Multi-layer Fractional Quantum Hall States

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Lattice Fractional Quantum Hall Effect

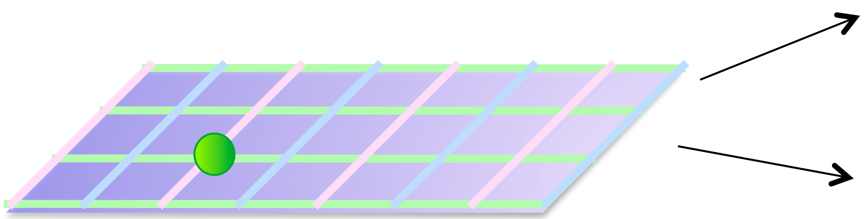
What are the FQH states that can form in lattice systems with bands of higher Chern numbers?

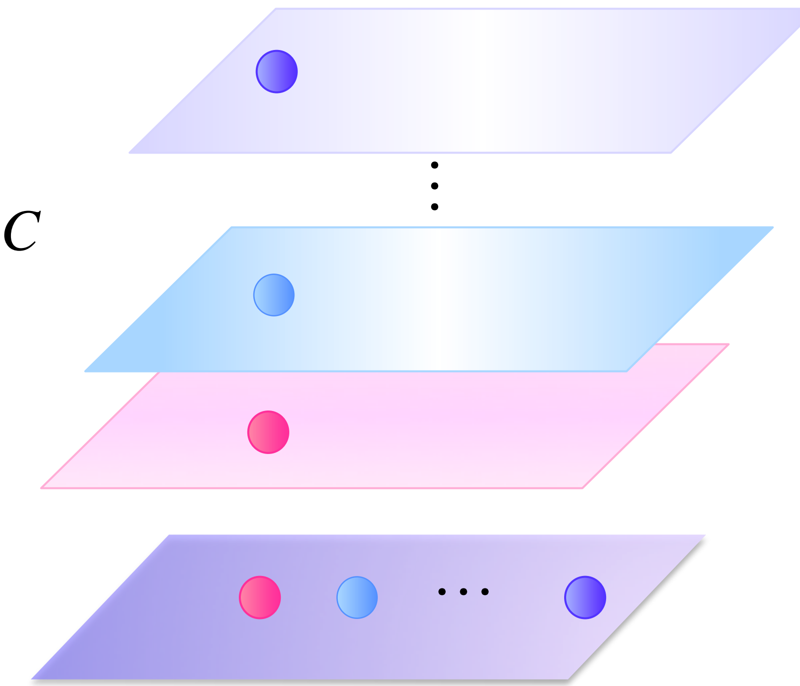
Examples: Fractional Chern Insulators,
Hofstadter bands

e.g. X.-L. Qi, PRL. 107, 126803 (2011)

Wannier basis: A band with Chern number C can be mapped to C Landau levels in the continuum.

Multi-layer picture





Multi-component/pseudospin picture

Lattice specific processes can change the nature of the state.

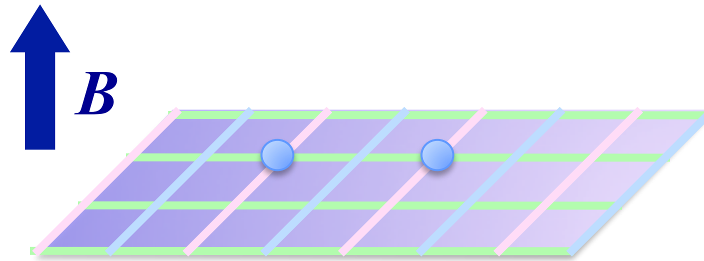
Example: umklapp scattering

*LH, G. Moller and S. H. Simon, PRL **108**, 256809 (2012)*

FQH in a $C = 2$ Hofstadter Band

LH, G. Moller and S. H. Simon, PRL ***108****, 256809 (2012)*

$$H = \sum_{i<j} \delta(z_i - z_j)$$



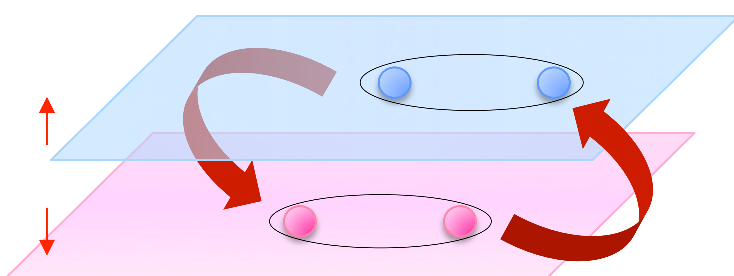
$B = 2n_e h/e \rightarrow \nu = 1$

$$\Psi_{gs} = Pf\left(\frac{1}{z_i - z_j}\right) \prod_{i<j \in N} (z_i - z_j)$$

→ *MR pfaffian state has good overlap with groundstate of H*

Cooper, Wilkin, Gun,
Phys. Rev. Lett. 87, 120405 (2001)

$C = 2$



Tunneling terms can be generated by umklapp scattering in the lattice

$$H = \sum_{i,j=1}^{N} \delta(z_i - z_j) \sum_{s \neq s' = \uparrow,\downarrow} \left(|ss\rangle\langle ss| + |ss'\rangle\langle ss'| + |ss'\rangle\langle s's| + t|ss\rangle\langle s's'| \right)_{ij}$$

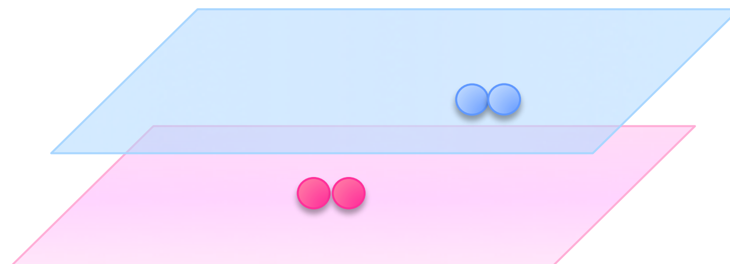
What is the groundstate of this Hamiltonian?

