Higgsless Superconductivity from Topological Defects in Compact BF Terms

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- 1982 FQE states: no symmetry breaking $\Rightarrow$ new quantum order, topological order (Wen):
  - gapped in the bulk;
  - gapless edge excitations.
- Low energy effective field theories for such states involve topological field theories
  - background independent;
  - ground state degeneracy;
  - quasiparticles have fractional statistics;
  - can break P and T symmetries.

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- Topological insulators are materials insulating in the bulk but support conducting edge excitations; exp. observed in 2d (Hsieh 2008) and 3d (Hasan 2009)
- Which is the topological field theory that describes this new phase of matter? Topological BF action

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- 3d: $S_{BF} = \frac{k}{2\pi}\int d^4x\; b_{\mu\nu}\epsilon_{\mu\nu\alpha\beta}\partial_\alpha a_\beta$, BF action
  - $j_\mu \propto \epsilon_{\mu\nu\alpha\beta}\partial_\nu b_{\alpha\beta}$ charge fluctuations;
  - $\phi_{\mu\nu} \propto \epsilon_{\mu\nu\alpha\beta}\partial_\alpha a_\beta$ magnetic fluctuations;
  - gauge invariance: $a_\mu \to a_\mu + \partial_\mu \xi \;, b_{\mu\nu} \to b_{\mu\nu} + \partial_\mu \eta_\nu - \partial_\nu \eta_\mu$;

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- How can we get a mass for a U(1) gauge theory $\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu}$ ((2+1) dimensions)?

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- Higgs mechanism: add coupling with a complex scalar: $(D_\mu\phi) * (D^\mu\phi) - V(|\phi|)$, with non trivial minimum in $< \phi >= v$
  - unbroken vacuum: complex scalar field with two real massive degrees of freedom and the gauge field with one massless excitation;
  - broken vacuum: there is an additional quadratic term $v^2 A_\mu A^\mu$ in the action, gauge symmetry is broken;
  - degrees of freedom: massive gauge boson, two degrees of freedom, one massive chargeless Higgs boson with mass $2e^2v^2$
  - phenomenological theory for "standard" superconductivity

- Topological mechanism (Deser, Jackiw, Templeton): add a CS term to the Lagrangian: $\frac{k}{4\pi} A_\mu \epsilon^{\mu\nu\alpha} \partial_\nu A_\alpha$
  - equation of motion: $(\partial_\mu \partial^\mu + (ke^2)^2) F_\mu = 0$ which describe the propagation of a single (transverse) degree of freedom with mass $m = |ke^2|$, gauge symmetry is not broken;
  - spin $s = \frac{k}{|k|}$;
  - Gauss law: $\partial_i E^i = mB$, magnetic field acts as source for electric field;
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  - P and T breaking.
- leads to anyon superconductivity

Anyon Superconductivity

- non-relativistic fermions interacting with a fictitious statistical gauge field α_μ:

  $\mathcal{L} = \psi^\dagger i\left(\partial_0 + i\alpha_0\right)\psi + \frac{1}{2m}\psi^\dagger\left(\partial_i + i\alpha_i\right)^2\psi + \frac{k}{4\pi}\alpha_\mu\epsilon^{\mu\nu\sigma}\partial_\nu\alpha_\sigma$

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- all states of the Jain hierarchy with $\nu = m/(mp+1)$, p an even integer, are also gapped

- these states are described by the Lagrangian density (Wen, Zee):

$$\mathcal{L} = \frac{1}{4\pi}\mathbf{a}_\mu^T K \epsilon^{\mu\nu\sigma}\partial_\nu \mathbf{a}_\sigma + j^\mu A_\mu$$

with $\mathbf{a}_\mu = \left(a_\mu^1 \ldots a_\mu^m\right)$ and
$j^\mu = \sum_{i=1}^m \left(j^\mu\right)^i = (1/2\pi)\sum_{i=1}^m \ \epsilon^{\mu\nu\sigma}\partial_\nu a_\sigma^i$ conserved electron current

