Angular-momentum-projection method to approach nuclear many-body problems

Yang Sun
Shanghai Jiao Tong University, China

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Nuclear structure models

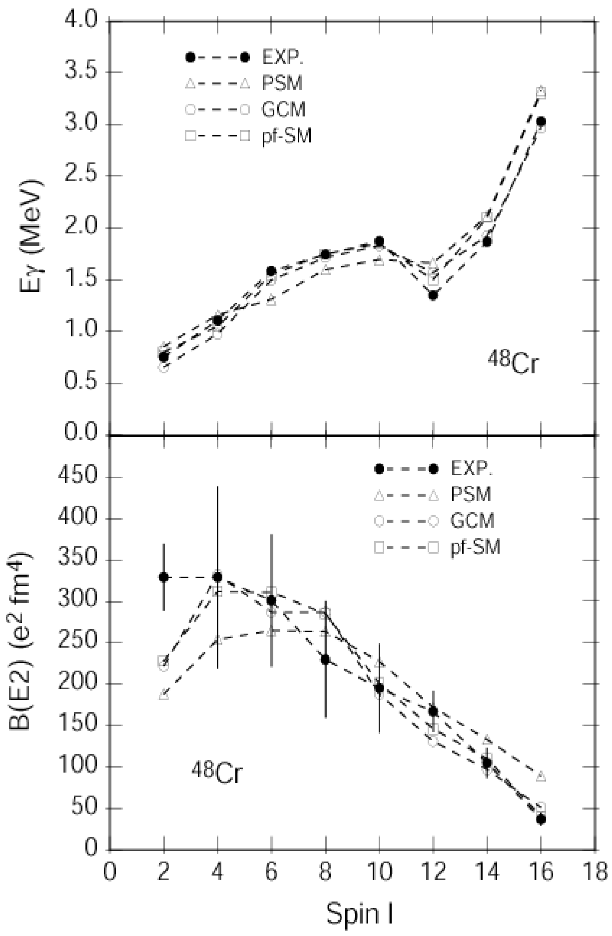
- Shell-model diagonalization method
  - Based on quantum mechanical principles
  - Growing computer power helps extending applications
  - A single configuration contains no physics
  - Huge basis dimension required, severe limit in applications
- Mean-field approximations
  - Applicable to any size of systems
  - Fruitful physics around minima of energy surfaces
  - No configuration mixing, results depend on quality of mean-field
  - States with broken symmetry, cannot study transitions
- Algebraic models
  - Based on symmetries, simple and elegant
  - Serve as important guidance for complicated calculations

How to treat deformed nuclei

- Most nuclei in the nuclear chart are deformed. To describe a deformed nucleus, a spherical shell model loses advantages.
- One can start from a deformed basis by breaking the rotational symmetry spontaneously.
- Then apply angular-momentum-projection technique to recover the symmetry.
  - important correlations prepared through a better mean-field
  - intrinsic states classified with well-defined physical meanings
  - these states transformed to the laboratory frame
  - diagonalization performed in the (angular-momentum) projected basis
  - results may be interpreted by algebraic models

Deformed basis vs spherical basis

- Rotational spectrum in ^{48}Cr
  - Exp. data:
    - Brandolini et al, *NPA* 642 (1998) 387
  - PSM:
    - Hara, Sun and Mizusaki, *PRL* 83 (1999) 1922
    - Deformed basis with a.-m. projection; Basis states ~ 50
  - pf-SM:
    - Caurier *et al.*, *PRL* 75 (1995) 2466
    - Conventional M-scheme spherical shell model; Basis states ~ 2 million



A method related to mean-field and shell models

- Angular-momentum projection method based on deformed mean-field solutions
  - Start from intrinsic bases (e.g. solutions of deformed mean-field) and select most relevant configurations
  - Use angular momentum projection technique to transform them to laboratory basis (many-body technique)
  - Diagonalize Hamiltonian in the projected basis (configuration mixing, a shell-model concept)
- It is an efficient way, and probably the only way to treat heavy, deformed nuclei microscopically in a shell model concept
- Example: Projected Shell Model
    - K. Hara, Y. Sun, *Int. J. Mod. Phys.* E 4 (1995) 637

Projected Shell Model (PSM)

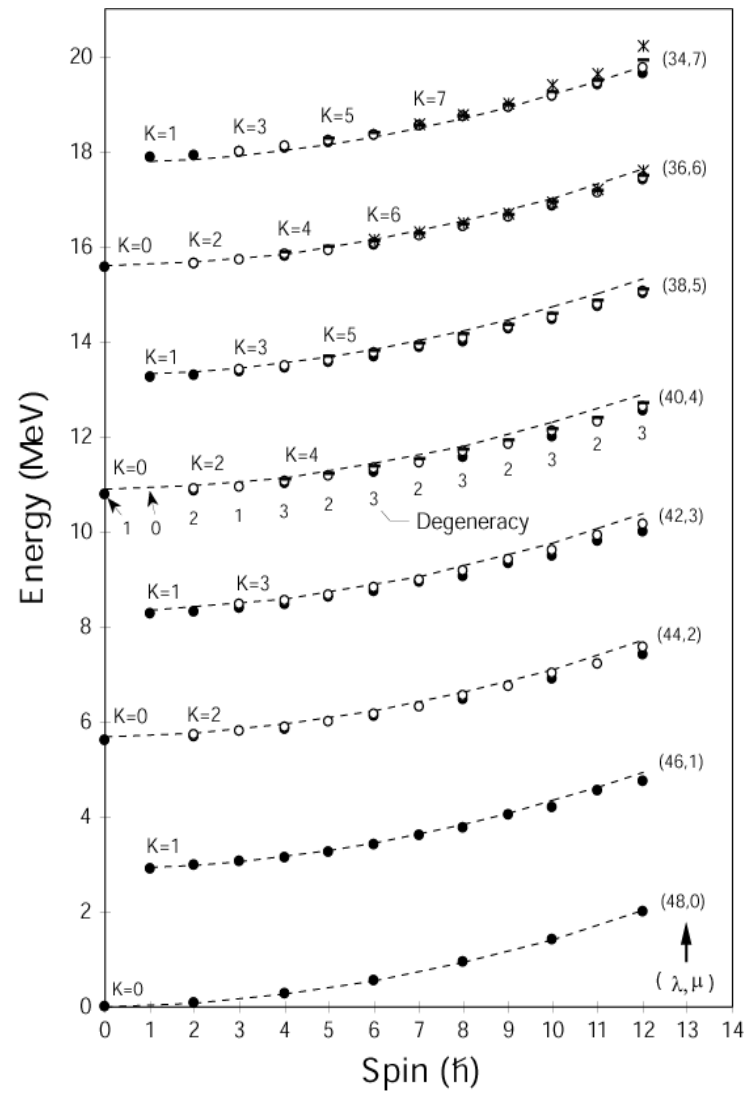
- Take a set of deformed (quasi)particle states (e.g. solutions of HF, HFB or Nilsson + BCS)
- Select configurations (qp vacuum + multi-qp states near the Fermi level)
- Project them onto good angular momentum (if necessary, also parity, particle number) to form a basis in lab frame
- If necessary, superimpose configurations belonging to different qp representations (the GCM-concept)
- Diagonalize a two-body Hamiltonian in projected basis

