Entanglement of Identical Particles

Amílcar de Queiroz

IF - UnB (Brasília - Brasil)

DFT - UZ (Zaragoza - España)

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Quantum Engineering of States and Devices
Nordita
Stockholm - Sweden

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Leibnitz and Indistinguishability

My friend Prof. A. Polito (IF-UnB) works on the nature of space in Leibnitz. I learned from him about a key principle due to Leibnitz.

Identity of indiscernibles: two things are identical if and only if they have the very same (intrinsic) properties regardless of their distinct positions in space.

I think this principle seems to lead to:

- the conceptual distinction between internal (intrinsic) versus external (extrinsic) properties;
- the principle of symmetry; in particular, gauge symmetry.

A. Polito, C. Polito, "Relações entre física e filosofia: estudo do problema da natureza do espaço em Leibniz",

http://www.sbf1.sbfisica.org.br/eventos/snef/xx/programa/resumo.asp?insId=175&traId=1

The Gibbs Paradox

Statement:

- The mixing of two identical gases does not produce entropy: $\Delta S = 0$.
- The difference between the entropy of two separated distinct gases and the entropy of the mixing of these gases is

$$\begin{aligned} \Delta S &= S_A + S_B - S_{AB} \\ &= -N_A \log N_A - N_B \log N_B + (N_A + N_B) \log(N_A + N_B). \end{aligned}$$

- If $A = B$, then $\Delta S = 2N_A \ln 2$.

E. T. Jaynes, in Maximum Entropy and Bayesian Methods, Kluwer Academic, Dordrecht (1992). N. G. van Kampen, in Essays in Theoretical Physics: in honour of Dirk ter Haar, (1984). J. Leinaas and J. Myrheim, Il Nuovo Cimento B Series 11, 37 (1977).

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The Gibbs Paradox

Another statement:

- For ideal gases, $E = \frac{3}{2}NT + \text{cte.}$, $PV = NT$, then from $T\Delta S = \Delta E + P\Delta V$ one gets

$$S(P,T) = \frac{5}{2}N \log T - N \log P + C(N).$$

- The function $C(N)$ does not depend on P, T. But the way it is usually fixed as a full-constant leads to wrong results if not used consistently.

The resolution of the paradox: the counting of identical particle microstates is different from the counting of distinct particle microstates. A factor of $N!$ from the group of permutations S_N has to be accounted for.

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Configuration Space

- Set $\mathbb{R}^k, \quad k \geq 3$.
- Let $N \geq 2$ identical spinless particles.
- Let $x_j \in \mathbb{R}^k$ denote the position of the j-th particle.
- A configuration of such N-particle system is given by the unordered set

$$\tilde{q} = \{x_1, x_2, ..., x_N\} = \{x_2, x_1, ..., x_N\} \in \widetilde{Q}$$

If two particles cannot occupy the same position in space, then we must remove the diagonal $\Delta = \{x_i = x_j | i \neq j\}$. The configuration space is $Q = \widetilde{Q} \backslash \Delta$. Complicated topology.

J. Leinaas and J. Myrheim, Il Nuovo Cimento B Series 11, 37 (1977). A. P. Balachandran, G. Marmo, B. S. Skagerstam, A. Stern, Classical Topology and Quantum States. World Scientific (1991).

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Configuration Space: Topology

The fundamental group of Q is $\pi_1(Q) = S_N$, the group of permutation of N objects. This leads to fermion and boson statistics.

A transposition in $s_{ij} \in S_N$ may be represented by a loop $\gamma_{ij}(t)$ in $\widetilde{Q}$ that permutes particle x_i and x_j.

Note that if $\mathbb{R}^2$, then $\pi_1(Q) = B_N$, the group of braids. This leads to anyons or fractional statistics.

Hilbert Space

To quantize a system of identical particles, we would like to consider a Hilbert space on top of Q. This is hard to directly construct.

The best strategy is to consider a Hilbert space $\mathcal{H}$ on top of a simply-connected space $\overline{Q} \supseteq Q$.