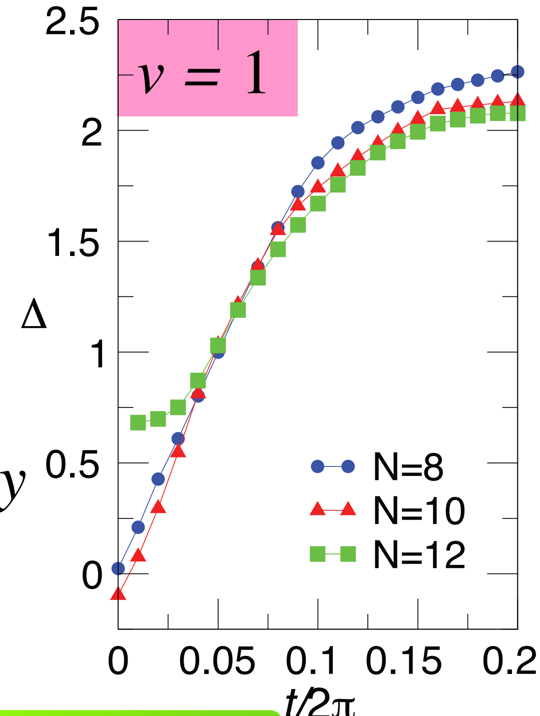
FQH in a $C = 2$ Hofstadter Band

2-body contact interaction in a system with $C = 2$

$$H = \sum_{i,j=1}^{N} \delta(z_i - z_j) \sum_{s \neq s' = \uparrow,\downarrow} \big(|ss\rangle\langle ss| + |ss'\rangle\langle ss'| + |ss'\rangle\langle s's| + \; t|ss\rangle\langle s's'| \big)_{ij}$$

→ *gap opens up*





→ *tunneling terms cancel the repulsion penalty*

→ *paired states can form in each layer*

→ *effective 3-body-like terms in each layer!*

Exact result for $|t| = 1$!

$$\Psi_{gs} = \sum_{N^\uparrow=1}^{N} \alpha_{N^\uparrow} Pf\left(\frac{1}{z_i^\uparrow - z_j^\uparrow}\right) \prod_{i<j=1}^{N^\uparrow} (z_i^\uparrow - z_j^\uparrow)\, Pf\left(\frac{1}{z_i^\downarrow - z_j^\downarrow}\right) \prod_{i<j=1}^{N^\downarrow} (z_i^\downarrow - z_j^\downarrow) \prod_{i=1}^{N^\uparrow} \prod_{j=1}^{N^\downarrow} (z_i^\uparrow - z_j^\downarrow)$$

CFT Description – Review of MR state

Reminder: Single layer Moore-Read pfaffian state …

Moore and Read, Nucl. Phys. B360 362 (1991)

$$\Psi_{gs} = Pf\left(\frac{1}{z_i - z_j}\right)\prod_{i<j=1}^{N}(z_i - z_j)$$

Ising CFT: $\{1, \sigma, \psi\}$

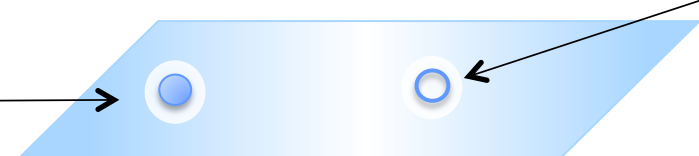
Neutral part: topological properties

$\mathrm{U}(1)_4$: $\{1, e^{i\phi_c}, e^{i\phi_c/2}, e^{-i\phi_c/2}\}$

Charged part

Electron operators

$$\psi_e(z) = \psi\, e^{i\phi_c}$$



$$\psi_{qh}(\eta) = \sigma\, e^{\pm i\phi_c/2}$$

Smallest-charge quasihole operator $q_{qh} = e/2$

$$\Psi_{gs}(\{z_i\}) = \left\langle \prod_{i=1}^{N}\psi_e(z_i)\right\rangle$$

$$\Psi_{qh}(\{z_i\},\{\eta_j\}) = \left\langle \prod_{i=1}^{N}\psi_e(z_i)\prod_{j=1}^{n}\psi_{qh}(\eta_j)\right\rangle$$

Not all 12 sectors of Ising x $\mathrm{U}(1)_4$ lead to valid wavefunctions

Final topological theory: $\mathrm{SU}(2)_2$

CFT Description – Coupled MR state

$$\Psi_{gs} = Pf\left(\frac{1}{z_i^{\uparrow} - z_j^{\uparrow}}\right)\prod_{i<j=1}^{N^{\uparrow}}(z_i^{\uparrow} - z_j^{\uparrow})\, Pf\left(\frac{1}{z_i^{\downarrow} - z_j^{\downarrow}}\right)\prod_{i<j=1}^{N^{\downarrow}}(z_i^{\downarrow} - z_j^{\downarrow}) \prod_{i=1}^{N^{\uparrow}}\prod_{j=1}^{N^{\downarrow}}(z_i^{\uparrow} - z_j^{\downarrow})$$

