On the entropy for quantum unstable states

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Outline of the talk

- The entropy for decaying states in non-relativistic quantum mechanics .
- Friedrichs model.
- Formal steps:
  - The Hamiltonian (spectral representation with resonances and Gamow states).
  - The partition function (path integration over coherent states).
- Consistency: the path integration yields results which are correct within the framework of the thermal perturbation theory.
- Results: Expression for the canonical entropy of a quantum decaying system composed of fermion- and boson-states.

Introduction

- Kobayashi and Shimbori (PRE 63 (2001)056101), and Kobayashi (Int. J.Theor.Phys. 42 (2003)2265), have elaborated on the notion of entropy for complex-energy systems.

  Comments: In these papers the real and imaginary parts of the energy of a resonance are treated independently, and the canonical partition function for the resonance is given as a product of canonical partition functions for the real and imaginary parts of the energy, so that the total entropy is the sum of both contributions. Then, decaying processes transfer entropy from the imaginary part to the real part and the rate of this transference depends on time, and on a complex temperature

- Another point of view has been advocated by the Brussels group (I. Prigogine et al, Int. J. Quantum Chem. 98 (2004) 69)

  Comments: Misra, Prigogine and Courbage (Proc.Natl.Acad.Sc.76 (1979)4768) have introduced the notion of an entropy operator for Gamow states. This operator belongs to a family of operators defined on an extended space.

The Friedrichs Model

The Hamiltonian

$$H_0 = \omega_0 \, a^\dagger a + \int_0^\infty d\omega \, \omega \, b_\omega^\dagger b_\omega \, ,$$

$$V = \int_0^\infty d\omega \, f(\omega) \, \left(a^\dagger b_\omega + a b_\omega^\dagger\right) .$$

$$|1\rangle = a^\dagger |0\rangle \; ; \qquad |\omega\rangle = b_\omega^\dagger |0\rangle \, .$$

$$a|0\rangle = b_\omega |0\rangle = 0 \, .$$

$$g(z) = \langle 1 | \frac{1}{z - H} | 1 \rangle \, .$$

The Friedrichs Model

The representation

$$\eta(z) = \omega_0 - z - \int_0^\infty d\omega \, \frac{\lambda^2 f^2(\omega)}{\omega - z} \, .$$

$$A^\dagger_{IN} := \int_\gamma d\omega \, \frac{\lambda f(\omega)}{\omega - z_R} \, b^\dagger_\omega - a^\dagger$$

$$A_{OUT} := \int_\gamma d\omega \, \frac{\lambda f(\omega)}{\omega - z_R} \, b_\omega - a$$

$$B^\dagger_{\omega,IN} := b^\dagger_\omega + \frac{\lambda f(\omega)}{\tilde{\eta}^+(\omega)} \left\{ \int_0^\infty d\omega' \, \frac{\lambda f(\omega')}{\omega' - \omega - i0} \, b^\dagger_{\omega'} - a^\dagger \right\}$$

$$B_{\omega,OUT} := b_\omega + \frac{\lambda f(\omega)}{\tilde{\eta}^+(\omega)} \left\{ \int_0^\infty d\omega' \, \frac{\lambda f(\omega')}{\omega' - \omega - i0} \, b_{\omega'} - a \right\}$$

The Friedrichs Model

The eigenvectors

$$\frac{1}{\tilde{\eta}^{+}(\omega)} := \frac{1}{\eta^{+}(\omega)} + 2\pi i A \delta(\omega - z_R)\,,$$

$$[A_{OUT}, A^{\dagger}_{IN}] = 1\;; \qquad \frac{\eta^{+}(\omega)}{\eta^{-}(\omega)}\,[B_{\omega,OUT}, B^{\dagger}_{\omega',IN}] = \delta(\omega - \omega')\,.$$

$$H = z_R\, A^{\dagger}_{IN}\, A_{OUT} + \int_0^{\infty} d\omega\, \omega\, \frac{\eta^{+}(\omega)}{\eta^{-}(\omega)}\, B^{\dagger}_{\omega,IN}\, B_{\omega,OUT}\,.$$

$$|\psi^D\rangle = A^{\dagger}_{IN}|0\rangle\,, \qquad A_{OUT}|\psi^D\rangle = |0\rangle\,.$$