Decompose $\mathcal{H}$ into irreducible representations of $\pi_1(Q) = S_N$

$$\mathcal{H} = \bigoplus_l \mathcal{H}^{(l)}. \tag{1}$$

Note the strong analogy to working with gauge symmetry.

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Spin-Statistics and Spin-Locality

O. Greenberg distinguishes between Spin-Statistics and Spin-Locality.

- Statistics: a field operator $\hat{\phi}$ may be decomposed as

$$\hat{\phi}(x) = \sum_k \varphi_k(x)\hat{a}_k \qquad \text{or} \qquad \hat{\phi}(x) = \sum_k \psi_k(x)\hat{b}_k,$$

  i.e., in terms of either bosons $[\hat{a}_k, \hat{a}_j] = 0$ or fermions $\{\hat{b}_k, \hat{b}_j\} = 0$.

- Locality: for space-like distance $(\underline{x} - \underline{y})^2 < 0$, a field may be
  - local iff $[\phi(\underline{x}), \phi(\underline{y})] = 0$,
  - anti-local iff $\{\phi(\underline{x}), \phi(\underline{y})\} = 0$.

O. Greenberg, Phys. Lett. B416 144-149 (1998). G. Lüders and B. Zumino, Phys.Rev. 110 1450-1453 (1958). I. Duck and E. C. G. Sudarshan, Am. J. Phys. 66, 284 (1998).

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Spin-Statistics and Spin-Locality

O. Greenberg expands an example due to Res Jost. He takes a neutral scalar field ϕ.

- If ϕ is expanded in terms of fermion operators, then the corresponding observables are non-local.
  - Observe that even anti-commutators of such fields are non-local $\{\phi(\underline{x}), \phi(\underline{y})\} = \Delta^{(1)}(\underline{x} - \underline{y})$.
  - The field ϕ still satisfies the CPT-theorem.
- If ϕ is anti-local, like in Lüders-Zumino proof of spin-statistics theorem, then $\phi \equiv 0$.

Entanglement Entropy

In 1986, Bombelli, Koul, Lee and Sorkin proposed and solved the following problem modeled after a black hole:

Consider a (scalar) field on a space-like hypersurface Σ. Integrate out the fields on a region $R \subset \Sigma$. What is the entropy emerging out from this process?

Solution: $S \propto \text{Volume of } \partial R$.

Incidentally, Srednicki solved the same problem later in 1993.

L. Bombelli, R. Koul, J. Lee and R. Sorkin, Phys. Rev., 1986, D34, 373-383. M. Srednicki, Phys. Rev. Lett., 1993, 71, 666-669.

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Entanglement Entropy (Formal)

Consider $\mathcal{H} = \mathcal{H}_R \otimes \mathcal{H}_L$. From a state vector $|\psi\rangle \in \mathcal{H}$, we form the (pure) density matrix $\rho : \mathcal{H} \to \mathcal{H}$ as

$$\rho = |\psi\rangle\langle\psi|.$$

The reduced density matrix is defined by the partial trace

$$\rho_R = \mathrm{Tr}_L\ \rho.$$

The entanglement (a.k.a. von Neumann) entropy is

$$S = -\mathrm{Tr}\ \rho_R \log \rho_R.$$

If $S \neq 0$, ρ_R is a mixed state, that is, ρ_R cannot be represented as a state vector in $\mathcal{H}_R$.

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Extension to Space-time Region

Recently, R. Sorkin found a formula for entanglement entropy associated with a space-time region R of some manifold M.