Comparison with other models

- Comparison with spherical shell model
  - No problem with basis size
  - PSM basis constructed by physical guidance
- Comparison with mean-field models
  - Violated symmetries restored
  - Configuration mixing implemented
- Comparison with algebraic models
  - Do not require a symmetry to start with
  - Yet the PSM results can be discussed with symmetry ideas
- Comparison with the Tuebingen, Tokyo approaches
  - Different in preparation of basis and in effective interactions

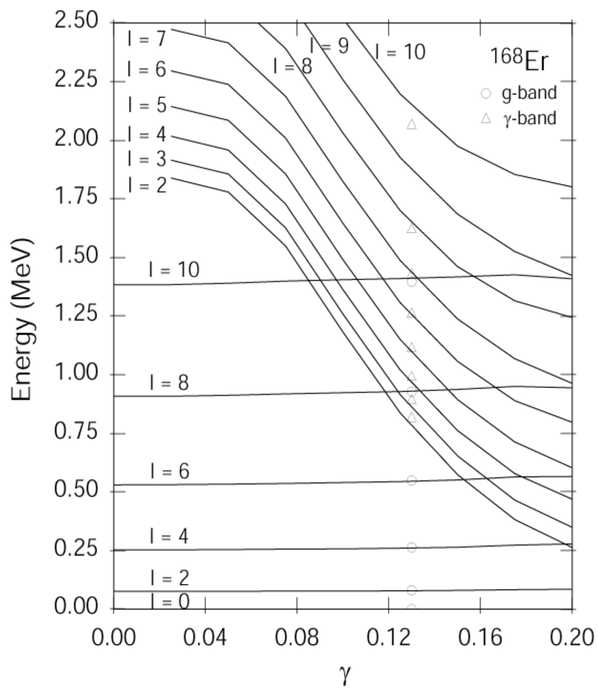
Emergence of SU(3) symmetry

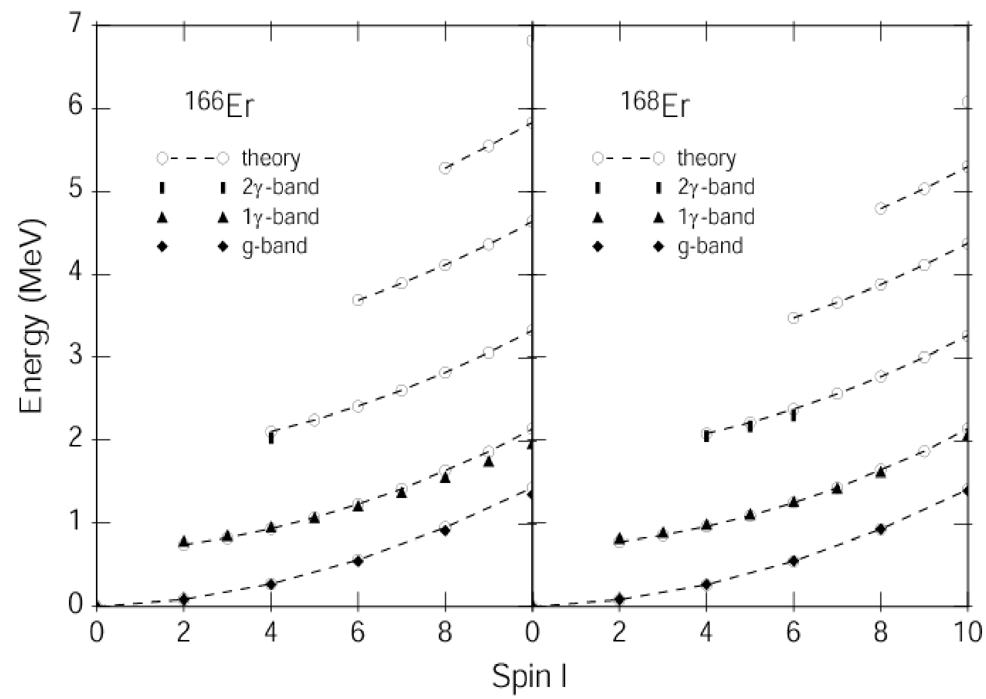
- Nearly perfect SU(3) symmetry emerges from a.-m.-projection
  - Project on separate BCS vacuum of $|\phi_\nu\rangle$ and $|\phi_\pi\rangle$, then couple the projected states $|I_\sigma\rangle = N^I \hat{P}^I |\phi_\sigma\rangle$ to form the basis $|(I_\nu \otimes I_\pi)I\rangle$
  - Diagonalize the Hamiltonian in the coupled basis
  - Multi-phonon scissors mode is predicted
  - Sun, Wu, Guidry *et al*., *PRL* 80 (1998) 672; *NPA* 703 (2002) 130



γ-vibrational states

- γ-vibration states cannot be obtained when axial symmetry in the basis states is assumed
- Need 3-dimensional angular-momentum projection performed on a triaxially deformed basis

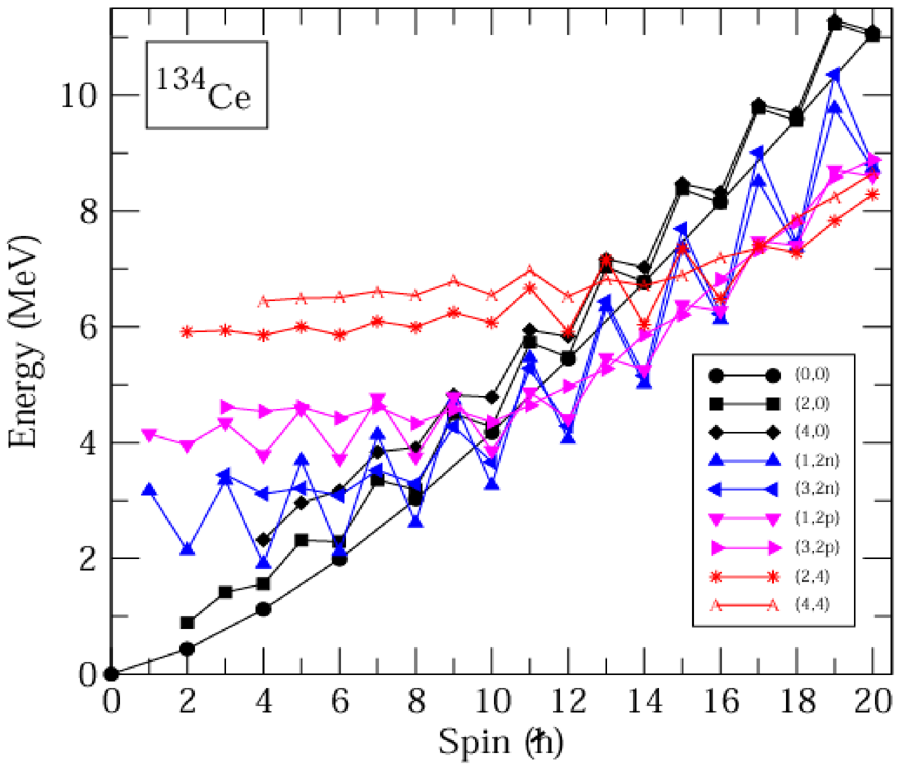




Y. Sun et al. *Phys. Rev.* C 61 (2000) 064323

γ-deformed multi-qp excitations

- 0-phonon (*K*=0), 1-phonon (*K*=2), 2-phonon (*K*=4) γ-vibrational bands
  - Y. Sun et al, Phys. Rev. C61 (2000) 064323
- Each phonon γ-vibrational mode can couple with qp states – generalization of the usual concept of γ-vibration
  - Sheikh *et al.*, Phys. Rev. C77 (2008) 034313; Nucl. Phys. A824 (2009) 58



