Eigenvalue and Eigenvector statistics for Bosonic Embedded Gaussian Ensembles

Computational Challenges in Nuclear and Many-Body Physics

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Outline of the talk

Success of Random Matrix Theory (RMT)

- Nuclear Physics
- Zeros of the Riemann ζ function
- Sound Waves in quartz crystals
- Quantum Chaos: Billiards, Quantum Dots
- Disordered mesoscopic systems
- Field theory and Statistical Mechanics
- Decoherence
  - T. Gorin et al 2008 New J. Phys. 10 115016
  - Phys. Rev. Lett. 99, 240405
- Bosonic Systems and Fidelity
  - Phys. Rev. E 81, 036218
  - Phys. Rev. E 83, 056216

A bit of History

- First appearance of RMT by John Wishart in Mathematical Statistics (1928)[1].
- Atomic nuclei energy levels study by Wigner (1951)[2].
- Mathematical Foundations of RMT established in Dyson's series of papers (1960's), starting with[3].
- *Bohigas-Giannoni-Schmidt conjexture:* Connection between Classical and Quantum Chaos[4].

Since then applied to many areas (and an area of active research!).

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[1] J. Wishart 1928 Biometrik A20 32-52.

[2] E.P. Wigner 1952 Proc. Cam. Phil. Soc. 47 790.

[3] F.J. Dyson 1962 J. Math. Phys. 140.

[4] O. Bohigas et al 1984 Phys. Rev. Lett. 51, 1. However, the conjecture was first given by M. Berry!

A bit of History (about Embedded Ensembles)

- Two-Body Random Ensemble (TBRE) (1970-71)[5]. TBRE fails in mathematical tractability [6].
- Embedded Gaussian Ensembles (EGE) facilitate the obtaining of analytical results[7].
- Bosonic extention of the EGE (BEGE) given by Kota[8].
- Some analytical results obtained by the Heildelberg group (on EGE and BEGE)[9].

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[5] B. French and S.S.M. Wong 1970 Phys. Lett. B 449(7).
O. Bohigas and J. Flores 1971 Phys. Lett. B 34(4), 261.

[6] T.A. Brody et al 1981 Rev. Mod. Phys. 53, 3.

[7] K.K. Mon and J.B. French 1975 Ann. Phys. 95:90-11.

[8] V.K.B. Kota and V. Potbhare 1980 Phys. Rev. C 21(6).

[8] L. Benet et al 2001 Ann. Phys. 292, (67-94).
T. Asaga et al 2001 Europhys. Lett. 56 (340).

What is the philosophy of RMT?

In Dyson's words:

"What is here required is a new kind of statistical mechanics, in which we renounce exact knowledge not of the state of the system but of the system itself. We picture a complex nucleus as a "black box" in which a large number of particles are interacting among each other according to unknown laws. The problem then is to define in a mathematically precise way an ensemble of systems in which **all possible laws of interaction are equally possible."**

- In RMT it's all about fluctuations.

An important issue: Ergodicity in RMT

"The statistical properties of individual members of the ensemble should almot coincide (to within a suitably narrow error bounds) with the ensemble average. Such behaviour ensures that ensemble predictions can be used for individual systems."[6]

Canonical Ensembles (GxE)

According to it's space-time symmetries obeyed by the treated system, the classification of Canonical Gaussian Ensembles is[9]:

$$P(H')dH' = P(H)dH$$

- GOE-Gaussian Orthogonal Ensemble ($\beta = 1$):

$$H' = O^T HO$$

- GUE-Gaussian Unitary Ensemble ($\beta = 2$):

$$H' = U^{-1} HU$$

At the end, it can be shown:

$$P(H) = \exp\left(-atr(H^2) + btr(H) + c\right),$$

$a > b$, and b and c are real numbers.

[9]We skip the GSE.

Correlations in the eigenvalues

- From the joint probability density function:

$$P_{N\beta}(E_1, ..., E_N) = C_{N\beta} \exp\left(-\frac{1}{2}\beta \sum_{j=1}^{N} E_j^2\right) \prod_{j<k} |E_j - E_k|^\beta.$$

$\Pi_{j<k}|E_j - E_k|^\beta$: Level repulsion!