From the eigenvalues of the operator $iL = \Delta^{-1}W$ on R, with

- $W(x, x')$: Wightman function;
- $i\Delta(x, x') = W(x, x') - \overline{W}(x', x) = 2\ \mathrm{ImW(x, x')}$,

He obtained:

$$S = \mathrm{Tr}\ L \log |L| \equiv \sum_{\lambda} \lambda \log \lambda. \tag{2}$$

R. Sorkin, arXiv:1205.2953. R. Sorkin *et al.*, arXiv:1207.7101.

Cf. Peschel, I. "Calculation of reduced density matrices from correlation functions" J.Phys.A: Math.Gen., 2003, 36,, L205

Black Holes

In physics of black holes, we have

- Bekenstein-Hawking entropy associated with a black hole $S = A_h/4$;
- in some cases, we have a counting formula, BUT we do not know what are being counted.

Problems:

- What should be counted? In which conditions?
- Why should we care to count something into an entropy formula?
- What does we learn from such counting?

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General Comments

- Y. S. Li, B. Zeng, X. S. Liu and G. L. Long, Phys. Rev. A 64 054302 (2001). J. Schliemann, J. I. Cirac, M. Kus, M. Lewenstein and D. Loss, Phys. Rev. A 64 022303 (2001). K. Eckert, J. Schliemann, D. Brussand M. Lewenstein, Annals of Physics 299 88-127 (2002).: Initial papers.
- P. Zanardi, Phys. Rev. A65 042101 (2002).: Towards fermionic lattices and entanglement relativity.
- F. Benatti, R. Floreanini, K. Titimbo, arXiv:1403.3178.: First (particle-based) versus Second (mode-based) quantization. Use of statistics notions as in O. Greenberg (they seem unaware of this fact, though.).
- F. Benatti, R. Floreanini and U. Marzolino, Benatti, Phys. Rev. A, 2014, 89, arXiv: 1403.1144. N. Killoran, M. Cramer and M. B. Plenio, Phys. Rev. Lett. 112, 150501 (2014).: Is entanglement of identical particles useful? Experimental proposes. Quantum metrology.
- Dyakonov, M. Quantum computing: a view from the enemy camp, Optics and Spectroscopy 95 261-267 (2003), cond-mat/0110326.: It will be impossible to construct a quantum computer. It requires the control of 10^5 particles.

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Fermions in a Double-Well

Imagine a double-well or two quantum-dots. At each well L or R one finds electrons (qubits) with either spin $+$ or spin $-$. The 1-particle Hilbert space is spanned by

$$\{|L,+\rangle,\ |L,-\rangle,\ |R,+\rangle,\ |R,-\rangle\}\,.$$

- Suppose initially one electron at each well.
- Problem: trace out d.o.f. associated to R-well. Equivalently, allow only observables associated to L-well.
- What is the entropy emerging out of this process?

J. Schliemann, J. I. Cirac, M. Kus, M. Lewenstein and D. Loss, Phys. Rev. A 64 022303 (2001). P. Zanardi, Phys. Rev. A65 042101 (2002). F. Benatti, R. Floreanini, K. Titimbo, arXiv:1403.3178.

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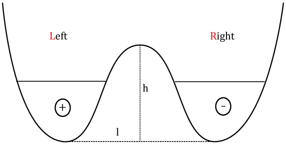
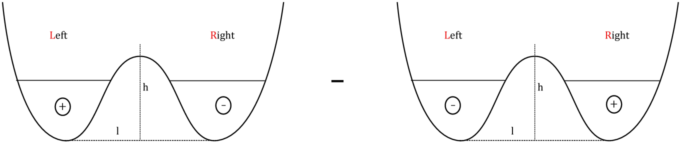
Identical Fermions



Wave-functions do not overlap.



Wave-functions overlap. Use Slater determinant.

K. Eckert, J. Schliemann, D. Brussand M. Lewenstein, Annals of Physics 299 88-127 (2002).

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Identical Fermion

In a N-tuple-well system, with $N > 2$, filled with fermions, a new question arises.

Can we trace out some wells respecting (fermion) statistics?

Most attempts based on extension of Schmidt decomposition, e.g. Slater-Schmidt decomposition. M. C. Tichy et al., J. Phys. B, **44**, 192001 (2011). K. Eckert, J. Schliemann, D. Bruss, and M. Lewenstein, Annals of Physics 299, 88 (2002).