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- A_μ is a real electromagnetic gauge potential, K is given by $K = \begin{pmatrix} p+1 & p & \cdots & p \\ p & p+1 & \cdots & p \\ \vdots & \vdots & \ddots & \vdots \\ p & p & \cdots & p+1 \end{pmatrix}$ with no zero eigenvalue, fully gapped

- allow statistical field to fluctuate:

$$\mathcal{L} \to \mathcal{L} = \frac{1}{4\pi} \mathcal{A}_\mu^T \Lambda \, \epsilon^{\mu\nu\sigma} \partial_\nu \mathcal{A}_\sigma + \frac{1}{2\pi} \mathcal{A}_\mu^T \epsilon^{\mu\nu\sigma} \partial_\nu \mathbf{A}_\sigma$$

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$$\mathcal{A}_\mu = \left(\alpha_\mu, a_\mu^1 \dots a_\mu^m\right) \; ; \mathbf{A} = (0, A_\mu \dots A_\mu) \; ; k = m/(mp+1)$$

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- $\Lambda = \begin{pmatrix} \frac{m}{mp+1} & -1 & \cdots & \cdots & -1 \\ -1 & p+1 & p & \cdots & p \\ -1 & p & p+1 & \cdots & p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & p & \cdots & \cdots & p+1 \end{pmatrix}$ has a zero eigenvalue

- the vanishing eigenvalue corresponds to the charge m
  eigenvector: $\left(\frac{\phi}{2\pi}, q^i, \ldots, q^m\right) = (mp+1, 1, \ldots, 1)$;
  $\phi \equiv \int d^2x\ \mathcal{B}$; $q^i \equiv \int d^2x\ (j^0)^i$
  - gapless mode is the anyon superfluid mode
  - protected from mixing with a continuum of particle-hole excitations by the gap of the average-field approximation
  - gives mass to the photon via the mixed $\mathcal{A}\mathbf{A}$ CS term, no Higgs boson
  - PT breaking

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- how can we distinguish it from Higgs mechanism?

Topologically massive theories in (d+1)

- BF theory:

$$S_{TM} = \int_{M_{d+1}} \frac{k}{2\pi} a_1 \wedge db_{d-1} \; (S_{BF})$$

$$\frac{-1}{2e^2} da_1 \wedge *da_1 + \frac{(-1)^{d-1}}{2g^2} db_{d-1} \wedge *db_{d-1}$$

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- BF theory provides a generalization of fractional statistics to arbitrary dimensions (Semenoff, Szabo), given by the adiabatic transport of hypersurface Σ_p around a Σ_{d-p} hypersurface:
  - statistical parameter $\frac{2\pi}{k} p(d-p)$;
  - 3+1: particles around vortex strings.

Doubled Anyon Superconductivity

- ($m = 1$) irreducible two-component spinors in 2d carry a pseudoscalar (vortex) degree of freedom $S_z = \pm 1/2$ (Boyanovsky, Blankenbecler, Yahalom) and charge $\rightarrow$ spin up electrons can described by a gauge field a^+: $(\phi^\mu)^+ = \frac{1}{2\pi}\epsilon^{\mu\nu\sigma}\partial_\nu a^+_\sigma$
  - it carries both vorticity and charge, $\phi^+ = \int d^2x (\phi^0)^+ = S_z^+ + (1/2)q^+$
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- $(1/2\pi)\epsilon^{\mu\nu\sigma}\partial_\nu\alpha_\sigma$ represents a background emergent charge, rather than magnetic flux, different from the intrinsic charge that couples to physical electromagnetic fields (Moessner, Sondhi)

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  $\mathcal{L}^{+} = \frac{1}{4\pi} (\mathcal{A}^{+})^{T}_{\mu} \Lambda^{+} \ \epsilon^{\mu\nu\sigma} \partial_{\nu} \mathcal{A}^{+}_{\sigma} + \frac{1}{2\pi} (\mathcal{A}^{+})^{T}_{\mu} \epsilon^{\mu\nu\sigma} \partial_{\nu} \mathbf{A}_{\sigma}$,
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  $\mathcal{A}^+_\mu = \left(\alpha_\mu, a^+_\mu\right)$
- $\Lambda^+ = \begin{pmatrix} \frac{1}{p+1} & -1 \\ -1 & p+1 \end{pmatrix}$
  - eigenvalues 0, superfluid mode
  - $\lambda^+ = ((p+1)^2 + 1)/(p+1))$ massive mode