Basic structure

- Ansatz of wavefunction: $\psi^I_M = \sum_\kappa f_\kappa \hat{P}^I_{MK_\kappa} |\phi_\kappa\rangle$

  with the projector: $\hat{P}^I_{MK} = \frac{2I+1}{8\pi^2}\int d\Omega\, D^I_{MK}(\Omega)\hat{D}(\Omega)$

- The eigenvalue equation: $\sum_{\kappa'} \left(H^I_{\kappa\kappa'} - EN^I_{\kappa\kappa'}\right) f_{\kappa'} = 0$

  with matrix elements: $H^I_{\kappa\kappa'} = \langle \phi_\kappa | \hat{H}\hat{P}^I_{KK'} | \phi_{\kappa'} \rangle$ $\quad N^I_{\kappa\kappa'} = \langle \phi_\kappa | \hat{P}^I_{KK'} | \phi_{\kappa'} \rangle$

- The Hamiltonian is diagonalized in the projected basis

$$\left\{ \hat{P}^I_{MK} | \phi_\kappa \rangle \right\}$$

a.-m.-projected multi-quasi-particle states based on a fixed deformation

- Even-even nuclei:

$$\left\{\hat{P}^I_{MK}|0\rangle, \hat{P}^I_{MK}\alpha^+_\nu\alpha^+_\nu|0\rangle, \hat{P}^I_{MK}\alpha^+_\pi\alpha^+_\pi|0\rangle, \hat{P}^I_{MK}\alpha^+_\nu\alpha^+_\nu\alpha^+_\pi\alpha^+_\pi|0\rangle, \ldots\right\}$$

- Odd-odd nuclei:

$$\left\{\hat{P}^I_{MK}\alpha^+_\nu\alpha^+_\pi|0\rangle, \hat{P}^I_{MK}\alpha^+_\nu\alpha^+_\nu\alpha^+_\nu\alpha^+_\pi|0\rangle, \hat{P}^I_{MK}\alpha^+_\nu\alpha^+_\pi\alpha^+_\pi\alpha^+_\pi|0\rangle, \hat{P}^I_{MK}\alpha^+_\nu\alpha^+_\nu\alpha^+_\nu\alpha^+_\pi\alpha^+_\pi\alpha^+_\pi|0\rangle, \ldots\right\}$$

- Odd-neutron nuclei:

$$\left\{\hat{P}^I_{MK}\alpha^+_\nu|0\rangle, \hat{P}^I_{MK}\alpha^+_\nu\alpha^+_\pi\alpha^+_\pi|0\rangle, \hat{P}^I_{MK}\alpha^+_\nu\alpha^+_\nu\alpha^+_\nu\alpha^+_\pi\alpha^+_\pi|0\rangle, \ldots\right\}$$

- Odd-proton nuclei:

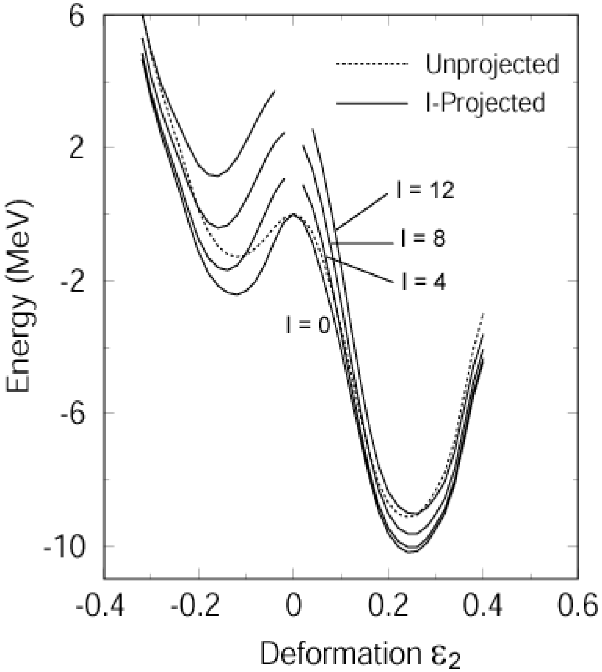
$$\left\{\hat{P}^I_{MK}\alpha^+_\pi|0\rangle, \hat{P}^I_{MK}\alpha^+_\nu\alpha^+_\nu\alpha^+_\pi|0\rangle, \hat{P}^I_{MK}\alpha^+_\nu\alpha^+_\nu\alpha^+_\pi\alpha^+_\pi\alpha^+_\pi|0\rangle, \ldots\right\}$$

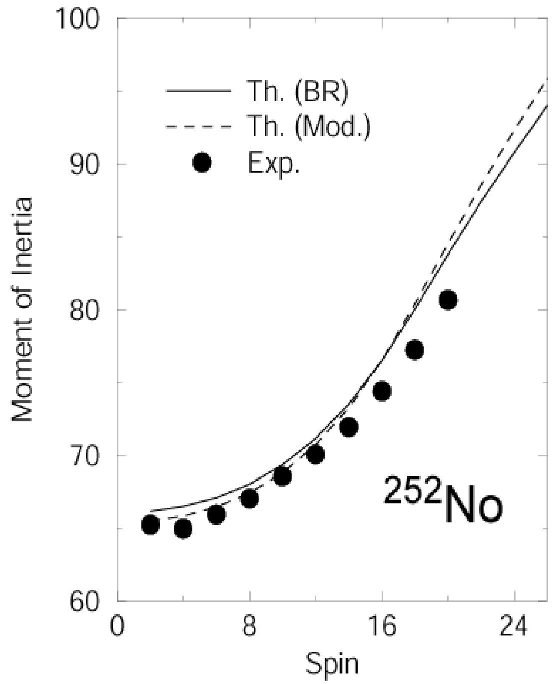
Hamiltonian and single particle space

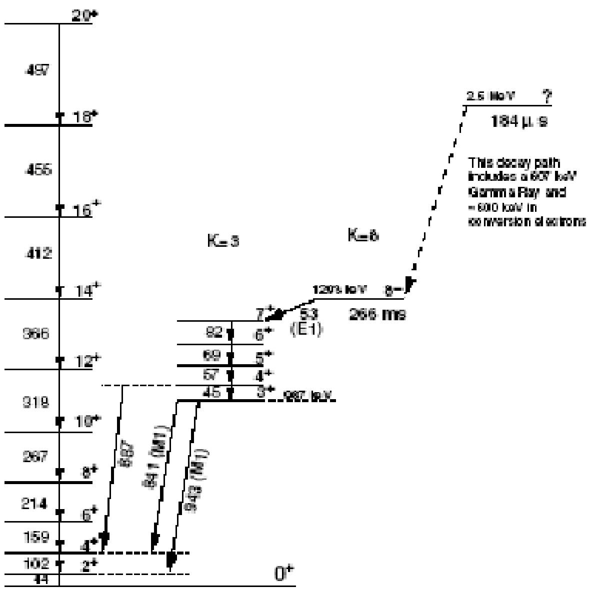
- Hamiltonian $H = H_0 - \sum_{\lambda} \frac{\chi_\lambda}{2} \sum_{\mu} Q^{+}_{\lambda\mu} Q_{\lambda\mu} - G_M P^+ P - G_Q \sum_{\mu} P^{+}_{\mu} P_{\mu}$
- Interaction strengths
  - χ is related to deformation ε by $\chi_{\tau\tau'} = \frac{2/3\ \varepsilon\hbar\omega_\tau \hbar\omega_{\tau'}}{\hbar\omega_n \langle Q_0 \rangle_n + \hbar\omega_p \langle Q_0 \rangle_p}$
  - G_M is determined by observed even-odd mass difference
  - G_Q is assumed to be proportional to G_M with a ratio ~ 0.20
- Single particle space
  - Three major shells for neutrons or protons (normally deformed)
    four major shells for neutrons or protons (super-deformed)
  - For example, for rare-earth nuclei, N = 4, 5, 6 for neutrons
    N = 3, 4, 5 for protons