Main problems:

- Non-natural values for entropy: e.g. $S \neq 0$ for separable cases.
- Non-universal criteria: different criteria for different statistics.
- Focus on bosons and fermions: e.g. no anyons.

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Identical Bosons

Imagine a two-photons system in a Bell-like state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|h\rangle \otimes |v\rangle + |v\rangle \otimes |h\rangle\Big), \tag{3}$$

with $|h\rangle$ and $|v\rangle$ standing for horizontal and vertical polarization.

- If the photons are distinguishable (e.g. different momenta though same frequency), then the Bell-like state is entangled.
- If the photons are indistinguishable, the Bell-like state seems to be separable.

Y. S. Li, B. Zeng, X. S. Liu and G. L. Long, Phys. Rev. A 64 054302 (2001).

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Restriction to Subalgebra

Partial trace fails to respect in a natural way correlations due to the indistinguishability (or identity) of particles.

There is an equivalent operation generalizing the notion of partial trace. Moreover it allows the treatment of entanglement of identical or non-identical particles on an equal footing:

Restriction of a state to a subalgebra.

A. P. Balachandran, T. R. Govindarajan, AQ and A. Reyes-Lega, Phys. Rev. Lett. 110 080503 (2013), arXiv:1205.2882. A. P. Balachandran, T. R. Govindarajan, AQ and A. Reyes-Lega, Phys.Rev. A88 022301 (2013), arXiv:1301.1300..

State on a Algebra of Observables

Instead of density matrix $\rho : \mathcal{H} \to \mathcal{H}$, we regard a state as a linear functional on the algebra of observables $\mathcal{A}$. Indeed, from the expectation value $\langle \mathcal{O} \rangle$ of the observable $\mathcal{O}$

$$\omega_\rho (\mathcal{O}) \equiv \langle \mathcal{O} \rangle = \mathrm{Tr}\ \rho \mathcal{O}, \tag{4}$$

we abstract the notion of

a state on an algebra of observable $\mathcal{A}$

as a linear funcional

$$\omega : \mathcal{A} \to \mathbb{C},$$

such that $\omega(\mathbb{1}) = 1$ and $\omega\left(\mathcal{O}^* \mathcal{O}\right) \geq 0$, for any $\mathcal{O} \in \mathcal{A}$.

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Restriction to Subalgebra

The initial data to describe a quantum system is therefore

$$(\mathcal{A}, \omega)\,.$$

Consider a subalgebra $\mathcal{A}_0 \subset \mathcal{A}$. Instead of partial trace, consider

$$\omega_0 \equiv \omega|_{\mathcal{A}_0}\,, \tag{5}$$

that is, the restriction of state ω on $\mathcal{A}$ to the subalgebra $\mathcal{A}_0$.

Therefore,

entanglement of a subalgebra $\mathcal{A}_0 \subset \mathcal{A}$ with the algebra $\mathcal{A}$ for a state ω.

Cf. with entanglement relativity due to P. Zanardi. P. Zanardi, Phys. Rev. Lett. 87 077901 (2001). P. Zanardi, Phys. Rev. A 65 042101 (2002).

GNS Construction: Hilbert Space

Gelfan'd-Naimark-Segal (GNS) construction gives a Hilbert space $\mathcal{H}_\omega$ out of $(\mathcal{A}, \omega)$ where $\mathcal{A}$ is represented on.

1. Inner product in $\mathcal{A}$ out of ω: $\langle \alpha | \beta \rangle \equiv \omega(\alpha^* \beta)$.
2. It may exist null states: $\mathcal{N} = \{0 \neq n \in \mathcal{A} \mid \langle n | n \rangle = 0\}$.
3. Removal of null states: $\mathcal{H}_\omega = \mathcal{A}/\mathcal{N}$, that is, set of classes of equivalence
$$\widetilde{\alpha} = \alpha + \mathcal{N}.$$
4. Action of observable $\alpha \in \mathcal{A}$ on $\mathcal{H}_\omega$: $\alpha | \widetilde{\beta} \rangle = | \widetilde{\alpha\beta} \rangle$.