Liouville representation

The density of Gamow States

$$\rho = |\psi^D\rangle\langle\psi^D|$$

$$L = H \otimes I - I \otimes H$$

$$L\rho = -i\Gamma\,|\psi^D\rangle\langle\psi^D|$$

$$\rho(t) = e^{-itL}\,|\psi^D\rangle\langle\psi^D| = e^{-t\Gamma}\,|\psi^D\rangle\langle\psi^D|$$

$$[H,\rho(t)]\,|f\rangle = e^{-t\Gamma}\,(H|\psi^D\rangle\langle\psi^D|f\rangle - |\psi^D\rangle\langle\psi^D|Hf\rangle)$$

$$= e^{-t\Gamma}(z_R\,|\psi^D\rangle\langle\psi^D|f\rangle - z_R^*\,|\psi^D\rangle\langle\psi^D|f\rangle)$$

$$= -i\Gamma\,e^{-t\Gamma}\,(|\psi^D\rangle\langle\psi^D|f\rangle) = -i\Gamma\,\rho(t)|f\rangle$$

$$\frac{d}{dt}\,\rho(t) = i[H,\rho(t)]$$

Entropy of a Gamow density

The entropy operator as a Lyapunov variable (Prigogine)

$$i[H, M] = D \geq 0$$

and

$$[M, D] = 0$$

The suggested entropy operator was defined as

$$M = |\psi^G\rangle\langle\psi^G|$$

Then

$$M|\phi\rangle = (\langle\psi^G|\phi\rangle)|\psi^G\rangle$$

Properties of M

- Positivity: For any test vector we have

$$\langle\phi|M\phi\rangle = \langle\phi|\psi^G\rangle\langle\psi^G|\phi\rangle = |\langle\psi^G|\phi\rangle|^2 \geq 0$$

- M is a Liapunov variable:

$$\langle\phi(t)|M\phi(t)\rangle = \langle e^{-itH}\phi|\psi^G\rangle\langle\psi^G|e^{-itH}\phi\rangle$$

$$= \langle\phi|e^{itH}|\psi^G\rangle\langle\psi^G|e^{-itH}\phi\rangle = e^{t\Gamma}|\langle\phi|\psi^D\rangle|^2$$

- M is well defined:

$$i[H,M] = i(H|\psi^G\rangle\langle\psi^G| - |\psi^G\rangle\langle\psi^G|H) = -\Gamma M = D$$

$$[D,M] = 0$$

The entropy

For a given state, the entropy is defined as

$$S := \mathrm{tr}\,\{\rho^\dagger M \rho\}$$

$$\rho^\dagger M \rho = |\psi^D\rangle\langle\psi^D|\psi^G\rangle\langle\psi^G|\psi^D\rangle\langle\psi^D|$$

In the Friedrichs model

$$\langle\psi^D|\psi^G\rangle$$

is well defined, so that it can be normalized to one. In this case, we conclude that

$$\rho^\dagger M \rho = \rho\,.$$

The trace

$$\mathrm{tr}\,\rho = \langle 1|\psi^D\rangle\langle\psi^D|1\rangle + \int_0^\infty \langle\psi^D|\omega\rangle\langle\omega|\psi^D\rangle\, d\omega$$

$$\mathrm{tr}\,\rho \approx \left[1 + \int_0^\infty d\omega\, \frac{\lambda^2 f^2(\omega)}{[z_R-\omega]_+[z_R^*-\omega]_-}\right]$$

$$\rho(t)M\rho(t) = e^{-itL}\rho M e^{-itL}\rho = e^{-2t\Gamma}\rho(0) = e^{-t\Gamma}\rho(t)$$