Example of a deformed rotor

- Angular-momentum-projected energy calculation shows a deep prolate minimum for a superheavy nucleus
  - A very good rotor with axially-symmetric deformed shape
  - Quasi-particle excitations based on the same deformed potential



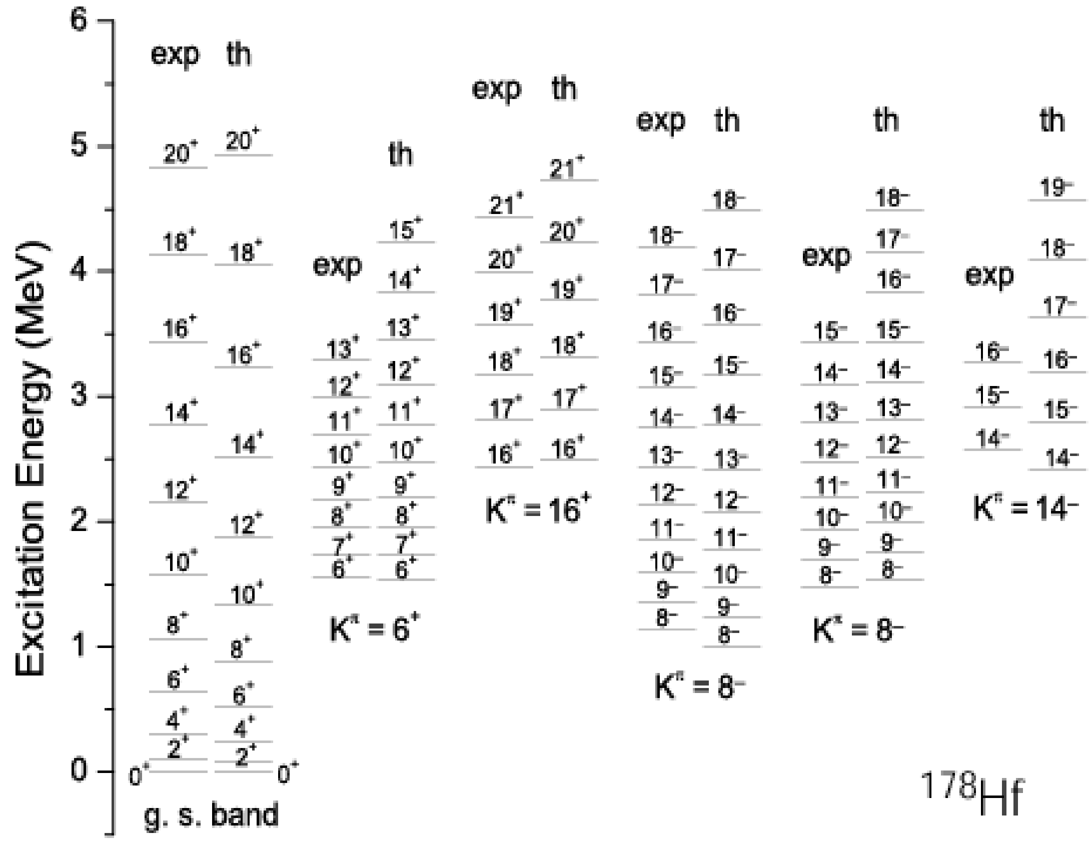




Herzberg *et al.*, *Nature* 442 (2006) 896

Multi-quasiparticle excitations

- 0-, 2-, 4-qp states of ^{178}Hf
- Data:
  - S.M. Mullins *et al*, *Phys. Lett.* B 393 (1997) 279
- Theory:
  - Y. Sun *et al*, *Phys. Lett.* B 589 (2004) 83



Calculation of matrix elements for multi-quasiparticle states

- If a multi-quasiparticle state is written as $|\Phi_\kappa\rangle$, then the central task is to calculate

$$\mathcal{H}_{\kappa\kappa'} = \langle\Phi_\kappa|\hat{H}[\Omega]|\Phi_{\kappa'}\rangle,$$

$$\mathcal{N}_{\kappa\kappa'} = \langle\Phi_\kappa|[\Omega]|\Phi_{\kappa'}\rangle,$$

with

$$[\Omega] = \frac{\hat{R}(\Omega)}{\langle\Phi|\hat{R}(\Omega)|\Phi\rangle}$$

- For example, a norm matrix element

$$\mathcal{N}_{\kappa\kappa'} = \langle\Phi|a_1\cdots a_n[\Omega]a^\dagger_{1'}\cdots a^\dagger_{n'}|\Phi\rangle$$

can be written as combinations of

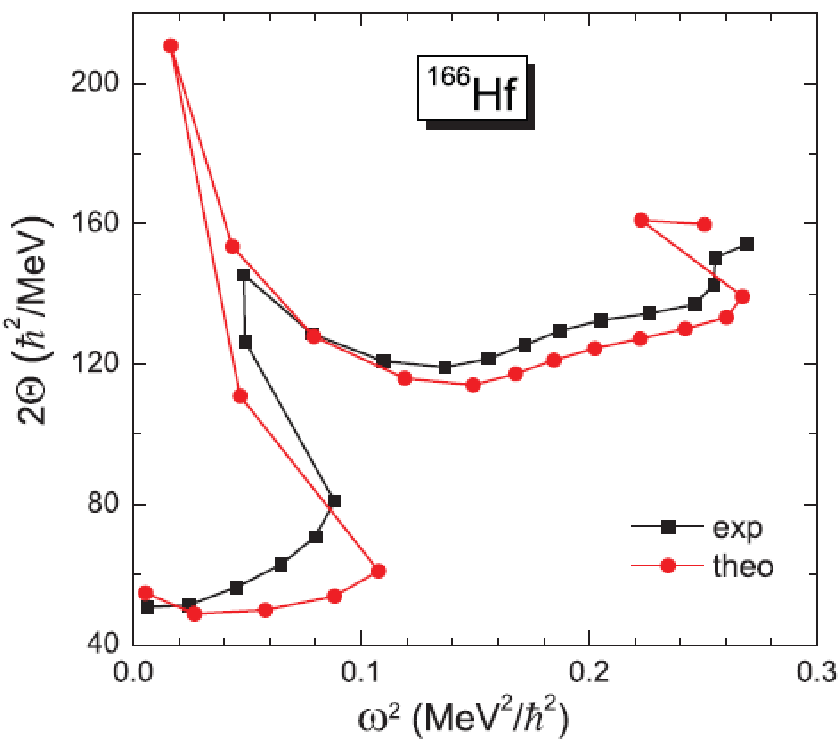
$$A_{\nu\nu'}(\Omega) \equiv \langle\Phi|[\Omega]a^\dagger_\nu a^\dagger_{\nu'}|\Phi\rangle = (V^*(\Omega)U^{-1}(\Omega))_{\nu\nu'}\ ,$$

$$B_{\nu\nu'}(\Omega) \equiv \langle\Phi|a_\nu a_{\nu'}[\Omega]|\Phi\rangle = (U^{-1}(\Omega)V(\Omega))_{\nu\nu'}\ ,$$