Density Matrix

The vector $|\widetilde{\mathbb{1}}\rangle$, where $\widetilde{\mathbb{1}} = \mathbb{1} + \mathcal{N}$, is dubbed cyclic vector.

A dense subset of $\mathcal{H}_\omega$ may be generated by the action of all $\alpha \in \mathcal{A}$.

Also, from the density matrix

$$\rho_\omega = |\widetilde{\mathbb{1}}\rangle\langle\widetilde{\mathbb{1}}|, \tag{6}$$

we obtain

$$\omega(\alpha) = \mathrm{Tr}\left(\rho_\omega \alpha\right). \tag{7}$$

The Hilbert space $\mathcal{H}_\omega$ may be reducible w.r.t. $\mathcal{A}$, so that $\mathcal{H}_\omega = \bigoplus_i \mathcal{H}_i$. Thus, there exist projectors P^i, such that

$$|\widetilde{\mathbb{1}}\rangle = \sum_i P^i |\widetilde{\mathbb{1}}\rangle = \sum_i |\widetilde{P^i}\rangle \tag{8}$$

Density Matrix

A density matrix associated to state ω writes

$$\rho_\omega = \sum_i |\widetilde{P^i}\rangle\langle\widetilde{P^i}| = \sum_i \rho^i, \tag{9}$$

with corresponding entropy

$$S(\rho_\omega) = -\mathrm{Tr}\ \rho_\omega \log \rho_\omega. \tag{10}$$

Equivalently, set normalized rank-1 density matrices

$$\hat{\rho}^i = \frac{1}{\lambda_i}\rho^i, \qquad \lambda^i = \omega(P^i), \tag{11}$$

so that

$$S(\rho_\omega) = -\sum_i \lambda_i \log \lambda_i. \tag{12}$$

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$M_2(\mathbb{C})$: 2×2 Matrices

A general element $a \in M_2(\mathbb{C})$ expands as

$$a = \sum_{i,j=1,2} a_{ij}|i\rangle\langle j| \equiv \sum_{ij} a_{ij} e_{ij} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{13}$$

Set a state on the algebra $M_2(\mathbb{C})$ as

$$\omega_\lambda(a) = \lambda a_{11} + (1-\lambda) a_{22}, \qquad 0 \leq \lambda \leq 1. \tag{14}$$

Observe,

$$\omega_\lambda(\mathbb{1}) = 1 \tag{15}$$

$$\omega_\lambda(a^\dagger a) = \sum_k \left(\lambda |a_{k1}|^2 + (1-\lambda)|a_{k2}|^2\right) \geq 0. \tag{16}$$

$M_2(\mathbb{C})$. The $\lambda = 0$ case.

1. Inner product in $\mathcal{A}$ out of ω: $\langle a|b\rangle = \omega_0(a^\dagger b) = \sum_k \bar{a}_{k2} b_{k2}$, $a, b \in \mathcal{A} \equiv \mathbb{C}^4$.

2. Null states: solutions $0 \neq a \in \mathcal{A}$ of $\omega_0(a^\dagger a) = 0$ are spanned by a_{k1}. Thus, $\begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix} \in \mathcal{N}_{\omega_0} \equiv \mathbb{C}^2$.

3. The GNS Hilbert space $\mathcal{H}_{\omega_0} = \mathcal{A}/\mathcal{N}_{\omega_0} \equiv \mathbb{C}^2$ spanned by vectors
$$|\widetilde{e}_{k2}\rangle = |e_{k2} + \mathcal{N}_{\omega_0}\rangle.$$

4. Action of $\mathcal{A}$ on $\mathcal{H}_{\omega_0}$: $a|\widetilde{b}\rangle = |\widetilde{ab}\rangle$.

5. $|1\rangle = |\widetilde{e}_{22}\rangle$. The GNS Hilbert space is irreducible w.r.t. $\mathcal{A}$. The density matrix is $\rho_{\omega_0} = |\widetilde{e}_{22}\rangle\langle\widetilde{e}_{22}|$. It has entropy *zero*.