Correlations are obtained from the n-point correlation function:

$$R_n(E_1, \dots, E_n) = \frac{N!}{(N-n)!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} P_N(E_1, \dots, E_N) dE_{n+1}, \dots, dE_N.$$

- Density of eigenvalues (Wigner's semicircle law):

$$\rho(E) = \frac{1}{\pi}(2N - x^2)^{1/2}, |x| < (2N)^{1/2},$$

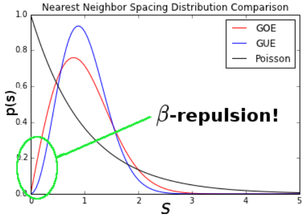
(or $\rho(E) = 0$, if $|x| > (2N)^{1/2}$).

Correlation measures

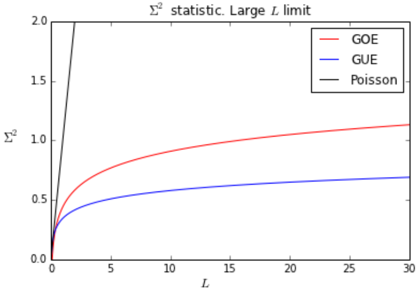
After a suitable "unfolding procedure":

- Short range correlations: Nearest Neighbor Spacing Distribution
  - Wigner's surmise (exact for 2×2 matrices):

$$p_\beta(s) = a_\beta s^\beta e^{-b_\beta s^2}$$



- Long range correlations: Σ^2-statistic (or Number Variance of Levels).
  In an interval of length L, there exists $L \pm \sqrt{\Sigma^2(L)}$ levels.



Eigenvector statistics: Inverse Participation Ratio (IPR)

Definition

For the μth eigenvector of a matrix H, the IPR (base-dependent) is

$$P_\mu = \sum_{n=1}^{N} |c_{\mu n}|^4$$

- Canonical RMT (large matrices): $P_{\mu,\beta} \approx C(\beta)/N$.
  Wave functions delocalized.
- Integrable systems: Localization of wave functions.

Up to know, we have just talked of the GxE's and their correlation measures.
But... what is exactly the "Bosonic Embedded Gaussian Ensemble"?

Bosonic Embedded Gaussian Ensemble

The problem

Consider a system composed of n-particles distributed over l-single particle states (or harmonic traps, like in the Bose-Hubbard model). The interaction is random, thus the interaction hamiltonian is defined as:

$$V_k^{(\beta)} = \sum_{\alpha,\gamma} v_{k;\alpha,\gamma}^{(\beta)} \Upsilon_{k,\alpha}^{\dagger} \Upsilon_{k,\gamma},$$

where

$$\Upsilon_{k,\alpha}^{\dagger} = \Upsilon_{j_1,\ldots,j_k} = \frac{1}{\mathcal{N}} \prod_{s=1}^{k} b_{j_s}^{\dagger}.$$

The coefficients (matrix elements):

$$v_{k;\alpha,\gamma}^{(\beta)},$$

are independent and identical distributed gaussian **random** variables with zero mean and finite variance, i.e. $v_{k;\alpha,\gamma}^{(\beta)}$ is a member of GxE $(\beta = 1, 2)$.[10]

[10] The b and $b^{\dagger}$ are the usual creation and anihilation bosonic operators.

- If $\beta = 1$ we say that we study the Bosonic Embedded Gaussian Orthogonal Ensemble or BEGOE.
- If $\beta = 2$ we say that we study the Bosonic Embedded Gaussian Unitary Ensemble or BEGUE.

Conserved quantities in the model

- Total Energy.
- Total number of particles.

