XY Model revisted

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A simples case: 2 sites

- Let us consider a simple system of two qubits with the XY type interaction[1]. The Hamiltonian is

$$H(h,\gamma) = -\frac{(1+\gamma)}{2}\sigma_1^x\sigma_2^x - \frac{(1-\gamma)}{2}\sigma_1^y\sigma_2^y - \frac{h}{2}(\sigma_1^z + \sigma_2^z). \qquad (1)$$

- γ - anisotropy factor
- h - external magnetic field in the z direction.
- σ_i^α - Pauli matrices of the i-th qubit, with $\alpha = x, y, z$.
- Simple example of the general XY model, for N qubits

$$H = -\sum_{i=1}^{N}\left[\left(\frac{1+\gamma}{2}\right)\sigma_i^x\sigma_{i+1}^x + \left(\frac{1-\gamma}{2}\right)\sigma_i^y\sigma_{i+1}^y + h\sigma_i^z\right], \qquad (2)$$

[1] 2009,Oh Physics Letters A 373 (2009) 644-647

In the matrix form, we have

$$H(h,\gamma) = -\begin{pmatrix} h & 0 & 0 & \gamma \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & 0 & -h \end{pmatrix} = H_{even} + H_{odd} \tag{3}$$

The basis chosen is
$|00\rangle, |01\rangle = \sigma_2^+ |00\rangle, |10\rangle = \sigma_1^+ |00\rangle, |11\rangle = \sigma_1^+ \sigma_2^+ |00\rangle$, where

- 0 means empty,
- 1 means occupied.

H_{even}

$$H_{even} = \begin{pmatrix} h & \gamma \\ \gamma & -h \end{pmatrix} \tag{4}$$

- Subspace spanned by $|00\rangle, |11\rangle$
- Eigenstates

$$E_{\pm}^{\theta} = \pm\sqrt{\gamma^2 + h^2} \tag{5}$$

$$\left|E_{-}^{\theta}\right\rangle = \cos\frac{\theta}{2}|00\rangle + \sin\frac{\theta}{2}|11\rangle \tag{6}$$

$$\left|E_{+}^{\theta}\right\rangle = -\sin\frac{\theta}{2}|00\rangle + \cos\frac{\theta}{2}|11\rangle \tag{7}$$

where $\tan\theta = \gamma/h$

H_{odd}

$$H_{odd} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{8}$$

- Subspace spanned by $|01\rangle, |10\rangle$
- Eigenstates

$$E^0_\pm = \pm 1 \tag{9}$$

$$\left|E^0_\pm\right\rangle = \frac{1}{\sqrt{2}}(|01\rangle \mp |10\rangle) \tag{10}$$

Level Crossing

Note that when $\gamma^2 + h^2 = 1$, there is a level crossing

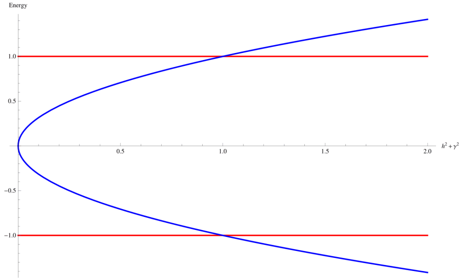

Figure : Level-crossing

1. What happen if we generalize to N sites, is there still a level crossing?
2. Which observables can measure this level crossing?[2]

[2] A. P. Balachandran, T. R. Govindarajan, Amilcar R. de Queiroz, A. F. Reyes-Lega, PRL110,080503 (2013)

Looking at XY model

- Spin chain with N sites.
- a: distance between the sites.
- $L = Na$: length of the chain.
- Periodic boundary conditions: $N + j \equiv j$, where $j = 1, ..., N$ are the label of the sites.

Writing again the Hamiltonian[3]

$$H(h, \gamma) = -\frac{1}{2} \sum_{j=1}^{N} \left[\left(\frac{1+\gamma}{2} \right) \sigma_j^x \sigma_{j+1}^x + \left(\frac{1-\gamma}{2} \right) \sigma_j^y \sigma_{j+1}^y + h \sigma_j^z \right].$$

[3] F. Franchini, *Notes on Bethe Ansatz Techniques*, Trieste, Italy, 2011.