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$M_2(\mathbb{C})$. The $\lambda \neq 0, 1$ case.

1. Inner product on $\mathcal{A}$ out of ω: for $a, b \in \mathcal{A}$,

$$\langle a|b\rangle = \omega_\lambda(a^\dagger b) = \sum_k \left(\lambda \bar{a}_{k1} b_{k1} + (1-\lambda)\bar{a}_{k2} b_{k2}\right).$$

2. There is no non-trivial null states.
3. The GNS Hilbert space is $\mathcal{H}_{\omega_\lambda} = \mathcal{A} \equiv \mathbb{C}^4$.
4. The action of $a \in \mathcal{A}$ on $|b\rangle \in \mathcal{H}_{\omega_\lambda}$ is $a|b\rangle = |ab\rangle$. Now, there exist subspaces $\mathcal{H}_i \equiv \mathbb{C}^2$, $i = 1, 2$, such that

$$a \cdot \mathcal{H}_i \subseteq \mathcal{H}_i \qquad \forall a \in \mathcal{A},$$

$$\mathcal{H}_{\omega_\lambda} = \mathcal{H}_1 \oplus \mathcal{H}_2.$$

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Entanglement Entropy

The Hilbert space $\mathcal{H}_\omega$ may be reducible w.r.t. representations of $\mathcal{A}_0$:

$$\mathcal{H}_\omega = \bigoplus_i \mathcal{H}_i. \quad (17)$$

Set $P_i : \mathcal{H}_\omega \to \mathcal{H}_i$ as orthogonal projectors. Then

$$|\widetilde{\mathbb{1}}_{\mathcal{A}}\rangle = \sum_i P_i |\widetilde{\mathbb{1}}_{\mathcal{A}}\rangle, \quad (18)$$

$$\mu_i = ||P_i|\widetilde{\mathbb{1}}_{\mathcal{A}}\rangle||. \quad (19)$$

The entanglement entropy is

$$S(\omega, \mathcal{A}_0) = -\sum_i \mu_i^2 \log \mu_i^2. \quad (20)$$

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Identical Particles

- One-particle Hilbert space: $\mathcal{H}^{(1)} = \mathbb{C}^d$. The group $U(d)$ acts naturally on $\mathbb{C}^d$.
- Algebra of observables: $M_d(\mathbb{C}) \cong \mathbb{C}U(d)$.
- k-particle Hilbert space: $\mathcal{H}^{(k)} = \bigotimes_{A,S}^k \mathcal{H}^{(1)}$.
- For $g \in U(d)$, the Coproduct $\Delta(g) = g \otimes g$ allows one to represent one-particle observables in the k-particle sector. Furthermore, it takes care of statistics.
  - Cf. Addition of angular momentum.
- Anyons or other statistics can be considered, as well. A. P. Balachandran, T. R. Govindarajan, AQ and A. Reyes-Lega, Phys.Rev. A88 022301 (2013), arXiv:1301.1300.

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Identical Particles

There are two main types of subalgebras we may be interested in a many-particles system:

1. The subalgebra of one-particle observables.
2. A subalgebra of partial one-particle observables. For instance, only spin or only position degrees of freedom.

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Two-fermions, $\mathcal{H}^{(1)} = \mathbb{C}^3$

- Basis of $\mathcal{H}^{(1)} = \mathbb{C}^3$: $\{|e_i\rangle, \quad i = 1,2,3\}$.
- Basis of $\mathcal{H}^{(2)} = \mathbb{C}^3 \wedge \mathbb{C}^3$: $\left\{|f^i\rangle = \epsilon^{ijk}|e_j \wedge e_k\rangle, \quad i = 1,2,3\right\}$.
- 2-particle algebra of observables: $\mathcal{A}^{(2)} \cong M_3(\mathbb{C})$. Indeed, $\mathcal{A}^{(1)} = U(3)$ and $3 \otimes 3 = 6 \oplus \bar{3}$.