- $k = \frac{1}{p+1}; p =$ even $\Rightarrow$ statistics parameters $\theta = \pi(1 - 1/k) = -p\pi$ and $\theta = \pi(1 + 1/k) = (2 + p)\pi$ are equivalent since they differ by exactly 2π

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- parity-reversed model corresponding to 2d electrons with spin down $\mathcal{L}^- = \frac{1}{4\pi}(\mathcal{A}^-)^T_\mu \Lambda^- \, \epsilon^{\mu\nu\sigma}\partial_\nu \mathcal{A}^-_\sigma - \frac{1}{2\pi}(\mathcal{A}^-)^T_\mu \epsilon^{\mu\nu\sigma}\partial_\nu \mathbf{A}_\sigma$, $\mathcal{A}^-_\mu = \left(\alpha_\mu, a^-_\mu\right)$

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- $$\mathcal{L} = \frac{1}{4\pi}\mathcal{A}_\mu^T \Lambda \, \epsilon^{\mu\nu\sigma}\partial_\nu \mathcal{A}_\sigma + \frac{1}{2\pi}\mathcal{A}_\mu^T \epsilon^{\mu\nu\sigma}\partial_\nu \mathbf{A}_\sigma$$

$\mathcal{A}_\mu = \left(\alpha_\mu, a_\mu^+, a_\mu^-\right)$, $\mathbf{A} = (0, A_\mu, -A_\mu)$

PT invariant model

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  $\mathcal{A}_\mu = \left(\alpha_\mu, a^+_\mu, a^-_\mu\right)$, $\mathbf{A} = (0, A_\mu, -A_\mu)$
- $\Lambda = \begin{pmatrix} 0 & -1 & -1 \\ -1 & p+1 & 0 \\ -1 & 0 & -(p+1) \end{pmatrix}$
  - eigenvalues 0, superfluid mode
  - two massive modes $\pm\lambda = \sqrt{(p+1)^2 + 2}$

Physical content

- gapless mode gauge field $\varphi_\mu = (p+1)\alpha_\mu + a^+_\mu - a^-_\mu$:
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- charged massive vector, no scalar Higgs boson

- superconductivity: low-energy sector below the mass of the vectors $\omega_\mu^\pm$; include Maxwell action for the photon and the dynamical term for the gapless mode
$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{2}{2\pi} \varphi_\mu \epsilon^{\mu\nu\sigma} \partial_\nu A_\sigma - \frac{1}{4g^2} f_{\mu\nu} f^{\mu\nu}$
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Mutual Anyon Superconductivity

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- what is the origin of the statistical gauge field?

- ground state degeneracy on a torus for multi-component Chern-Simons terms $(1/4\pi)\mathbf{a}_\mu M \epsilon^{\mu\nu\sigma}\partial_\nu \mathbf{a}_\sigma$ governed by $|\det M| = K$ if M has only integer entries (Polychronakos)

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- formulate the model as a unique multicomponent Chern-Simons model: $\mathcal{L} = \frac{1}{4\pi}\mathcal{J}_\mu^T Q \, \epsilon^{\mu\nu\sigma}\partial_\nu \mathcal{J}_\sigma$, with $\mathcal{J}_\mu = (A_\mu, a_\mu, b_\mu, -\alpha_\mu)$

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- $Q = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & p+1 & 1 \\ 2 & p+1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ all entries of Q are integer

  and $(\det\ \mathrm{Q}) = 4 \Rightarrow$ topological order with degeneracy parameter $K = 4$

"Conventional" superconductors

- BCS: gap arises from SSB $U(1) \to Z_N$, Z_2 for Cooper pairs:
  - residual Aharonov-Bohm interaction between charges and vortices (Bais et al.) $\Rightarrow$ effective BF theory with $k = N$; $k = 2$ for Cooper pairs ($k^2 = 4$ on the torus (Hansson et al.))