Phase Diagram

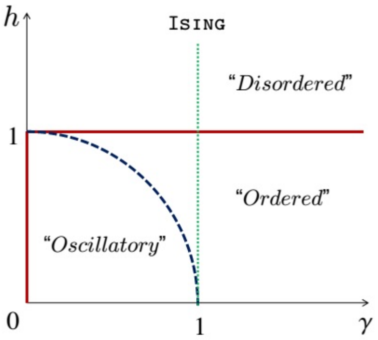

Figure : Phase diagram for ferromagnectic XY spin chain[4]

- $\gamma = 0$ - XX model,
- $\gamma = 1$ - Ising model.

[4] F. Franchini, *Notes on Bethe Ansatz Techniques*, Trieste, Italy, 2011.

Second quantization picture

After a Jordan-Wigner transformation, we arrive to a second quantized version of the XY Hamiltonian

$$H_{bulk} = \frac{-1}{2}\sum_{j=1}^{N-1}\Big(\psi_j^\dagger\psi_{j+1} + \psi_{j+1}^\dagger\psi_j + \gamma(\psi_j^\dagger\psi_{j+1}^\dagger + \psi_{j+1}\psi_j) \\ -2h\psi_j^\dagger\psi_j\Big) - \frac{hN}{2} \qquad (11)$$

$$H_{boundary} = \frac{T_N}{2}(\psi_N^\dagger\psi_1 + \psi_1^\dagger\psi_N + \gamma(\psi_N^\dagger\psi_1^\dagger + \psi_1\psi_N) - 2h\psi_N^\dagger\psi_N) \qquad (12)$$

where

$$T_N = \prod_{i=1}^{N}\sigma_i^z \qquad (13)$$

The T_N operator and boundary conditions

- From Jordan-Wigner, one can see there is a relation between spin in z-direction and number of fermions.

$$\sigma^z = 1 - 2\psi^\dagger\psi \tag{14}$$

- For $\gamma \neq 0 \longrightarrow [H, \sigma^z] \neq 0$ - Number of fermions is not conserved.
- Since fermions are created in pair, the 'parity' of number of fermions is conserved.
- T_N is the operator that counts such parity

$$T_N = (-1)^{Q_N}, \quad Q_N = \sum_{i=1}^{N} \psi_i^\dagger \psi_i \equiv \text{Number of fermions} \tag{15}$$

- $T_N = +1$ if Q_N even, $T_N = -1$ if Q_N odd

- The sign of the boundary term of the Hamiltonian (**??**) varies depending on the number of fermions.
- In order to correct the sign of the Hamiltonian, we have to impose differents boundaries conditions for each sector
- For the even sector
$$\psi_{j+N} = -\psi_j \tag{16}$$
- For the odd sector
$$\psi_{j+N} = \psi_j \tag{17}$$
- Since $T_N^2 = 1$, one can build up a projector and separate the theory in two sectors.
$$H = \frac{1+T_N}{2} H_{even} + \frac{1-T_N}{2} H_{odd}. \tag{18}$$
- H_{even} is H with antiperiodic boundary conditions.
- H_{even} is H with periodic boundary conditions.

Diagonalizing the Hamiltonian

One can write ψ e $\psi^\dagger$ in normal modes using the discrete Fourier transform and applying the correct boundary conditions to each sector

$$\psi_j = \frac{1}{\sqrt{aN}} \sum_n e^{ik_n ja}\ b_{k_n}, \tag{19}$$

$$-\frac{N-1}{2} \leq n \leq \frac{N-1}{2}$$

$$\psi_j^\dagger = \frac{1}{\sqrt{aN}} \sum_n e^{-ik_n ja}\ b_{k_n}^\dagger. \tag{20}$$

where

$$k_n = \frac{2\pi}{L}\left(n + \frac{P}{2}\right), \tag{21}$$

and

$$P = 1,\ \text{antiperiodic boundary conditions,}$$
$$P = 0,\ \text{periodic boundary conditions.}$$