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Two-fermions, $\mathcal{H}^{(1)} = \mathbb{C}^3$; Case 1: $\mathcal{A}_0 = \mathcal{A}^{(1)} \subset \mathcal{A}^{(2)}$

For any $|\psi\rangle \in \mathcal{H}^{(2)}$, the pure state $\omega_\psi = |\psi\rangle\langle\psi|$ restricted to $\mathcal{A}_0 = \mathcal{A}^{(1)} \subset \mathcal{A}^{(2)}$ gives zero entropy.

Indeed, the $U(3)$ representation $\bar{3}$ is irreducible.

However the entropy computed by partial trace is equal to $\log 2$ for any $|\psi\rangle$! G. Ghirardi and L. Marinatto, Phys. Rev. A, **70**, 012109 (2004).

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Two-fermions, $\mathcal{H}^{(1)} = \mathbb{C}^3$; Case 2: Partial Observations

- $\mathcal{A}_0 \subset \mathcal{A}^{(2)}$ generated by

$$M^{ij} = |f^i\rangle\langle f^j|, \quad i,j = 1,2, \quad \text{and} \quad \mathbb{1}_{3\times 3}.$$

- (Pure) states on $\mathcal{A}^{(2)}$:

$$\omega_\theta = |\psi_\theta\rangle\langle\psi_\theta|, \quad \text{with} \tag{21}$$

$$|\psi_\theta\rangle = \cos\theta|f^1\rangle + \sin\theta|f^2\rangle. \tag{22}$$

Two-fermions, $\mathcal{H}^{(1)} = \mathbb{C}^3$; Case 2: Partial Observations

We obtain the following results:

1. $0 < \theta < \frac{\pi}{2}$: The null vector space is trivial and

$$S(\theta) = -\cos^2\theta \log\cos^2\theta - \sin^2\theta \log\sin^2\theta \qquad (23)$$

2. $\theta = 0$: Null vector space is non-trivial, $\dim \mathcal{H} = 2$ and entropy is zero.

3. $\theta = \frac{\pi}{2}$: Null vector space is non-trivial, $\dim \mathcal{H} = 1$ and entropy is zero.

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Double-Well, $\mathcal{H}^{(1)} = \mathbb{C}^4$

- $\mathcal{H}^{(1)}$ for a fermion with two external d.o.f. (position) and two internal d.o.f. (spin):

$$a_\sigma, a_\sigma^\dagger : \text{ “left” with spin } \sigma = +, - \tag{24}$$

$$b_\sigma, b_\sigma^\dagger : \text{ “right” with spin } \sigma = +, - \tag{25}$$

- $\mathcal{H}^{(2)}$ spanned by $a_+^\dagger a_-^\dagger |\Omega\rangle$, $b_+^\dagger b_-^\dagger |\Omega\rangle$, $a_\sigma^\dagger b_\rho^\dagger |\Omega\rangle$, where $|\Omega\rangle$ is the vacuum.

- 2-particles algebra of observables $\mathcal{A} = M_6(\mathbb{C})$.

Double-Well, $\mathcal{H}^{(1)} = \mathbb{C}^4$

- Consider the family of (pure) states $\omega_\theta = |\psi_\theta\rangle\langle\psi_\theta|$ on $\mathcal{A}$ with

$$|\psi_\theta\rangle = \left(\cos\theta\ a_+^\dagger b_-^\dagger + \sin\theta\ a_-^\dagger b_+^\dagger\right)|\Omega\rangle \qquad (26)$$