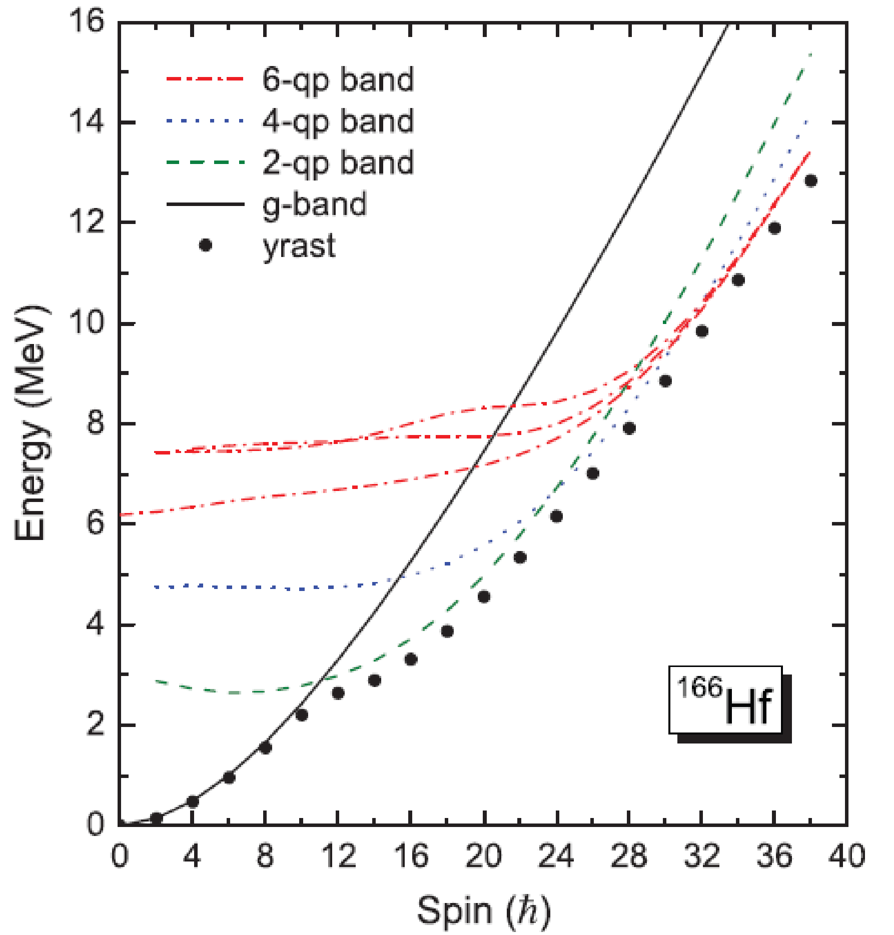
$$C_{\nu\nu'}(\Omega) \equiv \langle\Phi|a_\nu[\Omega]a^\dagger_{\nu'}|\Phi\rangle = (U^{-1}(\Omega))_{\nu\nu'}\ ,$$

Multi-quasiparticle computation using the Pfaffian algorithm

- Calculation of projected matrix elements usually uses the generalized Wick theorem
- A matrix element having n (n') qp creation or annihilation operators respectively on the left- (right-) sides of the rotation operator contains $(n + n - 1)!!$ terms in the expression – a problem of combinatorial complexity
- Use of the Pfaffian algorithm:
  - L.M. Robledo, Phys. Rev. C 79 (2009) 021302(R).
  - L.M. Robledo, Phys. Rev. C 84 (2011) 014307.
  - T. Mizusaki, M. Oi, Phys. Lett. B 715 (2012) 219.
  - M. Oi, T. Mizusaki, Phys. Lett. B 707 (2012) 305.
  - T. Mizusaki, M. Oi, F.-Q. Chen, Y. Sun, Phys. Lett. B 725 (2013) 175



A third band-crossing is described.