Central questions in the BEGE model

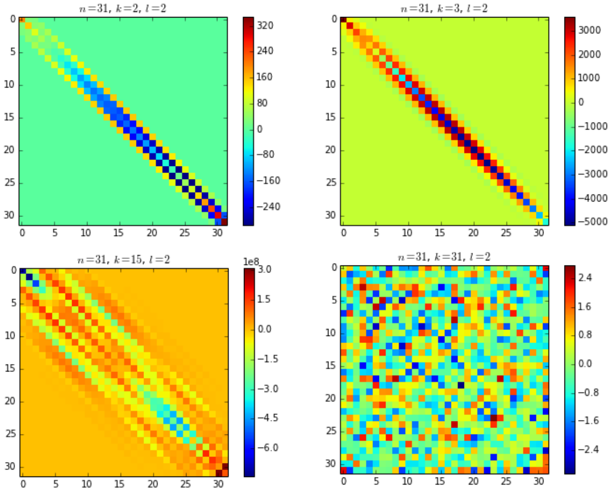
We want to understand the statistical properties of a bosonic system using the model of the BEGE.

- Are the spectra ergodic?
- What is the shape of the spectral density?
- What are the spectral fluctuation properties?
- What can we say about the statistics of the eigevectors in the ensemble?
- Can we use the BEGE to describe, e.g. the Bose-Hubbard hamiltonian?

Limits that we study

- Bosonic dense limit: $n >> 1$, with k and l fixed (no fermionic analog). Mimics a Bose-Einstein condensate.
- When $k \to n$: We recover the GOE or GUE.

A typical member of BEGOE (matrix structure)



The simplest case $l = 2$

$$V_k^{(\beta)} = \sum_{l_1, l_2 = 1}^{k} v_{k; l_1, l_2}^{(\beta)} \frac{(b_1^\dagger)^{l_1} (b_2^\dagger)^{k - l_1} b_1^{l_2} b_2^{k - l_2}}{\sqrt{l_1!(k - l_1)! l_2!(k - l_2)!}}$$

The simplest case $l = 2$

$$v^{(\beta)}_{k;l_1,l_2}(b_1^\dagger)^{l_1}(b_2^\dagger)^{k-l_1}b_1^{l_2}b_2^{k-l_2}$$

Suppose $\beta = 1$ **and two-body forces:** $k = 2$**:**

$$\begin{pmatrix} v_{2;1,1}(b_1^\dagger)^2 b_1^2 & v_{2;1,2}(b_1^\dagger)^2 b_1 b_2 & v_{2;1,3}(b_1^\dagger)^2 b_2^2 \\ h.c. & v_{2;2,2} b_1^\dagger b_2^\dagger b_1 b_2 & v_{2;2,3} b_1^\dagger b_2^\dagger b_2^2 \\ h.c. & h.c. & v_{2;3,3}(b_2^\dagger)^2 b_2^2 \end{pmatrix}.$$

- Dimension of the matrix is $\binom{k+l-1}{k} = k+1 = 3$, then **only 6 elements are independent!**

The simplest case $l = 2$

Suppose now $n = 1023$, then, if we choose the ocupation number basis $|\mu, n-\mu\rangle \Rightarrow$ the dimension of the hamiltonian matrix is

$$\binom{1023+2-1}{1023} = 1024.$$

Because $V_k^{(1)} = (V_k^{(1)})^\dagger \Rightarrow$ **There are in principle** $(1024 * 1025)/2$ **independent matrix elements.**

These 524800-independent elements are generated only using the 6-independent matrix elements of $v_{2;m,n}$!

Moreover, many matrix elements are zero.

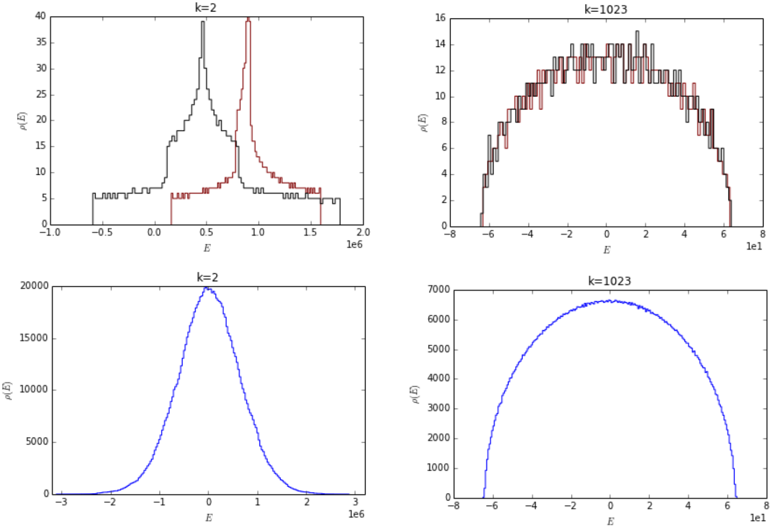
- **Highly correlated ensemble**

... as a function of k.