- Subalgebra $\mathcal{A}_0$: one-particle observations at the left position. It is generated by

$$\begin{aligned}
&\mathbb{1}_{\mathcal{A}},\\
&n_{+-} = a_+^\dagger a_+ a_-^\dagger a_-,\\
&N_a = a_+^\dagger a_+ + a_-^\dagger a_-,\\
&T_i = \frac{1}{2} a_\sigma^\dagger \left(\sigma_i\right)^{\sigma\sigma'} a_{\sigma'}
\end{aligned}$$

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Two Fermions, $\mathcal{H}^{(1)} = \mathbb{C}^4$

We obtain the following results:

1. $0 < \theta < \frac{\pi}{2}$: non-trivial null vector space, $\dim \mathcal{H}_\theta = 4$ and

$$S(\theta) = -\cos^2\theta \ \log\cos^2\theta - \sin^2\theta \ \log\sin^2\theta.$$

2. $\theta = 0$: non-trivial null vector space, $\dim \mathcal{H}_\theta = 2$, entropy is zero.

3. $\theta = \frac{\pi}{2}$: non-trivial null vector space, $\dim \mathcal{H}_\theta = 2$, entropy is zero.

The last two cases should be contrasted with $S = \log 2$ found in the literature. M. C. Tichy *et al.*, J. Phys. B, **44**, 192001 (2011).

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Two Bosons, $\mathcal{H}^{(1)} = \mathbb{C}^3$

- Basis of $\mathcal{H}^{(1)} = \mathbb{C}^3$: $\{|e_i\rangle, \quad i = 1, 2, 3\}$.

- Basis of $\mathcal{H}^{(2)} = \mathbb{C}^3 \vee \mathbb{C}^3$:

$$|e_i \vee e_j\rangle = \begin{cases} \frac{1}{\sqrt{2}}\Big(|e_i\rangle \otimes |e_j\rangle + |e_j\rangle \otimes |e_i\rangle\Big), & i \neq j, \\ |e_i\rangle \otimes |e_i\rangle & \end{cases}$$

- 2-particle algebra of observables: $\mathcal{A}^{(2)} \cong M_6(\mathbb{C})$. Indeed, $\mathcal{A}^{(1)} = U(3)$ and $3 \otimes 3 = 6 \oplus \bar{3}$.

- A pure state on $\mathcal{A}^{(2)}$:

$$|\theta, \varphi\rangle = \sin\theta\cos\varphi\ |e_1 \vee e_2\rangle + \sin\theta\sin\varphi\ |e_1 \vee e_3\rangle + \cos\theta\ |e_3 \vee e_3\rangle$$

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Two Bosons, $\mathcal{H}^{(1)} = \mathbb{C}^3$

- $\mathcal{A}_0$ are generated by $|e_i\rangle\langle e_j|$, with $i = 1, 2$ and $\mathbb{1}_6$. This is equivalent to restriction to one-particle observables associated with only $|e_1\rangle$ and $|e_2\rangle$. The observables are equivalent to $U(2)$ (or better $SU(2)$).

- We may split the $\mathcal{H} = \mathbb{C}^6$ into invariant subspaces w.r.t. $SU(2)$ as $\mathbb{C}^6 = \mathbb{C}^3 \oplus \mathbb{C}^2 \oplus \mathbb{C}$, or equivalently $(1) \oplus (1/2) \oplus (0)$.

- For $(\theta, \varphi) \neq (0, 0)$, the GNS construction leads to a cyclic state $|\widetilde{\mathbb{1}_{\mathcal{A}}}\rangle = |\widetilde{e_1 \vee e_1}\rangle + |\widetilde{e_1 \vee e_3}\rangle + |\widetilde{e_3 \vee e_3}\rangle$, with entropy

$$S(\theta, \varphi) = -2\cos^2\theta \log\cos\theta \\ - 2\sin^2\theta \left[\cos^2\varphi \log(\sin\theta\cos\varphi) + \sin\varphi \log(\sin\theta\sin\varphi)\right]$$

As Bal says:

Think quantumly,
Act Planckly!

UnB

Muito obrigado!
Tack så mycket!