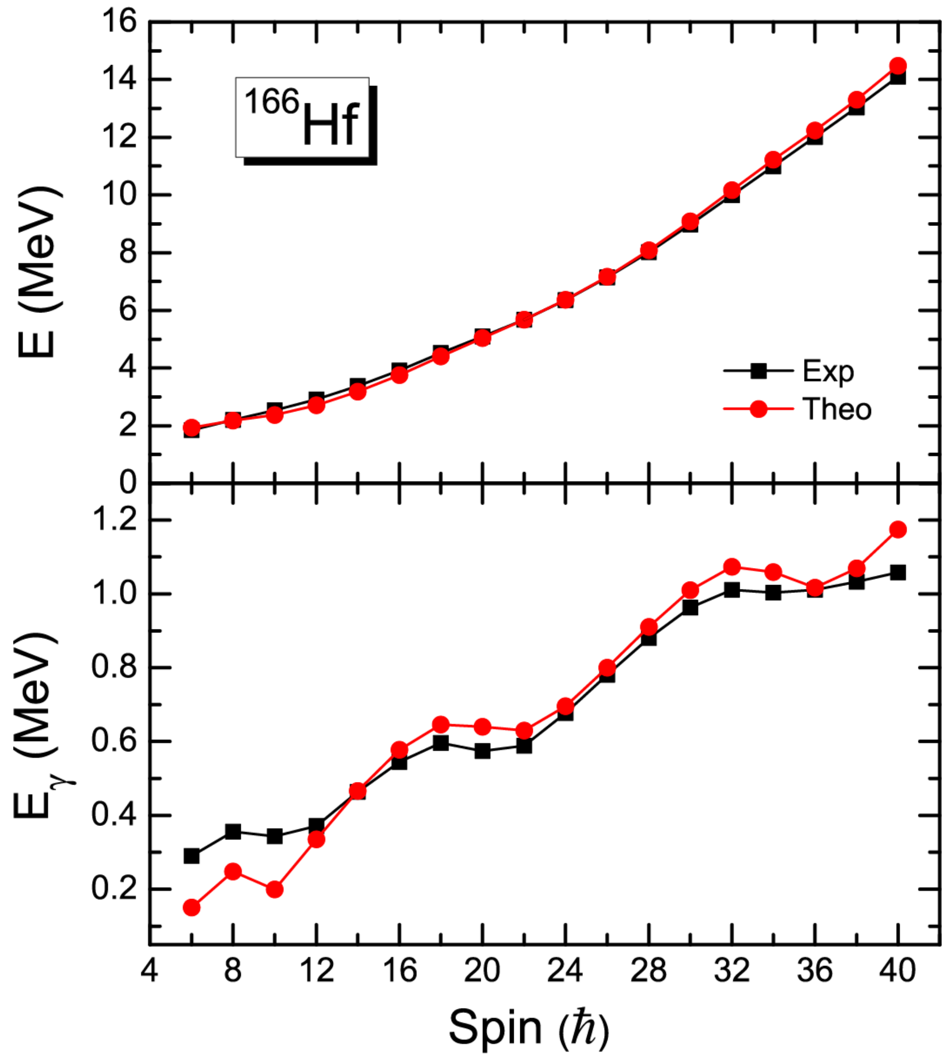


$$\{|\Phi\rangle,\ a^\dagger_{\nu_i}a^\dagger_{\nu_j}|\Phi\rangle,\ a^\dagger_{\pi_i}a^\dagger_{\pi_j}|\Phi\rangle,\ a^\dagger_{\nu_i}a^\dagger_{\nu_j}a^\dagger_{\pi_k}a^\dagger_{\pi_l}|\Phi\rangle,$$
$$\times\ a^\dagger_{\nu_i}a^\dagger_{\nu_j}a^\dagger_{\nu_k}a^\dagger_{\nu_l}|\Phi\rangle,\ a^\dagger_{\pi_i}a^\dagger_{\pi_j}a^\dagger_{\pi_k}a^\dagger_{\pi_l}|\Phi\rangle,$$
$$\times\ a^\dagger_{\nu_i}a^\dagger_{\nu_j}a^\dagger_{\nu_k}a^\dagger_{\nu_l}a^\dagger_{\nu_m}a^\dagger_{\nu_n}|\Phi\rangle,\ a^\dagger_{\pi_i}a^\dagger_{\pi_j}a^\dagger_{\pi_k}a^\dagger_{\pi_l}a^\dagger_{\pi_m}a^\dagger_{\pi_n}|\Phi\rangle,$$
$$\times\ a^\dagger_{\pi_i}a^\dagger_{\pi_j}a^\dagger_{\nu_k}a^\dagger_{\nu_l}a^\dagger_{\nu_m}a^\dagger_{\nu_n}|\Phi\rangle,\ a^\dagger_{\nu_i}a^\dagger_{\nu_j}a^\dagger_{\pi_k}a^\dagger_{\pi_l}a^\dagger_{\pi_m}a^\dagger_{\pi_n}|\Phi\rangle\}$$

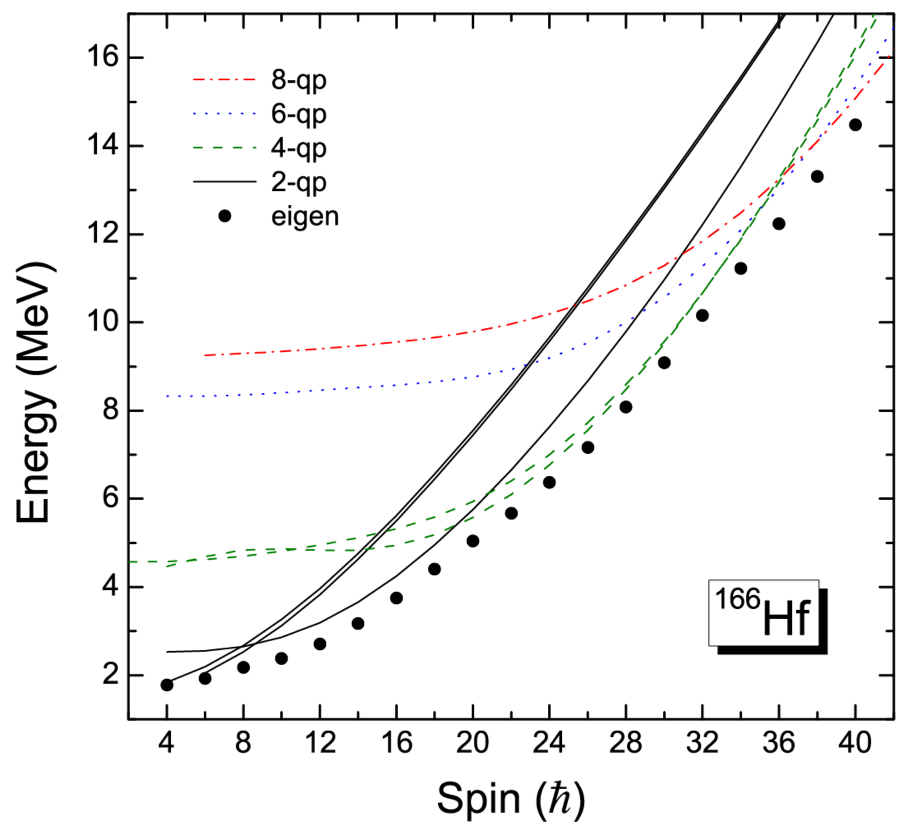
Extension of configuration space to 6-qps.

L.-J. Wang et al. *Phys. Rev.* C90 (2014) 011303(R)