What is the shape of the spectral density? (BEGOE $l = 2$)

- Parameters: $n = 1023$, $k = 2, 1023$, $l = 2$, 1000-realizations.

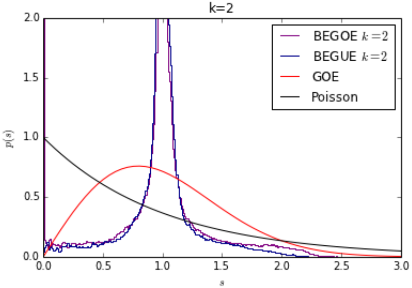
In general, **the ensemble is not ergodic!**



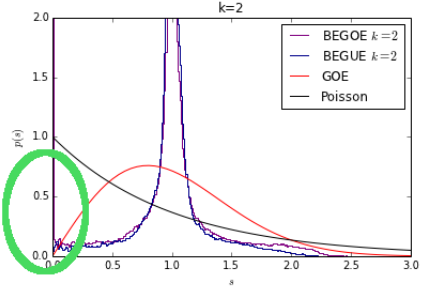
What can we learn for example, from the $p(s)$?

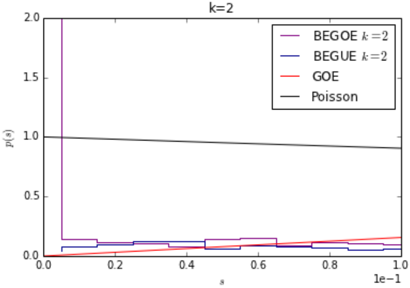
- Comparison of the $p(s)$-statistic BEGOE vs BEGUE.

Parameters: $n = 1023$, $k = 2$, $l = 2$, 1000-realizations.



Let's have a closer look!





- Huge peak at zero for $\beta = 1 \Rightarrow$ Quasi-degenerate spectrum!

If you want to know more: [Phys. Rev. E 81, 036218].

BEGOE $l = 3$

Little is known about the BEGOE with $l = 3$. Generically it's a chaotic system.

- As with $l = 2$, occurs a transition in the statistics as a function of k.

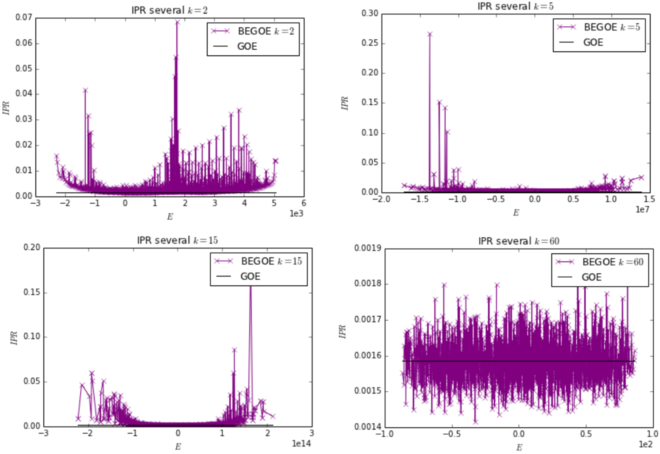
Parameters: $n = 60$, $l = 3$, 1000-realizations. All histograms are normed to 1.

- Eigenvalue Density ($BEGOE\_l3 - dens.mp4$).
- Nearest Neighbor Spacing Distribution ($BEGOE\_l3 - ps.mp4$).
- Σ^2 statistic ($BEGOE\_l3 - sigma2.mp4$).

Inverse Participation Ratio (IPR)

Taking a random realization in the ensemble.

Do we have Shnirelman doublets for this case? We don't know it yet!