$$S(t) = \mathrm{tr}\,\left[\rho(t)M\rho(t)\right] = e^{-2\Gamma t}\,\mathrm{tr}\,\rho(0)$$

$$\approx e^{-2\Gamma t}\,\left[1 + \int_0^\infty d\omega\, \frac{\lambda^2 f^2(\omega)}{[z_R-\omega]_+[z_R^*-\omega]_-}\right]$$

$$S(t) = e^{-2\Gamma t}\,|N|^2\,\left[1 + \int_0^\infty d\omega\, \frac{\lambda^2 f^2(\omega)}{(z_R-\omega)(z_R^*-\omega)}\right]$$

Coherent states

$$|\alpha\rangle := \exp\{\alpha\, A^\dagger_{IN} - \alpha^*\, A_{OUT}\}\, |0\rangle$$

$$\langle\alpha| := \langle 0|\exp\{\alpha^*\, A_{OUT} - \alpha\, A^\dagger_{IN}\}$$

$$A_{OUT}|\alpha\rangle = \alpha|\alpha\rangle$$

$$\langle\alpha|A^\dagger_{IN} = \alpha^*\langle\alpha|$$

$$\int_C \frac{d^2\alpha}{\pi}\, |\alpha\rangle\langle\alpha| = 1$$

$$d^2\alpha = (d\mathrm{Real}\,\alpha)(d\mathrm{Im}\,\alpha)$$

The path integral

$$\langle \alpha_i | \rho | \alpha_f \rangle = \lim_{N->\infty} \rho_N(\alpha_i, \alpha_f)$$

$$\rho_N(\alpha_i, \alpha_f) = \int \prod_{k=1}^{N} \left(\frac{d^2 \alpha_k}{\pi} \right)$$

$$\exp \left\{ -\tau \left[\sum_{n=1}^{N} H_+(\alpha_{n-1}, \alpha_n) \;\; + \sum_{n=1}^{N+1} \left(\frac{\alpha_n^* - \alpha_{n-1}^*}{2\tau} \right) \alpha_n - \alpha_{n-1}^* \;\; \left(\frac{\alpha_n - \alpha_{n-1}}{2\tau} \right) \right] \right.$$

$$H_+(\alpha, \alpha') = \frac{\langle \alpha | H | \alpha' \rangle}{\langle \alpha | \alpha' \rangle}$$

$$\langle \alpha | \alpha' \rangle = \exp \left\{ -\frac{|\alpha|^2}{2} - \frac{|\alpha'|^2}{2} + \alpha^* \alpha' \right\}$$

$$H_+(\alpha, \alpha') = z_R \, \alpha^* \alpha'$$

The partition function

$$\rho_N(\alpha_i, \alpha_f) := \exp\{-\beta\, z_R\, \alpha_i^* \alpha_f\}$$

$$Z = \int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi}\, \rho(\alpha, \alpha) = \int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi}\, \exp\{-\beta\, z_R\, |\alpha|^2\}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} dx\, e^{-\beta z_R x^2} \int_{-\infty}^{\infty} dy\, e^{-\beta z_R y^2} = \frac{1}{\beta z_R}$$

The canonical entropy

$$S = k(1 - \log(\beta\, z_R)) = k\left[1 - \ln\left(\beta\sqrt{E_R^2 + \frac{\Gamma^2}{4}}\right) - i\arctan\left(\frac{\Gamma}{2E_R}\right)\right]$$

Conclusions

- The presence of a complex entropy, for the case of Gamow vectors, requires of some interpretation on the meaning of its imaginary part.
- Note that the resonance in the Friedrichs model is caused by the interaction of the system with the background, which plays the role of the thermodynamical bath. Then, we suggest that the real part of the entropy is the entropy of the system and that the imaginary part of it is the entropy transferred from the system to the background.
- Should the thermodynamical entropy be identified with the modulus of the obtained expression, one concludes that the total entropy for a decaying state is bigger than the entropy of a stable system.