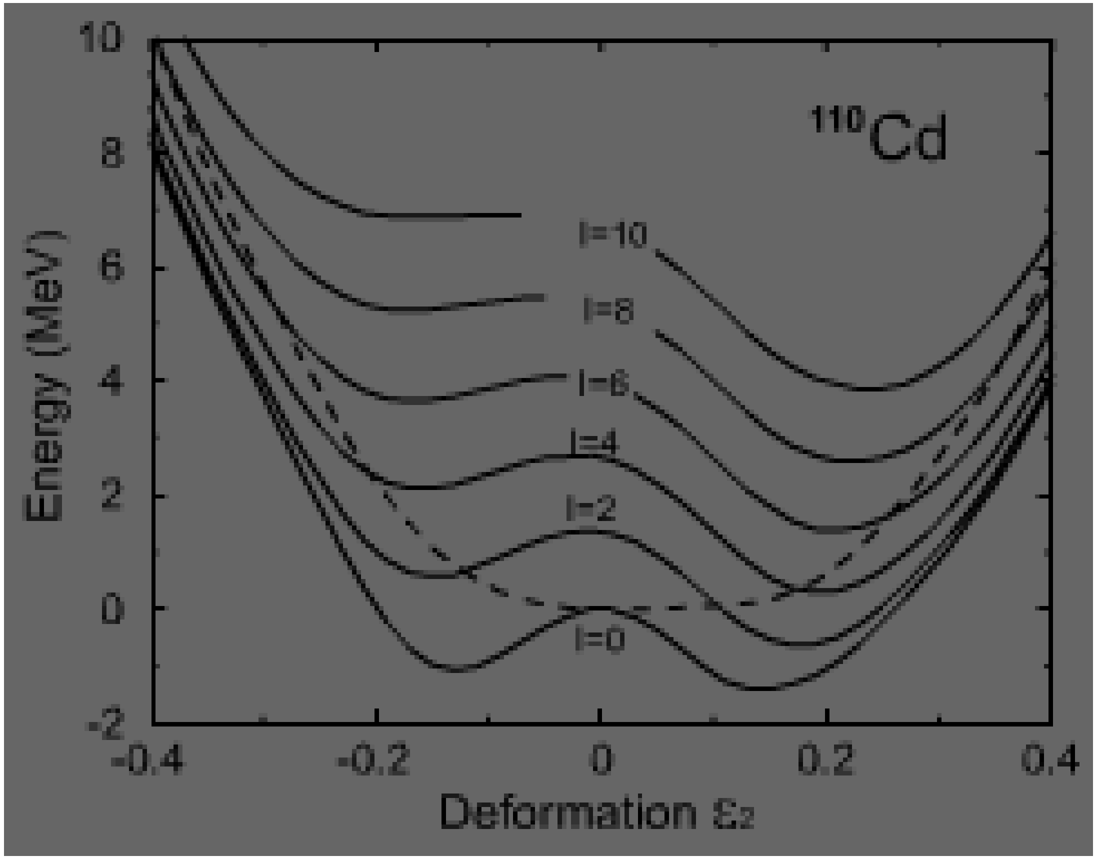
Example for very high-spin states



Calculation including 8-qps
based on a fixed deformation



Example of softness – no definite shapes

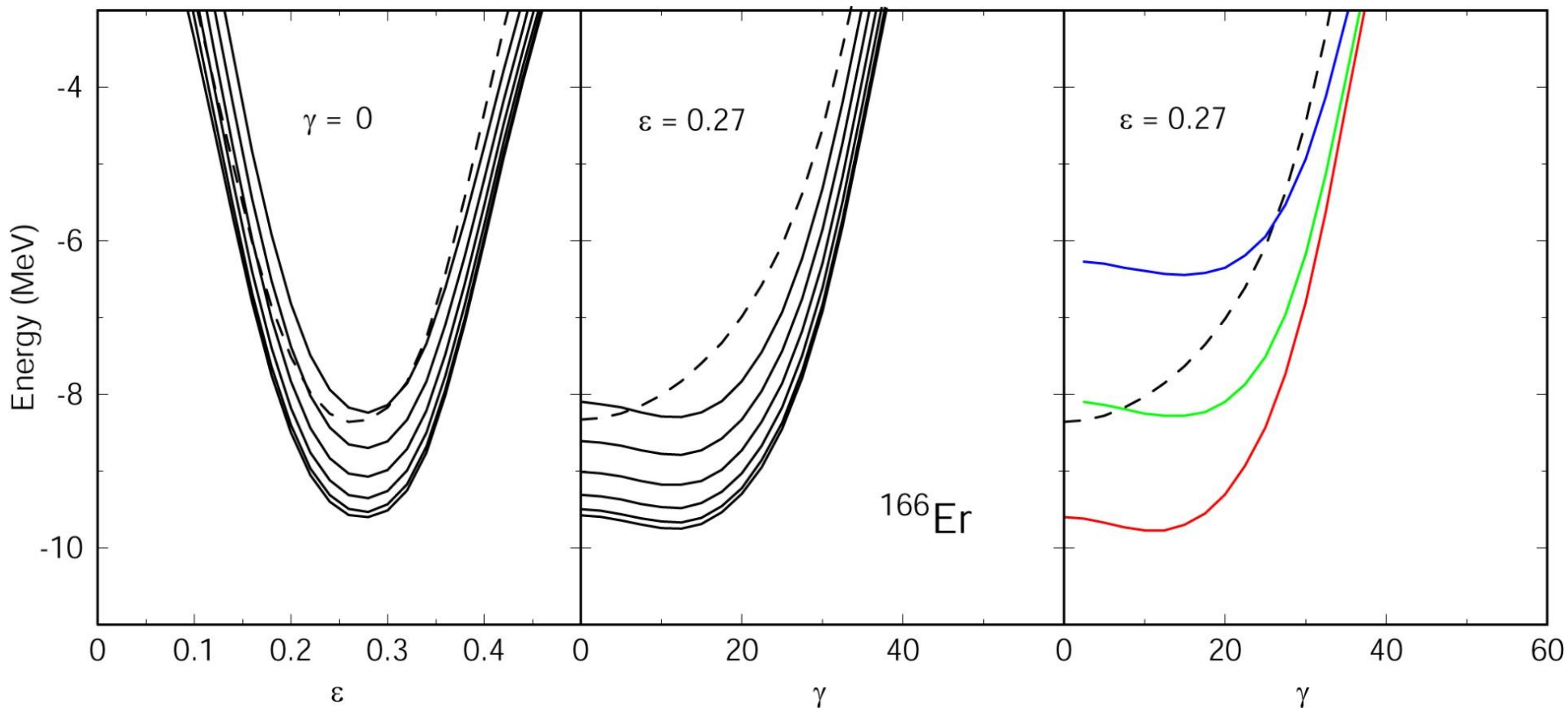


Mean-field calculation shows a spherical shape.

Projected calculation shows shallow minima separated by a low energy barrier.

Shapes may be developed with rotation.

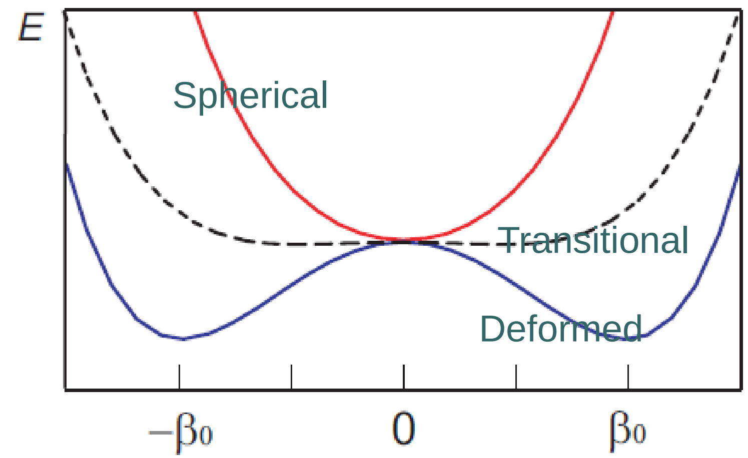
γ-softness in well-deformed nuclei



Angular-momentum-projected energy surfaces as functions of ε and γ

Description of a system with soft potential surfaces

- A spherical nucleus described by spherical shell model.
- A deformed nucleus described by deformed shell model.
- Transitional ones are *difficult*. A better wavefunction is a superposition of many states of deformation parameter β.



Schematic energy potential for spherical (red), transitional (dashed), and deformed (blue) nuclei.

$$|\Psi^I\rangle = \int f^I(\beta)\,|\Phi^I(\beta)\rangle\, d\beta$$

$$|\Phi^I(\beta)\rangle = \hat{P}^I\,|\phi(\beta)\rangle$$

$$\{\beta\} = \{\beta_1, \beta_2, \beta_3, \ldots\}$$

Generate Coordinate Method (GCM)

- GCM starts with a general ansatz for a trail wave function

$$|\Psi\rangle = \int da\, f(a)\, |\Phi(a)\rangle$$

with $\{a\} = a_1, a_2, \ldots, a_i$ being generate coordinates

- $f(a)$ is a weight function, determined by solving the Hill-Wheeler Equation

$$\mathscr{H} f = E \mathscr{N} f$$

with the overlap functions

$$\begin{aligned}\mathscr{H}(a,a') &= \left\langle \Phi(a) \middle| \hat{H} \middle| \Phi(a') \right\rangle , \\ \mathscr{N}(a,a') &= \langle \Phi(a) | \Phi(a') \rangle\end{aligned}$$

Projected Generate Coordinate Method (PGCM)

- Choosing generate coordinate as ε_2, an improved wave function

$$\left|\Psi^{I,N}\right\rangle = \int d\varepsilon_2 f^{I,N}(\varepsilon_2)\left|\Phi^{I,N}(\varepsilon_2)\right\rangle$$

$$\left|\Phi^{I,N}(\varepsilon_2)\right\rangle = \hat{P}^I\hat{P}^N\left|\Phi_0(\varepsilon_2).\right\rangle$$