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- how can we distinguish?

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- Q_1 is closed $\Rightarrow$ $Q_1 = d\alpha_{d-1}$, summation over Q_1 $\rightarrow$ summation over α_{d-1} (gauge volume due to the additional symmetry $\alpha_{d-1} \rightarrow \alpha_{d-1} + \lambda_{d-1}$ subtracted)

- $S = \frac{k}{2\pi}\int_{M_{d+1}}(a_1 \wedge db_{d-1} + a_1 \wedge d\alpha_{d-1})$ generalization to any dimensions of the mutual CS for $p = 0$:
  - scalar gapless mode $\varphi_{d-1} = b_{d-1} - \alpha_{d-1}$
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- due to BF mass term Q_1 defects have short-range interaction
- the origin of the fictitious emergent gauge field lies in the condensation of topological defects, no Higgs boson, we call this superconductivity model "topological"

3d

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  - $b_{\mu\nu}$ and the fictitious gauge field $\alpha_{\mu\nu}$ are Kalb-Ramond antisymmetric (two-form) gauge field;
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- $\alpha_{\mu\nu}$ provides a uniform Kalb-Ramond emergent charge $F^0 = \epsilon^{ijk} \partial_i \alpha_{jk}$ attached to physical charges by the equation of motion for the gauge field a_μ, $j^0 = -\frac{1}{2\pi} F^0$

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- what drive the phase transition from a topological insulator into a Higgsless superconductor?
  - 2d: the dynamical terms for the emergent gauge fields are infrared irrelevant $\Rightarrow$ physics determined by the topological terms;
  - 3d: Maxwell term is marginal and must be included in the action
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  - cutoff theory valid only in the perturbative scaling regime (masses of irrelevant operators beyond the ultraviolet cutoff) otherwise we can have non-perturbative effects, like phase transitions
- $S_{\rm top.ins.} = \int d^4x \left[\frac{1}{4e^2} f_{\mu\nu} f_{\mu\nu} - \frac{i}{2\pi} a_\mu \epsilon^{\mu\nu\sigma\rho} \partial_\nu b_{\sigma\rho} + \right.$ $\left. + \frac{1}{12g^2} h_{\mu\nu\alpha} h_{\mu\nu\alpha} - \frac{i}{2\pi} a_\mu Q_\mu \right]$
  $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \; , h_{\mu\nu\sigma} = \partial_\mu b_{\nu\sigma} + \partial_\nu b_{\sigma\mu} + \partial_\sigma b_{\mu\nu}$

- f_μ = dual Kalb-Ramond field strength $= (1/6)\epsilon_{\mu\nu\sigma\rho}\partial_\nu b_{\sigma\rho} = \pi j_\mu$ conserved charge current $\Rightarrow$ Kalb-Ramond dynamical term = charge-charge interaction with strength g

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- integrate out the gauge fields a_μ and $b_{\mu\nu}$; effective action for the topological excitations Q_μ (l lattice spacing):

$$Z_{\rm top} = \sum_{\{Q_\mu\}} \exp\left(-S_{\rm top}\right)$$

$$S_{\rm top} = \sum_{\mathbf{x}} \frac{e^2}{2l^2}\, Q_\mu \frac{\delta_{\mu\nu}}{m^2 - \nabla^2} Q_\nu \quad , m = eg/\pi$$

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- when the range of the screened Coulomb interaction becomes smaller than a critical value $(\sqrt{2\mu}/e)l$ topological insulators develop a transition to Higgsless topological superconductor

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• Dynamical statistical gauge field	
• isolated gapless mode	isolated gapless mode
• anyon: charge+ flux	intrinsic + emergent charge

Anyon superconductivity:	Higgsless superconductivity:
Statistical gauge field	Emergent gauge field
vector	peseudovector
fluxes + charges	emergent + intrinsic charges
Average field approximation	
quantum Hall state	topological insulator
gapped, PT breaking	gapped, PT invariant
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