Each layer is like a Moore-Read pfaffian state …

2 copies of *Ising* CFT: $\{1,\ \sigma^{\uparrow},\ \psi^{\uparrow}\}\times\{1,\ \sigma^{\downarrow},\ \psi^{\downarrow}\}$

Neutral part: topological properties

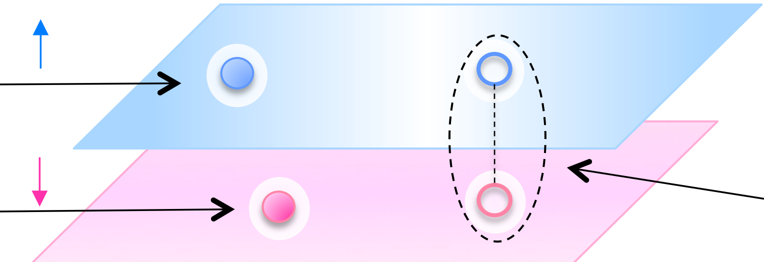
$U(1)_4$: $\{1,\ e^{i\phi_c},\ e^{i\phi_c/2},\ e^{-i\phi_c/2}\}$

Charge

Electron operators

$\psi_e^{\uparrow}(z^{\uparrow}) = \psi_{\uparrow}\, e^{i\phi_c}$

$\psi_e^{\downarrow}(z^{\downarrow}) = \psi_{\downarrow}\, e^{i\phi_c}$



Smallest-charge $q_{qh} = e/2$ quasihole operators with ***valid*** wavefunctions:

$\psi_{qh}(\eta) = \underline{\sigma_{\uparrow}\sigma_{\downarrow}}\, e^{\pm i\phi_c/2}$

same position

Excitation with σ or σ separated are not energetically favorable.

→ Sectors with odd numbers of σ or σ operators are *confined*.

Topological Properties – neutral part

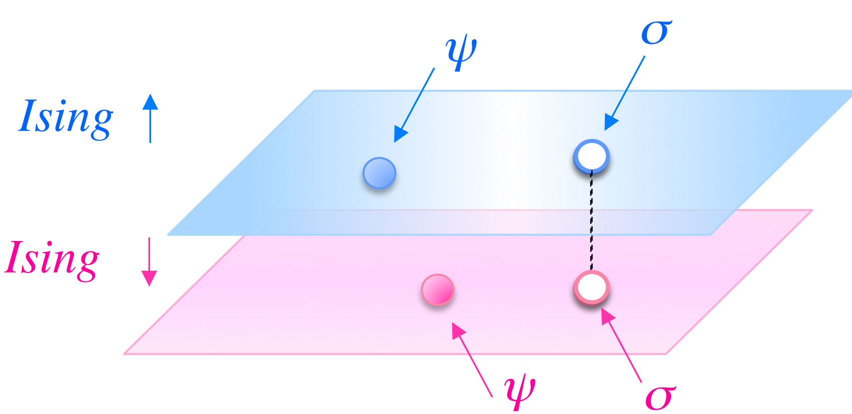
Let us ignore the charge part and just focus on the neutral part.

Topological sectors of *Ising* x *Ising*:

↓ \ ↑	1	σ	ψ
1	1 1	~~1 σ~~	1 ψ
σ	~~σ 1~~	σ σ	~~σ ψ~~
ψ	ψ 1	~~ψ σ~~	ψ ψ

Visible in edge spectrum!

If we cross out the sectors that involve an odd number of σ or σ :



			U(1)$_4$
1 1 $h = 0$ $\psi\,\psi$ $h = 1$	→	1	1
1 ψ $h = 1/2$ ψ 1 $h = 1/2$	→	ψ	$e^{i\phi}$
$\sigma\,\sigma$ $h = 1/8$	→	a	$e^{i\phi/2}$
	↘	a^*	$e^{-i\phi/2}$

In this reduced subspace certain sectors are topologically identical.

$\sigma\,\sigma$ sector has to split

Condensing $\psi\,\psi$

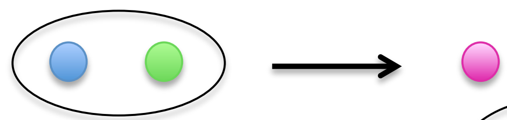
Bais and Slingerland, Phys. Rev. B 79, 045316 (2009)

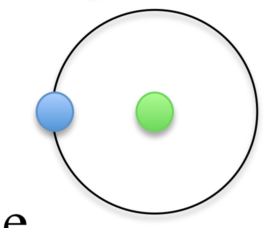
Topological Bose Condensation

- Start with an anyon theory

Primary fields of a CFT

1. Anyon labels: $\{1, a, b, c \ldots\}$  $\cdots$

2. Fusion rules: $a \times b = c + ?$ 

3. Braiding rules: $e^{2\pi i(h_c - h_a - h_b)}$ 

Conformal dimensions of the primary fields of the CFT

- Identify a boson $\longrightarrow$ $h_b = \text{integer}$

- Condense the boson $\longrightarrow$ $b = 1$

- Identify the remaining sectors such that they form a new consistent anyon theory