The inverse transformation is

$$b_n \equiv b_{k_n} = \sqrt{\frac{a}{N}} \sum_{j=1}^{N} e^{-ik_n ja} \ \psi_j, \tag{22}$$

$$b_n^\dagger \equiv b_{k_n}^\dagger = \sqrt{\frac{a}{N}} \sum_{j=1}^{N} e^{ik_n ja} \ \psi_j^\dagger. \tag{23}$$

The anticommutation relations for b

$$\{b_n, b_m\} = \{b_n^\dagger, b_m^\dagger\} = 0, \tag{24}$$

$$\{b_n, b_m^\dagger\} = a \ \delta_{n,m}. \tag{25}$$

Defining

- $b_{-n} \equiv d_n^\dagger$: creates an antiparticle with momentum k_n.
- $b_{-n}^\dagger \equiv d_n$: destroys an antiparticle with momentum k_n.

$$
\begin{aligned}
H = &-\frac{hN}{2} \\
&-\frac{1}{2a}\left[(\cos(k_0 a) - h)(b^\dagger_{k_0} b_{k_0} + d^\dagger_{k_0} d_{k_0}) + i\gamma \sin(k_0 a)(b^\dagger_{k_0} d_{k_0} + b_{k_0} d^\dagger_{k_0})\right] \\
&-\frac{1}{a}\sum_{n=1}^{\frac{(N-1)}{2}}\left[(\cos(k_n a) - h)(b^\dagger_n b_n + d_n d^\dagger_n) + i\gamma \sin(k_n a)(b^\dagger_n d_n + b_n d^\dagger_n)\right]
\end{aligned}
$$

- The hamiltoninan has three important terms

$$
H = \mu N + H_{k_0} + \widetilde{H}. \tag{26}
$$

- $\mu N \equiv -hN/2$ is a chemical potential.
- The H_{k_0} is a special case. For periodic boundary conditions it represents a zero mode.$(k_0 = 0)$

Diagonalizing - Bogoliubov transformation

- Because $\gamma \neq 0$ there is a term mixing b's e d's operators.
- Need to introduce new operators that are linear combination of b's and d's $\Longrightarrow$ Bogoliubov transformation.
- It takes Ψ_n to $\Phi_n^\dagger \equiv \begin{pmatrix} \chi_n^\dagger & \eta_n^\dagger \end{pmatrix}$, eigenvectors of $\mathcal{H}_n$.

$$\begin{pmatrix} b_n \\ d_n \end{pmatrix} = \begin{pmatrix} \cos\nu_n & \sin\nu_n \\ -\sin\nu_n & \cos\nu_n \end{pmatrix} \begin{pmatrix} \chi_n \\ \eta_n \end{pmatrix}, \tag{27}$$

with ν_n given by

$$\tan 2\nu_n = \frac{\sin(k_n a)}{h - \cos(k_n a)}. \tag{28}$$

- The Hamiltonian now becomes

$$\widetilde{H} = -\frac{1}{a} \sum_{n=1}^{\frac{(N-1)}{2}} \Psi_n^\dagger \mathcal{H}_n \Psi_n, \qquad \Psi_n^\dagger = \begin{pmatrix} b_n^\dagger & d_n^\dagger \end{pmatrix}$$

$$\mathcal{H}_n = \begin{pmatrix} \cos(k_n a) - h & i\gamma \sin(k_n a) \\ -i\gamma \sin(k_n a) & -(\cos(k_n a) - h) \end{pmatrix}. \tag{29}$$

- The eigenvalue of $\mathcal{H}_n$ are

$$\epsilon(k_n) = \sqrt{(h - \cos(k_n a))^2 + \gamma^2 \sin^2(k_n a)} \tag{30}$$

$k_0 \neq 0$: even sector

- The hamiltonian

$$H^+ = \frac{\epsilon(k_0)}{2a}(\chi^\dagger_{k_0}\chi_{k_0} + \eta_{k_0}\eta^\dagger_{k_0}) + \frac{1}{a}\sum_{n=1}^{\frac{N-1}{2}} \epsilon(k_n)\left\{\chi^\dagger_n\chi_n + \eta_n\eta^\dagger_n\right\}, \quad (31)$$

  with $P = 1$ in k_n.

