle|$$

in each such reduced subspace by,

$$\mathcal{J}_{\max}(S, \{M_n\}) \,, \qquad n > 1$$

Here n>1 because,

$$M_1 = \frac{1}{2}(L - 2S - \sum_{n=2}^{\infty}(2n)M_n)$$

is uniquely determined

We find that for each S-fixed subspace the maximum spin current absolute value is achieved for the reduced subspace for which,

$$M_n = 0\,, \qquad n > 1$$

Specifically, we find that the minimum value of the deviation,

$$\delta\mathcal{J}_{\text{min}}(S,\{M_n\}) = \mathcal{J}_{\text{max}}(S,\{M_n=0\}) - \mathcal{J}_{\text{max}}(S,\{M_n\}) \geq 0$$

is an increasing function of S with limiting behaviors,

$$\begin{aligned}\delta\mathcal{J}_{\text{min}}(S,\{M_n\}) &= 0\,, \qquad 2S \ll L \\ &= \frac{J}{L}\sum_{n=2}^{\infty}(n-1)\,M_n\,, \qquad (L-2S) \ll L\end{aligned}$$

This result allows us to fulfil the first step of our derivation of a rigorous upper bound for the spin stiffness

It accounts for replacing in the stiffness expression all matrix elements associated with a fixed S value,

$$|\langle l_{\rm r}, S, -S|\hat{J}|l_{\rm r}, S, -S\rangle|$$

by their maximum value,

$$\mathcal{J}_{\max}(S) \equiv \mathcal{J}_{\max}(S, \{M_n = 0\})$$

This procedure leads to a first rigorous stiffness upper bound,

$$D^*_{S^z}(T) = \frac{(2S^z)^2}{2LT} \sum_{S=|S^z|}^{L/2} \sum_{l_{\rm r}} p_{S,l_{\rm r},S^z} \left(\frac{\mathcal{J}_{\max}(S)}{2S}\right)^2$$

Furthermore, $\mathcal{J}_{\max}(m_S)/2S$ is found to be a smoothly decreasing function of S with limiting behaviors,

$$\begin{aligned}\mathcal{J}_{\max}(S)/2S &= J\frac{\pi}{4}, \qquad 2S \ll L \\ &= J\frac{(L-2S)}{2L}, \qquad (L-2S) \ll L\end{aligned}$$

Hence the maximum value of $\mathcal{J}_{\max}(m_S)/2S$ refers to the minimum S value,

$$S = |S^z| = m\,L/2$$

A second rigorous yet larger stiffness upper bound is then reached by replacing $\mathcal{J}_{\max}(m_S)/2S$ by its maximum value for the whole S range contributing to a given m,

$$\frac{\mathcal{J}_{\max}(S)}{2S} \rightarrow \frac{\mathcal{J}_{\max}(m\,L)}{(m\,L)}$$

Importantly, the stiffness expression state summations can then be performed exactly for all finite temperatures T>0. Indeed, the probability distribution in each fixed-spin-projection canonical ensemble is normalized as,

$$\sum_{S=|S^z|}^{L/2} \sum_{l_{\rm r}} p_{l_{\rm r},S,S^z} = 1$$

The following simple yet rigorous spin stiffness upper bound valid for all finite temperatures is then obtained,

$$D^{**}_{S^z}(T) = \frac{\mathcal{J}^2_{\rm max}(m\,L)}{2TL}\,, \qquad T > 0$$

Its limiting behaviors are,

$$\begin{aligned} D^{**}_{S^z}(T) \quad &= \quad \frac{\left(J\,\frac{\pi}{4}\right)^2\,m^2\,L}{2T}\,, \quad m \ll 1\,, \\ &= \quad \frac{\left(J\,\frac{1}{2}\right)^2\,(1-m)^2\,L}{2T}\,, \quad (1-m) \ll 1 \end{aligned}$$

The small-m expression confirms that the spin stiffness of the spin-1/2 XXX chain vanishes for all finite temperatures in the limit of zero spin density, alike at zero spin density

Such a result implies that in the thermodynamic limit and at finite temperatures the spin-1/2 isotropic Heisenberg chain has no ballistic spin transport at zero spin density

Summary and short discussion

Our rigorous results show that the spin stiffness of the spin-1/2 XXX chain vanishes in the thermodynamic limit for finite temperature and any fixed range or even distribution of the spin projection, or any distribution of the spin density m shrinking sufficiently fast that,

$$\langle m^2 \rangle L \to 0$$

Note that this leaves out, marginally, the grand canonical ensemble in which,

$$\langle m^2 \rangle = \mathcal{O}(1/L)$$

However, the large overestimate of the energy eigenstates spin currents we used in deriving the stiffness upper bound leads us to expect that our result remains valid in the grand canonical case, in accord with the usual expectation of the equivalence of ensembles in the thermodynamic limit

The spin stiffness of the spin-1/2 XXZ chain has for high temperature the following limiting values,

$$\Delta = 0 \quad \leftrightarrow \quad D = \frac{(J/4)^2}{T}$$

$$\Delta = 1 \quad \leftrightarrow \quad D = 0$$

(our result is also valid for any T including for high T)

The stiffness lower bound found by Prosen and Ilievski given above saturates the high-temperature spin stiffness in both these two limits. Combined with the equality of that lower bound to the spin stiffness found by Zotos at $\lambda = \pi/l'$, this most likely implies that the bound saturates the stiffness for the whole range, $0 \leq \Delta \leq 1$

T. Prosen, and E. Ilievski, Phys. Rev. Lett. 111, 057203 (2013)

X. Zotos, Phys. Rev. Lett. 82, 1764 (1999)

This provides strong evidence that the spin stiffness computed by Zotos is rigorous for any finite temperature

Recent rigorous results on the charge stiffness of the 1D half-filled Hubbard model reveal that it vanishes in the thermodynamic limit at finite temperatures

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This fully contradicts previous predictions

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Combination of the results on the stiffnesses of different integrable models indicate that the method used by Fujimoto and Kawakami fails to provide the correct T>0 stiffness when the stiffness vanishes at T=0 and is valid at T>0 for models whose stiffness is finite at T=0

The spin-1/2 XXZ chain belongs to the latter class, the divergences emerging at zero spin density in the integrands of Eqs. (24) and (25) of the following reference cancelling each other

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