Condensation of $(\psi\ \psi)$

$0 \quad (1\ 1) \longrightarrow \boxed{(1\ 1) = (\psi\ \psi)}$

$\boxed{1/16} \quad (1\ \sigma) \times (\psi\ \psi) = (\psi\ \sigma) \longrightarrow$ $(1\ \sigma)$, $(\psi\ \sigma)$ are *confined*

$1/2 \quad (1\ \psi) \times (\psi\ \psi) = (\psi\ 1) \longrightarrow \boxed{(1\ \psi) = (\psi\ 1)}$

$\boxed{1/16} \quad (\sigma\ 1) \times (\psi\ \psi) = (\sigma\ \psi) \longrightarrow$ $(\sigma\ 1) = (\sigma\ \psi)$ are *confined*

$1/8 \quad (\sigma\ \sigma) \times (\psi\ \psi) = (\sigma\ \sigma) \longrightarrow \boxed{(\sigma\ \sigma)_1}, \boxed{(\sigma\ \sigma)_2}$

$\boxed{9/16} \quad (\sigma\ \psi)$

$1/2 \quad (\psi\ 1)$

$\boxed{9/16} \quad (\psi\ \sigma)$

$1 \quad (\psi\ \psi)$

Ising fusion rules:

$\sigma \times \psi = \sigma$

$\psi \times \psi = 1$

$\sigma \times \sigma = 1 + \psi$

Splitting of $(\sigma\ \sigma)$

Before splitting

$$(\sigma\ \sigma) \times (\sigma\ \sigma) = (1\ 1) + (1\ \psi) + (\psi\ 1) + (\psi\ \psi)$$

$$(\sigma\ \sigma) \begin{array}{l} \nearrow (\sigma\ \sigma)_1 \\ \searrow (\sigma\ \sigma)_2 \end{array}$$

$$(\sigma\ \sigma)_1 \times (\sigma\ \sigma)_1 = (\sigma\ \sigma)_2 \times (\sigma\ \sigma)_2 = (1\ \psi) = (\psi\ 1)$$

$$(\sigma\ \sigma)_1 \times (\sigma\ \sigma)_2 = (\sigma\ \sigma)_2 \times (\sigma\ \sigma)_1 = (\psi\ \psi) = (1\ 1)$$

→ After splitting, the $(\sigma\ \sigma)$ sectors become Abelian!

Ising fusion rules:

$\sigma \times \psi = \sigma$

$\psi \times \psi = 1$

$\sigma \times \sigma = 1 + \psi$

Condensation of $(\psi\ \psi)$

$0 \quad (1\ 1) \longrightarrow (1\ 1) = (\psi\ \psi)$

$1/16 \quad (1\ \sigma) \times (\psi\ \psi) = (\psi\ \sigma) \longrightarrow$ $(1\ \sigma)$, $(\psi\ \sigma)$ are *confined*

$1/2 \quad (1\ \psi) \times (\psi\ \psi) = (\psi\ 1) \longrightarrow (1\ \psi) = (\psi\ 1)$

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$1/8 \quad (\sigma\ \sigma) \times (\psi\ \psi) = (\sigma\ \sigma) \longrightarrow (\sigma\ \sigma)_1,\ (\sigma\ \sigma)_2$

$9/16 \quad (\sigma\ \psi)$

$1/2 \quad (\psi\ 1)$

$9/16 \quad (\psi\ \sigma)$

$1 \quad (\psi\ \psi)$

$\Downarrow$

$U(1)_4$

Ising fusion rules:

$\sigma \times \psi = \sigma$

$\psi \times \psi = 1$

$\sigma \times \sigma = 1 + \psi$

Topological Order of the Coupled Pfaffian

Full theory: $U(1)_4$ x $U(1)_4$

Electron operators:

$$\boxed{\begin{aligned}\psi_e^{\uparrow}(z^{\uparrow}) &= \sqrt{2}\cos\phi_n(z^{\uparrow})e^{i\phi_c(z^{\uparrow})}\\ \psi_e^{\downarrow}(z^{\downarrow}) &= i\sqrt{2}\sin\phi_n(z^{\downarrow})e^{i\phi_c(z^{\downarrow})}\end{aligned}} \quad = \quad \boxed{\begin{aligned}\psi_e^{\uparrow} &= \psi_{\uparrow}\, e^{i\phi_c}\\ \psi_e^{\downarrow} &= \psi_{\downarrow}\, e^{i\phi_c}\end{aligned}}$$

Only those sectors of $U(1)_4$ x $U(1)_4$ that are local with respect to these electron operators will lead to valid wavefunctions.

→ The rest are confined.

Adding Charge – further reduction

Full theory: $U(1)_4$ x $U(1)_4$

$U(1)_4$ \ $U(1)_4$	1	a	a^*	ψ
1	$1\ 1$	$\cancel{1\ a}$	$\cancel{1\ a^*}$	$1\ \psi$
a	$\cancel{a\ 1}$	$a\ a$	$a\ a^*$	$\cancel{a\ \psi}$
a^*	$\cancel{a^*\ 1}$	$a^* a$	$a^* a^*$	$\cancel{a^*\ \psi}$
ψ	$\psi\ 1$	$\cancel{\psi\ a}$	$\cancel{\psi\ a^*}$	$\psi\ \psi$

Excluding the confined sectors, we end up with these 8:

In this reduced subspace, only 4 sectors are topologically distinct.