- *There is no zero mode*.
- Indeed, $\chi^\dagger_n$ creates a *quasiparticle* state.
- The ground state is

$$\chi_n \left|GS\right\rangle_+ = 0 \quad \eta^\dagger_n \left|GS\right\rangle_+ = 0, \quad n = 0, ..., (N-1)/2. \quad (32)$$

$k_0 = 0$: odd sector

- The hamiltonian

$$H^- = -\frac{1}{a}(1-h)b_0^\dagger b_0 - \frac{1}{a}\sum_{n=1}^{\frac{(N-1)}{2}} \epsilon(k_n)\left\{\chi_n^\dagger \chi_n + \eta_n \eta_n^\dagger\right\}, \qquad (33)$$

with $P = 0$ in k_n.

- *There is* a zero mode!

$$k_0 = \frac{2\pi 0}{L} = 0. \qquad (34)$$

- The rotation angle in the Bogoliubov transformation

$$\tan 2\nu_n = \frac{\sin k_n a}{h - \cos k_n a}, \qquad (35)$$

is zero, because $n = 0$ implies $\nu_0 = 0 \Rightarrow$ the zero mode is *invariant* under Bogoliubov transformation,

$$\chi_0 = b_0. \qquad (36)$$

The zero mode energy term implies two regimes:

1. When $h < 1$, weak external field

$$-(1-h) = \epsilon(0) \;\Rightarrow\; \text{sign}\,\epsilon(0) = \text{sign}\,\epsilon(k_n), \tag{37}$$

$$H^- = \frac{1}{a}\epsilon(0)\chi_0^\dagger\chi_0 + \frac{1}{a}\sum_{n=1}^{(N-1)/2}\epsilon(k_n)\left\{\chi_n^\dagger\chi_n + \eta_n\eta_n^\dagger\right\}, \tag{38}$$

2. When $h > 1$, strong external field

$$-(1-h) = -\epsilon(0) \;\Rightarrow\; \text{sign}\,\epsilon(0) = -\,\text{sign}\,\epsilon(k_n), \tag{39}$$

$$H^- = -\frac{1}{a}\epsilon(0)\chi_0^\dagger\chi_0 + \frac{1}{a}\sum_{n=1}^{(N-1)/2}\epsilon(k_n)\left\{\chi_n^\dagger\chi_n + \eta_n\eta_n^\dagger\right\}. \tag{40}$$

- The ground state should be the one which

$$\chi_n \left| GS' \right\rangle_- = 0 \quad \eta_n^\dagger \left| GS' \right\rangle_- = 0, \quad n = 1, ..., (N-1)/2, \tag{41}$$

  but this state is forbidden by the constrain in the sector: the number of excitations must be odd!
- The true ground state of this sector is

$$\left| GS \right\rangle_- = \chi_0^\dagger \left| GS' \right\rangle_- = b_0^\dagger \left| GS' \right\rangle_- . \tag{42}$$

Sketch

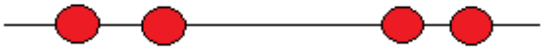

Figure : Particles created from $|GS\rangle$: Even sector

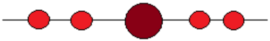

Figure : Particles created from $\chi_0^\dagger |GS\rangle$: Odd sector

Ground state energy in the termodynamic limit

- The ground state energy of the even sector is

$$E_0^+ = \sum_{n=0}^{(N-1)/2} \epsilon\left[\frac{2\pi}{N}\left(n+\frac{1}{2}\right)\right] \tag{43}$$

- The ground state energy of the odd sector for $h < 1$ is

$$E_0^- = -(1-h) + \sum_{n=0}^{(N-1)/2} \epsilon(k_n) = \sum_{n=0}^{(N-1)/2} \epsilon\left(\frac{2\pi n}{N}\right) \tag{44}$$

- The difference between ground state energies is

$$E_0^+ - E_0^- = O\left(\frac{1}{N}\right) \to 0,\ \ N \to \infty. \tag{45}$$

- $E_0^+ = E_0^-$, for $h < 1$, in the thermodynamic limit.
- The spectrum is degenerate for $h < 1$, but this degenerancy has a thermodynamic origin.