Ongoing projects

The Bose-Hubbard hamiltonian vs the BEGOE model

Bose-Hubbard hamiltonian

$$H_{BH} = \frac{U}{2} \sum_{i=1}^{l} n_j(n_j - 1) + J \sum_{i \neq j} b_i^{\dagger} b_j$$

Can Bose-Hubbard be modelated as an emsemble of random operators?

Let's see!

- For the two-body part, recall the matrix $v^{(\beta=1)}_{k;l_1,l_2}$, the term $n_j(n_j-1)$ is equivalent of having:

$$\begin{pmatrix} v_{2;1,1}(b_1^\dagger)^2 b_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & v_{2;3,3}(b_2^\dagger)^2 b_2^2 \end{pmatrix}.$$

We no longer manipulate the k-body interaction. But we can manipulate the mean and the variance of the $v^{(\beta=1)}_{k;i,i}$!

- $\bar{v}^{(\beta=1)}_{k;i,i} = \bar{U}$, and the variance we write it as $varU$.
- Similarly can be done with the one body-part, then the mean is $\bar{J}$ and the variance is $varJ$.

Therefore

The *Random Bose-Hubbard hamiltonian* is defined as

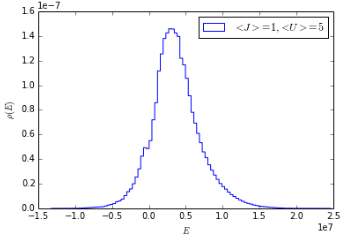
$$H'_{BH} = V'_{k=1}(\bar{J}, varJ) + V'_{k=2}(\bar{U}, varU),$$

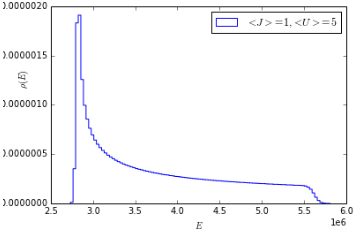
$(\beta = 1)$.

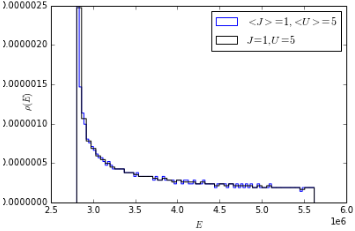
Random Bose-Hubbard Hamiltonian

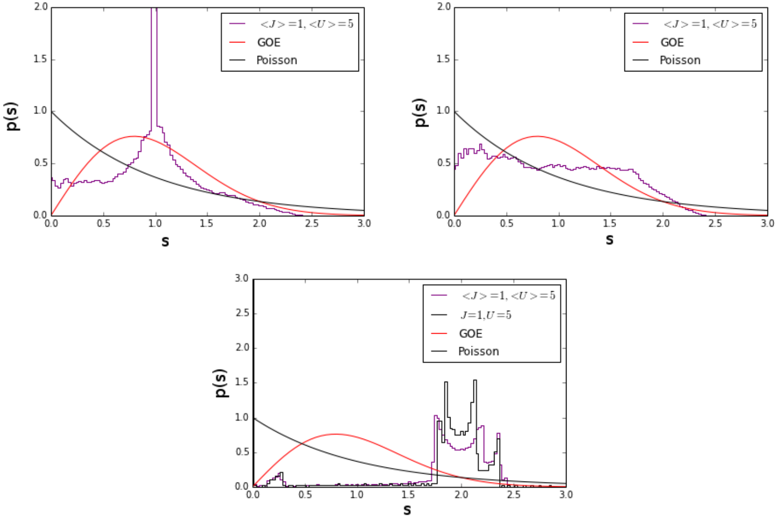
- $n = 1500$, $l = 2$, 1000-realizations.

The different plots correspond to variaces: 10, 10^{-3}, 10^{-12}.