			$\to U(1)_2$ x $U(1)_2$
$1\ 1$ $h=0$ $\psi\ \psi$ $h=1$	$\longrightarrow$	1	$1 \quad 1$
$1\ \psi$ $\psi\ 1$ $h=1/2$	$\longrightarrow$	ψ	$e^{\frac{i\phi_1}{\sqrt{2}}}\ e^{\frac{i\phi_2}{\sqrt{2}}}$
$a\ a$ $a^*\ a^*$ $h=1/4$	$\longrightarrow$	e	$e^{\frac{i\phi_1}{\sqrt{2}}}\ 1$
$a^*\ a$ $a\ a^*$ $h=1/4$	$\longrightarrow$	m	$1\ e^{\frac{i\phi_2}{\sqrt{2}}}$

Adding Charge – further reduction

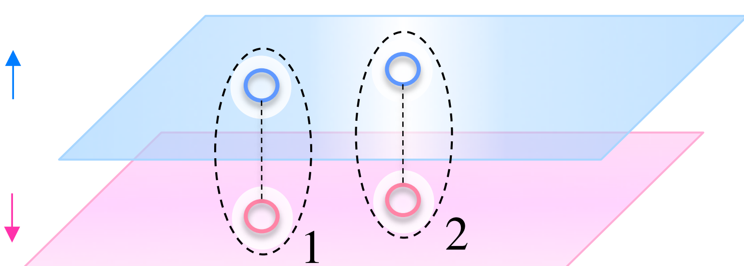
Full theory: $U(1)_4 \times U(1)_4$

$U(1)_4$ \ $U(1)_4$	1	a	a^*	ψ
1	$1\ 1$	$\cancel{1\ a}$	$\cancel{1\ a^*}$	$1\ \psi$
a	$\cancel{a\ 1}$	$a\ a$	$a\ a^*$	$\cancel{a\ \psi}$
a^*	$\cancel{a^*\ 1}$	$a^* a$	$a^* a^*$	$\cancel{a^*\ \psi}$
ψ	$\psi\ 1$	$\cancel{\psi\ a}$	$\cancel{\psi\ a^*}$	$\psi\ \psi$

Excluding the confined sectors, we end up with these 8: $\longrightarrow$ $U(1)_2 \times U(1)_2$

$\boldsymbol{\psi = e \times m}$

$$\psi^1_{qh}(\eta) = \left(\sigma_\uparrow \sigma_\downarrow\right)_1 e^{i\phi_c/2}$$



$$\psi^2_{qh}(\eta') = \left(\sigma_\uparrow \sigma_\downarrow\right)_2 e^{i\phi_c/2}$$

$1\ 1$, $\psi\ \psi$	$h=0$, $h=1$	$\longrightarrow$	1	$1 \quad 1$
$1\ \psi$, $\psi\ 1$	$h=1/2$	$\longrightarrow$	ψ	$e^{\frac{i\phi_1}{\sqrt{2}}} \quad e^{\frac{i\phi_2}{\sqrt{2}}}$
$a\ a$, $a^*\ a^*$	$h=1/4$	$\longrightarrow$	e	$e^{\frac{i\phi_1}{\sqrt{2}}} \quad 1$
$a^*\ a$, $a\ a^*$	$h=1/4$	$\longrightarrow$	m	$1 \quad e^{\frac{i\phi_2}{\sqrt{2}}}$

Adding Charge – further reduction

Full theory: $U(1)_4 \times U(1)_4$

$U(1)_4$ \ $U(1)_4$	1	a	a^*	ψ
1	$1\ 1$	$\cancel{1\ a}$	$\cancel{1\ a^*}$	$1\ \psi$
a	$\cancel{a\ 1}$	$a\ a$	$a\ a^*$	$\cancel{a\ \psi}$
a^*	$\cancel{a^*\ 1}$	a^*a	a^*a^*	$\cancel{a^*\ \psi}$
ψ	$\psi\ 1$	$\cancel{\psi\ a}$	$\cancel{\psi\ a^*}$	$\psi\ \psi$

$\boldsymbol{\psi = e \times m}$

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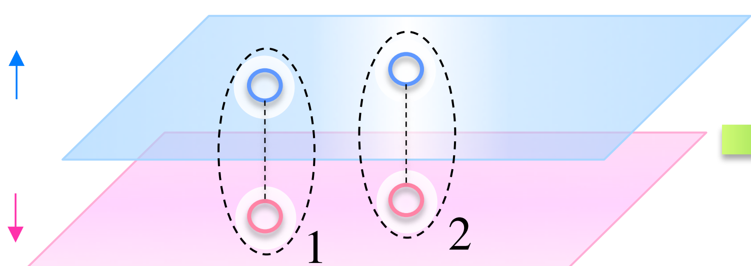
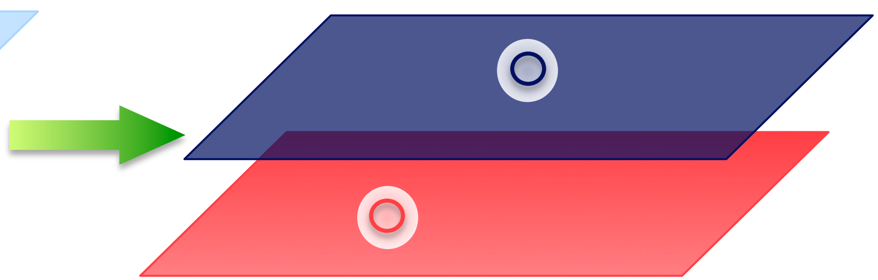


$$\psi^2_{qh}(\eta') = \left(\sigma_\uparrow \sigma_\downarrow\right)_2 e^{i\phi_c/2}$$