Ground state energy plotted

There is a level crossing!

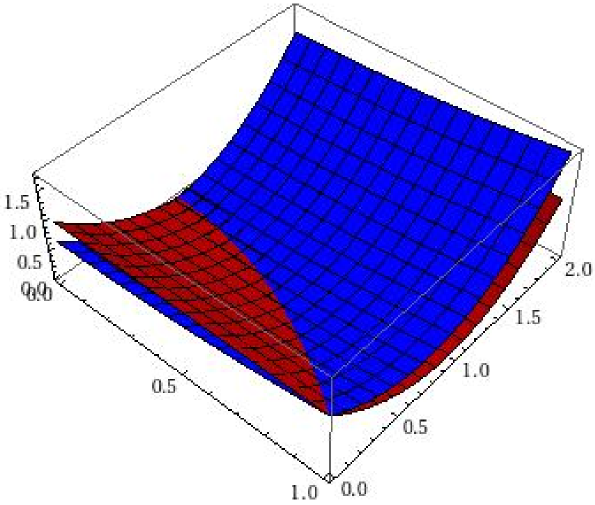

Figure : 3d plot of energy $\times h \times \gamma$

Ground state energy plotted

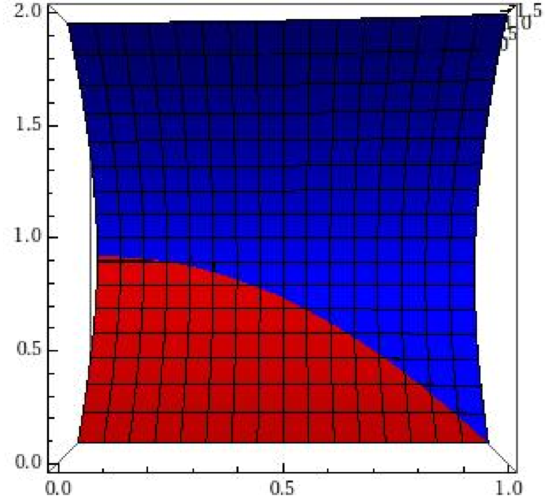

Figure : 3d plot of energy$\times h \times \gamma$

Ground state energy plotted

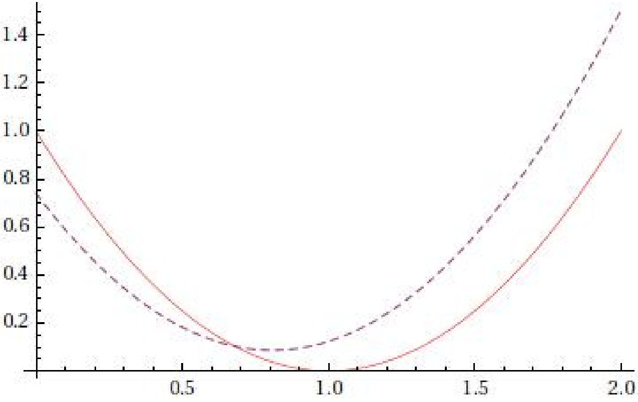

Figure : Energy versus h

Others boundary conditions

Now, let's analyze other possibles boundary conditions. One can choose, for each sector

$$\psi_{j+N} = \pm e^{i\theta}\psi_j, \tag{46}$$

a quasiperiodic boundary condition. Using Bloch's theorem, ψ_j can be written as

$$\psi_j = e^{i\theta j/N}\phi_j, \tag{47}$$

where $\phi_{j+N} = \pm\phi_j$, depending on the sector. Indeed,

$$\psi_j = e^{i\theta(j+N)/N}\phi_{j+N} = \pm e^{i\theta}(e^{i\theta j/N}\phi_j) = \pm e^{i\theta}\psi_j \tag{48}$$