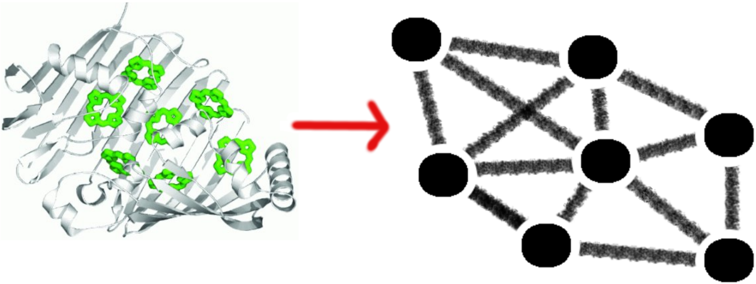


p(s)
s
$<J>=1, <U>=5$
GOE
Poisson
0.0
0.5
1.0
1.5
2.0
2.5
3.0
$<J>=1, <U>=5$
$J=1, U=5$
GOE
Poisson

Study of disordered networks

Motivation

- A biological network (taken from [Hoyer et al 2010 New J. Phys. 12 065041] and an abstract disordered network.



Study of efficiency in disordered networks

Definition of efficiency[11]

The efficiency $\mathcal{P}$ is defined as

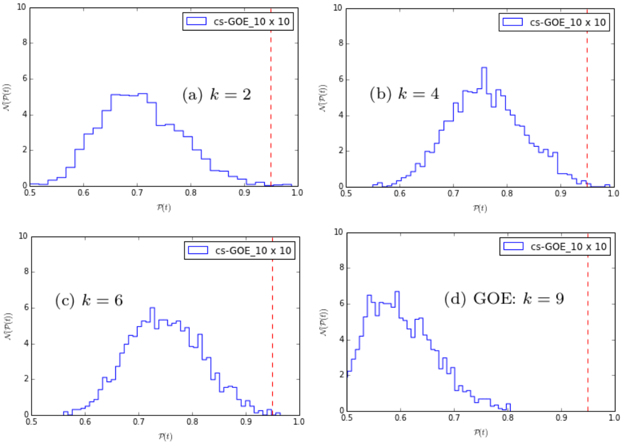
$$\mathcal{P}_{\mu,\nu} = \max_{[0,T_{Hor})} |\langle \mu, e^{-iV_k^{(\beta)}t}\nu\rangle|^2.$$

The maximum probability of finding the initial state $|\nu\rangle$ in the state $|\mu\rangle$ in the time interval $[0, T_{Hor})$, for all μ and ν such that $\mu > \nu$.

- In the biological networks, efficiency is about 95%.

[11] M. Walschaers et al 2012 PRL 111, 180601.

- Pretty small systems up to now: $n = 9$, $l = 2$ and 2000-realizations.

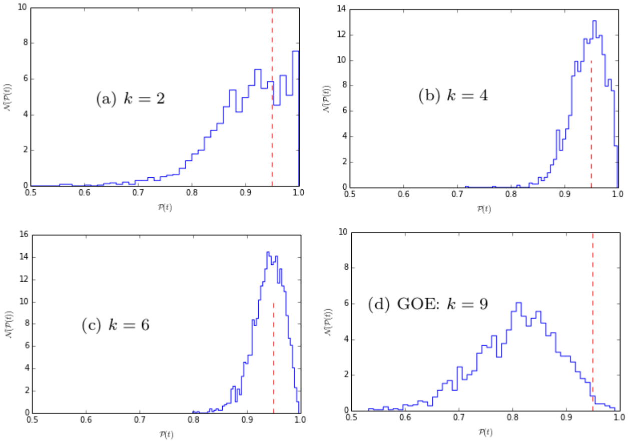


- In the BEGOE the efficiency is poor, we need an additional structural element to boost the efficiency!

- We introduce the centro-symmetry [11] at k-body level:

$$[v^{(1)}_{k;l_1,l_2}, J_K] = 0,$$

where $J_K = \delta_{i,N-j+1}$.



- **Efficiency is improved compared to the case presented at[11].** We believe that this is in part due to the correlations. Maybe the bosonic nature of the system is also responsible of this.

Summary

- The BEGE model are highly correlated ensembles. They are also non-ergodic in RMT sense.
- The BEGE with $l = 3$ is poorly studied. What can we learn from the statistics of such ensemble?
- We have a computational way to analize eigenvalue and eigenvector correlations.
- The Random Bose-Hubbard model displays a rich behaviour. We would also like to study fidelity in this ensemble.
- A disordered network generated by ocupation-number states is boosted by the centro-symmetry. What is the asymptotic behaviour, i.e. $n >> 1$?

For your attention

Many thanks!