(220) state in a different basis

$U(1)_2 \times U(1)_2$

$1 \quad 1$

$e^{\frac{i\phi_1}{\sqrt{2}}} \quad e^{\frac{i\phi_2}{\sqrt{2}}}$

$e^{\frac{i\phi_1}{\sqrt{2}}} \quad 1$

$1 \quad e^{\frac{i\phi_2}{\sqrt{2}}}$

Edge States of the Coupled Pfaffian

U(1)

k	*count*	
⋮	⋮	⋮
5	7	$a_{-5}/\ a_{-4}a_{-1}/\ a_{-3}a_{-2}\ /\ a_{-3}a_{-1}^{2}/\ a_{-2}^{2}a_{-1}/\ a_{-2}a_{-1}^{3}/a_{-1}^{5}\ \ \lvert 0\rangle$
4	5	$a_{-4}/\ a_{-3}a_{-1}\ /\ a_{-2}^{2}/\ a_{-2}a_{-1}^{2}\ /\ a_{-1}^{4}\ \ \lvert 0\rangle$
3	3	$a_{-3}/\ a_{-2}\ a_{-1}/\ a_{-1}^{3}\ \ \lvert 0\rangle$
2	2	$a_{-2}/\ a_{-1}^{2}\ \ \lvert 0\rangle$
1	1	$a_{-1}\ \ \lvert 0\rangle$
0	1	$\lvert 0\rangle$

a_{-n}^{m} places m bosons in an orbital with momentum n

Edge States of the Coupled Pfaffian

$U(1)_4$

Additional bosonic descendent fields should be included in the counting

$V_{4n} = e^{i2n\phi}$ $\quad k = 2n^2 \quad n = 1, 2, \ldots$

k	count	
.	.	.
.	.	.
.	.	.
5	7	$a_{-5}/\ a_{-4}a_{-1}/\ a_{-3}a_{-2}\ /\ a_{-3}a_{-1}^{2}/\ a_{-2}^{2}a_{-1}/\ a_{-2}a_{-1}^{3}/a_{-1}^{5}\ \vert 0\rangle$
4	5	$a_{-4}/\ a_{-3}a_{-1}\ /\ a_{-2}^{2}/\ a_{-2}a_{-1}^{2}\ /\ a_{-1}^{4}\ \vert 0\rangle$
3	3	$a_{-3}/\ a_{-2}\ a_{-1}/\ a_{-1}^{3}\ \vert 0\rangle$
2	2	$a_{-2}/\ a_{-1}^{2}\ \vert 0\rangle$
1	1	$a_{-1}\ \vert 0\rangle$
0	1	$\vert 0\rangle$

$a_{-n}^{\;m}$ places m bosons in an orbital with momentum n

Edge States of the Coupled Pfaffian

$\mathrm{U}(1)_4$

Additional bosonic descendent fields should be included in the counting

$V_{4n} = e^{i2n\phi}$ $\quad k = 2n^2 \quad n = 1, 2, \ldots$

k	count	
. . .	. . .	. . .
5	7 + 3	$a_{-5}/\ a_{-4}a_{-1}/\ a_{-3}a_{-2}\ /\ a_{-3}a_{-1}^{\ 2}/\ a_{-2}^{\ 2}a_{-1}/\ a_{-2}a_{-1}^{\ 3}/a_{-1}^{\ 5}/\ V_4\, a_{-3}/\ V_4\, a_{-2}\, a_{-1}^{\ 2}/\ V_4\, a_{-1}^{\ 2}\ \lvert 0\rangle$
4	5 + 2	$a_{-4}/\ a_{-3}a_{-1}\ /\ a_{-2}^{\ 2}/\ a_{-2}a_{-1}^{\ 2}\ /\ a_{-1}^{\ 4}\ /\ V_4\, a_{-2}/\ V_4\, a_{-1}^{\ 2}\ \ \lvert 0\rangle$
3	3 + 1	$a_{-3}/\ a_{-2}\, a_{-1}/\ a_{-1}^{\ 3}\ /\ V_4\, a_{-1}\ \ \lvert 0\rangle$
2	2 + 1	$a_{-2}/\ a_{-1}^{\ 2}/\ V_4\ \ \lvert 0\rangle$
1	1	$a_{-1}\ \ \lvert 0\rangle$
0	1	$\lvert 0\rangle$
k	count	

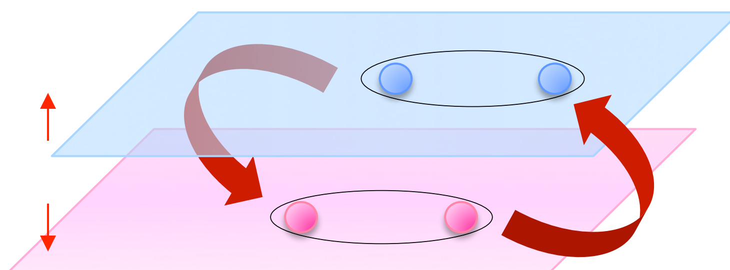
The addition of these operators corresponds to adding new particles to the system.

In a system with fixed total particle number *the level* does not appear!