We can substitute the operators ϕ_i on the fermionic Hamiltonian, and get

$$H = -\frac{1}{2}\sum_{j=1}^{N}\Big[\Big(e^{i\theta/N}\phi_j^\dagger\phi_{j+1} + e^{-i\theta/N}\phi_{j+1}^\dagger\phi_j\Big) + \gamma\Big(e^{-i(2j+1)\theta/N}\phi_j^\dagger\phi_{j+1}^\dagger + e^{i(2j+1)\theta/N}\phi_{j+1}\phi_j\Big) - 2h\phi_j^\dagger\phi_j\Big] = H_\theta \tag{49}$$

Note that we still have $[T_N, H_\theta] = 0$, therefore the equivalence below is valid

$$(H_0, bc_\theta) \quad \Leftrightarrow \quad (H_\theta, bc_0). \tag{50}$$

The ϕ operators obey a fermionic anticommutation relation.

Doing all the calculations done before, we see this 'quasiperiodic' boundary conditions leads to a redefinitions of the energy eigenvalues

$$\epsilon(\tilde{k}_n) = \sqrt{(h - \cos(k_n a + \theta/N))^2 + \gamma^2 \sin^2(k_n a + \theta/N)} \tag{51}$$

and of the Bogoliubov's angle

$$\tan 2\tilde{\nu}_n = \frac{\sin(k_n a + \theta/N)}{h - \cos(k_n a + \theta/N)}. \tag{52}$$

θ has the role of a flux in the chain and it can be tunned to measure the zero mode on the lab!

A more genreal case is to consider the boundary condition $\psi_{j+N} = e^{i\theta Q_N}\psi_j$, which will lead to the following dressed operators

$$\psi_j = e^{i\theta Q_{j-1}}\phi_j, \quad \psi_j^\dagger = e^{i\theta Q_{j-1}}\phi_j^\dagger. \tag{53}$$

Now, the anticommutations relations become different for $\phi_j, \phi_j^\dagger$

$$\phi_j^\dagger\phi_k + e^{i\epsilon\theta(N_j-1)}\phi_k\phi_j^\dagger = \delta_{jk} \tag{54}$$

$$\phi_j^\dagger\phi_k^\dagger + e^{i\epsilon\theta(N_j-1)}\phi_k^\dagger\phi_j^\dagger = \phi_j\phi_k + e^{i\epsilon\theta(N_j-1)}\phi_k\phi_j = 0 \tag{55}$$

$$\epsilon = \begin{cases} 1, & k > j \\ -1, & k < j \\ 0, & k = j \end{cases} \tag{56}$$

Now, the operators ϕ_i satisfy an "anyonic" anticommutation relation[5].

[5] M. D. Girardeau, PRL97,100402 (2006)

The anyonic Hamiltoninan then is

$$H = -\frac{1}{2}\sum_{j=1}^{N}\left[\left(\phi_j^\dagger \phi_{j+1} e^{i\theta N_j} + e^{-i\theta N_j}\phi_{j+1}^\dagger \phi_j\right)\right.$$

$$\left. +\gamma\left(e^{-2i\theta Q_{j-1}}\phi_j^\dagger \phi_{j+1}^\dagger e^{i\theta N_j} + e^{2i\theta Q_{j-1}} e^{-i\theta N_j}\phi_{j+1}\phi_j\right) - 2h\phi_j^\dagger \phi_j\right]$$

$$= H_\theta \tag{57}$$

Perspectives

- See these aspectcs in the point of view of correlation functions.
- We can play with differents boundaries conditions.
- Continuum limit: which is the QFT regularized by the XY model?[6]

[6] Manuel Asorey, Fernando Falceto, GermÃ¡n Sierra, Nuclear Physics B 622 [FS] (2002) 593â614