Edge States of the Coupled Pfaffian

Charged sector $U(1)_4$ x $U(1)_4$ Neutral sector

$$H(r,t) = \sum_{i,j=1}^{N} \delta(z_i - z_j) \sum_{s \neq s' = \uparrow,\downarrow} \Big(|ss\rangle\langle ss| + r(|ss'\rangle\langle ss'| + |ss'\rangle\langle s's|) + \boxed{t|ss\rangle\langle s's'|\Big)_{ij}}$$



Total Charge = fixed → No V_4 operators can be added → U(1)

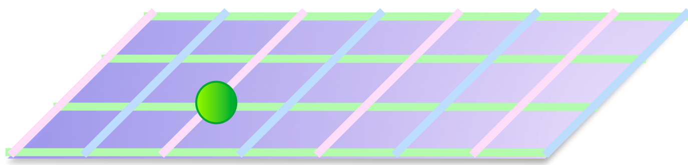
No restriction on the neutral sector → V_4 operators can be added → $U(1)_4$

→ Edge theory: U(1) x $U(1)_4$

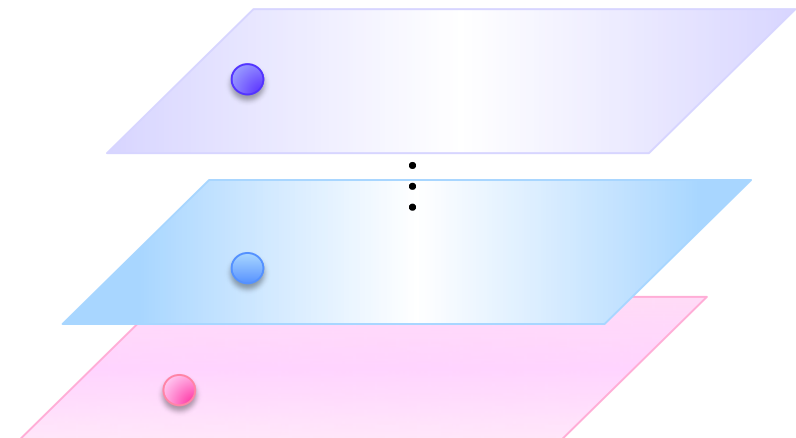
Multi-layer Systems

For a C layer system, with the Hamiltonian:

$$H = \sum_{i,j=1}^{N} \delta(z_i - z_j) \sum_{s \neq s'=0}^{C-1} \left(|ss\rangle\langle ss| + |ss'\rangle\langle ss'| + |ss'\rangle\langle s's| + t|ss\rangle\langle s's'| \right)_{ij}$$



Chern number C



The groundstate is of the form:

$$\Psi = \sum_{\{N^s\}\in N} \alpha_{N^s} \prod_{s=1}^{C} Pf\left(\frac{1}{z_i^s - z_j^s}\right) \prod_{i<j=1}^{N^s} (z_i^s - z_j^s) \prod_{s,s'=1}^{C} \prod_{j=1}^{N-N^s} \prod_{i=1}^{N^s} (z_i^s - z_j^{s'})$$

$\nu = 1$

The properties of the resulting state can be deduced from C Ising CFTs, coupled via a single U(1):

Neutral sector

$\text{Ising}^1 \times \text{Ising}^2 \times \ldots \times \text{Ising}^C \times U(1)_4$

Charged sector

Multi-layer Systems

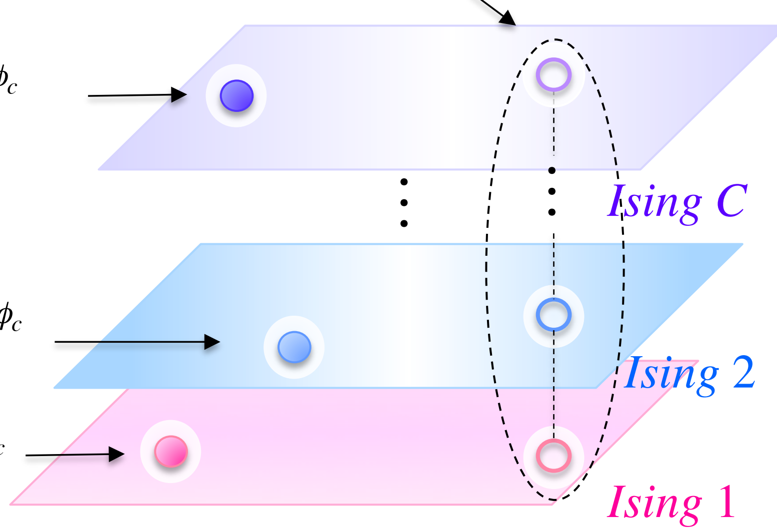
Electron operators:

$$\psi_{qh}(\eta) = \left(\sigma_1\sigma_2\ldots\sigma_C\right)e^{i\phi_c/2}$$

$$\psi_e^C(z^C) = \psi_C\, e^{i\phi_c}$$

$$\psi_e^2(z^2) = \psi_2\, e^{i\phi_c}$$

$$\psi_e^1(z^1) = \psi_1\, e^{i\phi_c}$$



Excitations that result in valid wavefunctions correspond to having (σ_1 σ_2 … σ_C) coincide.

→ All other σ sectors will be confined.

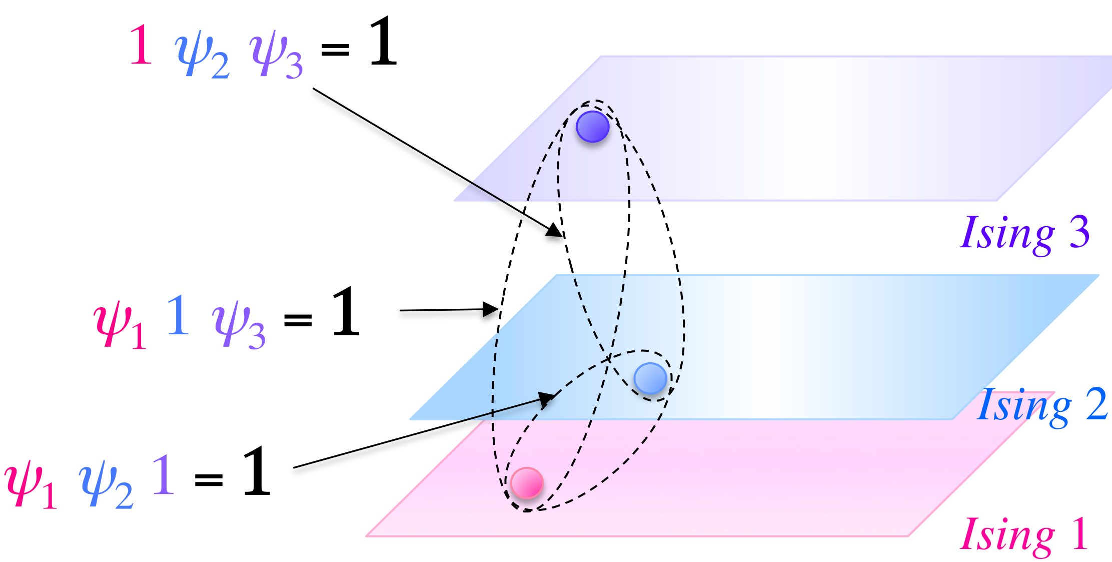
Equivalently: All pairs of the form $\psi_i\psi_j$ between Ising layers ***condense***.

Ising i *Ising j*

$$\boxed{\psi_i\psi_j \longrightarrow 1}$$

Bais and Slingerland, Phys. Rev. B 79, 045316 (2009)

Multi-layer Systems



Excitations that result in valid wavefunctions correspond to having $(\sigma_1\ \sigma_2\ \sigma_3)$ coincide.

→ All sectors with 1 or 2 σ operators will be confined.

Equivalently: All pairs of the form $\psi_i\psi_j$ between Ising layers ***condense***.

Ising i *Ising j*

$$\boxed{\psi_i\psi_j \longrightarrow 1}$$

Bais and Slingerland, Phys. Rev. B 79, 045316 (2009)

→ *16 distinct categories*

Kitaev, Ann. of Phys. 321, 2 (2006)

Neutral Sector: The 16-fold Way

Kitaev, Ann. of Phys. 321, 2 (2006)

C: odd

conf. dim. → h =
q. dim. → d =

	$\mathbf{1}$	$\boldsymbol{\sigma}$	$\boldsymbol{\psi}$	
$h =$	0	$C/16$	1/2	*Ising-like*
$d =$	1	$\sqrt{2}$	1	

$C = 2 \bmod 4$

	$\mathbf{1}$	$\boldsymbol{a}$	$\boldsymbol{a^*}$	$\boldsymbol{\psi}$	
$h =$	0	$C/16$	$C/16$	1/2	$\mathrm{U}(1)_4$-*like*
$d =$	1	1	1	1	

$\boxed{\mathrm{SO}(C)_1}$

$C = 0 \bmod 4$

	$\mathbf{1}$	$\boldsymbol{e}$	$\boldsymbol{m}$	$\boldsymbol{\psi}$	
$h =$	0	$C/16$	$C/16$	1/2	$\mathrm{U}(1)_2\mathrm{x}\mathrm{U}(1)_2$-*like*
$d =$	1	1	1	1	

Same structure
1D spin chains →

Mansson, Lahtinen, Suorsa and Ardonne, PRB 88, 041403(R) (2013)
Lahtinen, Mansson and Ardonne PRB 89, 014409 (2014)

Adding the Charge Sector: $U(1)_4$

$SO(C)_1$ *Neutral sector*

→ $SO(C)_1 \times U(1)$ *Probably visible in the edge spectrum*

Full theory: $SO(C)_1 \times U(1)_4$

Independent sectors that result in valid wavefunctions correspond to ***condensing*** new $\psi\psi$ pairs between the charge and neutral sectors.

$$\boxed{\begin{matrix}\psi\psi \\ 1\,1\end{matrix}} \longrightarrow 1$$

→ $\boxed{SO(C+2)_1 \quad \textit{final topological theory}}$

Summary

- Two-body Hamiltonian + pair tunneling in a C-layer system

$\longrightarrow$ *Exact groundstate* at $\nu = 1$ is the Coupled Pfaffian State

$$\Psi = \sum_{\{N^s\}\in N} \alpha_{N^s} \prod_{s=1}^{C} Pf\left(\frac{1}{z_i^s - z_j^s}\right) \prod_{i<j=1}^{N^s} (z_i^s - z_j^s) \prod_{s,s'=1}^{C} \prod_{j=1}^{N-N^s} \prod_{i=1}^{N^s} (z_i^s - z_j^{s'})$$

- Edge spectrum: $SO(C)_1$ x $U(1)$
- Final topological theory: $SO(C+2)_1$

C odd: Non-Abelian

C even: Abelian

- Topological phase transitions?