- Hamiltonian

$$\hat{H} = \hat{H}_0 - \frac{\chi}{2}\sum_\mu \hat{Q}^+_\mu\hat{Q}_\mu - G_M\hat{P}^+\hat{P} - G_Q\sum_\mu \hat{P}^+_\mu\hat{P}_\mu$$

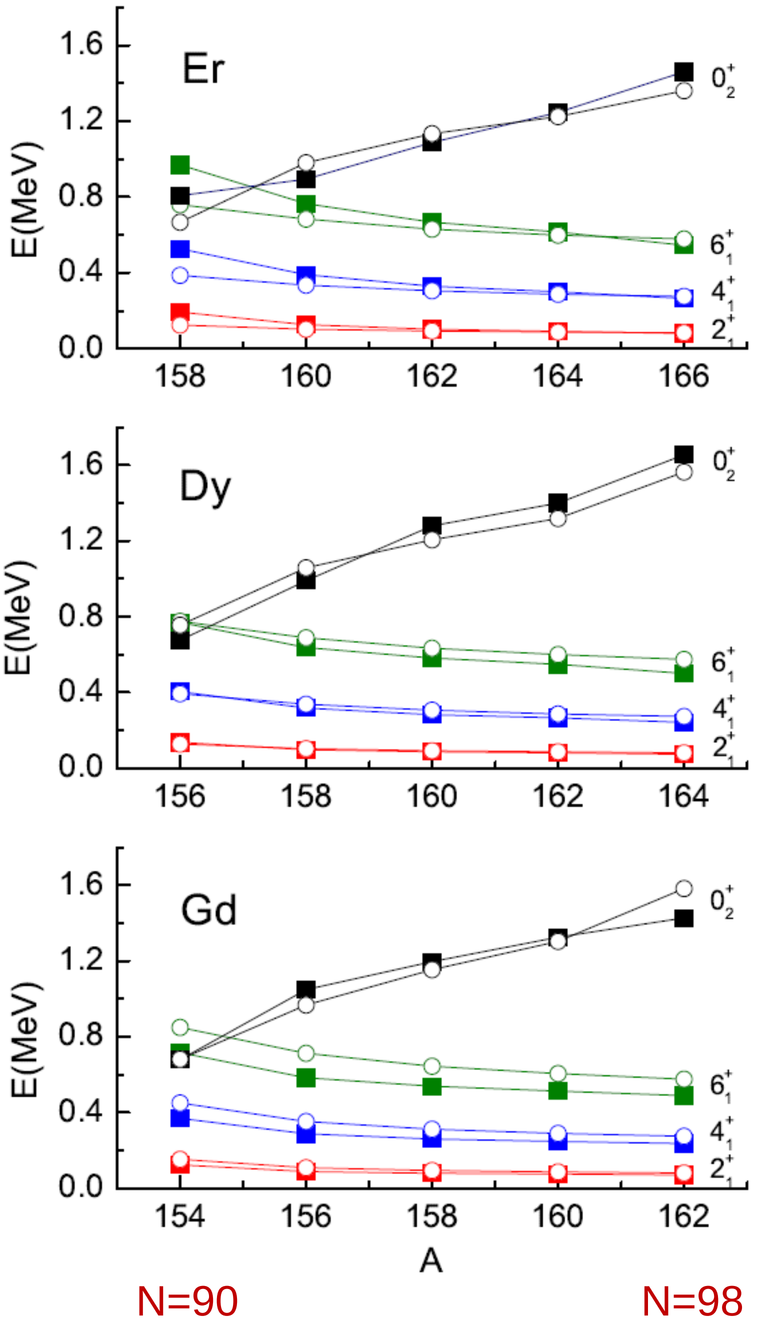
with a fixed set of parameters (fixed χ, G_M, and G_Q) is diagonalized for a chain of isotopes.

F.-Q. Chen, Y. Sun, P. Ring, *Phys. Rev.* C88 (2013) 014315

Energy levels

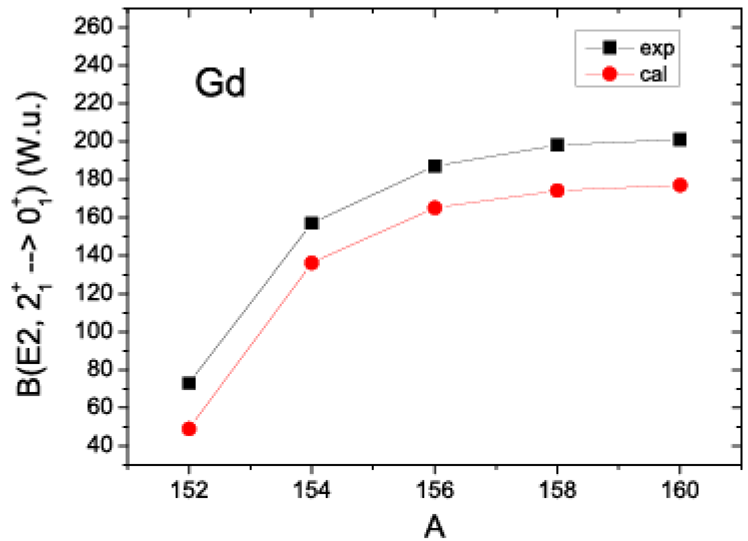
- Comparison of energy levels of 2_1^+, 4_1^+, and 6_1^+ of ground band and excited 0_2^+ state
  - Exp data (filled squares)
  - Calculations (open circles)

for isotopes from N=90 (transitional) to N=98 (well-deformed) nuclei

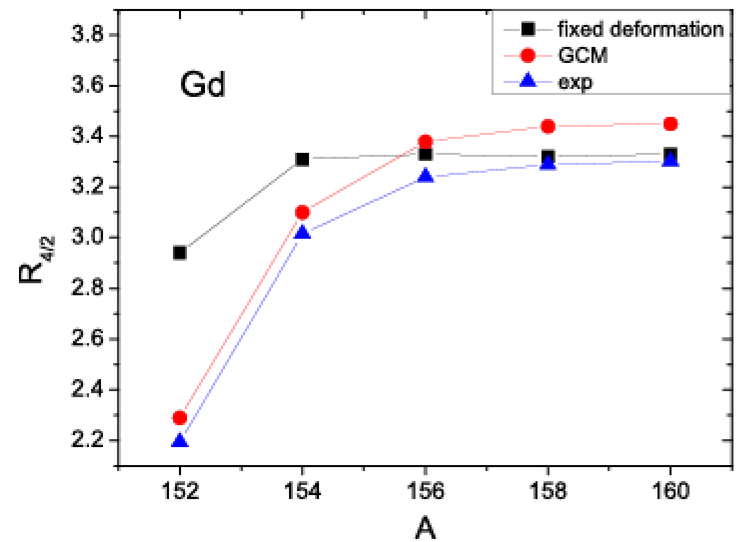


Spherical-deformed shape phase transition

- Drastic changes in electric quadrupole transition B(E2, $2^+ \rightarrow 0^+$) from vibrator ^{152}Gd (N=88), to critical point ^{154}Gd (N=90), to rotor $^{156\text{-}160}$Gd (N>90).



- Black squares show if use only one fixed deformation ε_2 in the calculation, transitional feature cannot be reproduced.



Distribution function

- The Hill-Wheeller Equation diagonalizes the Hamiltonian in a non-orthogonal basis, and therefore, $f(\varepsilon_2)$ is not a proper quantity to analyze the GSM wave function.

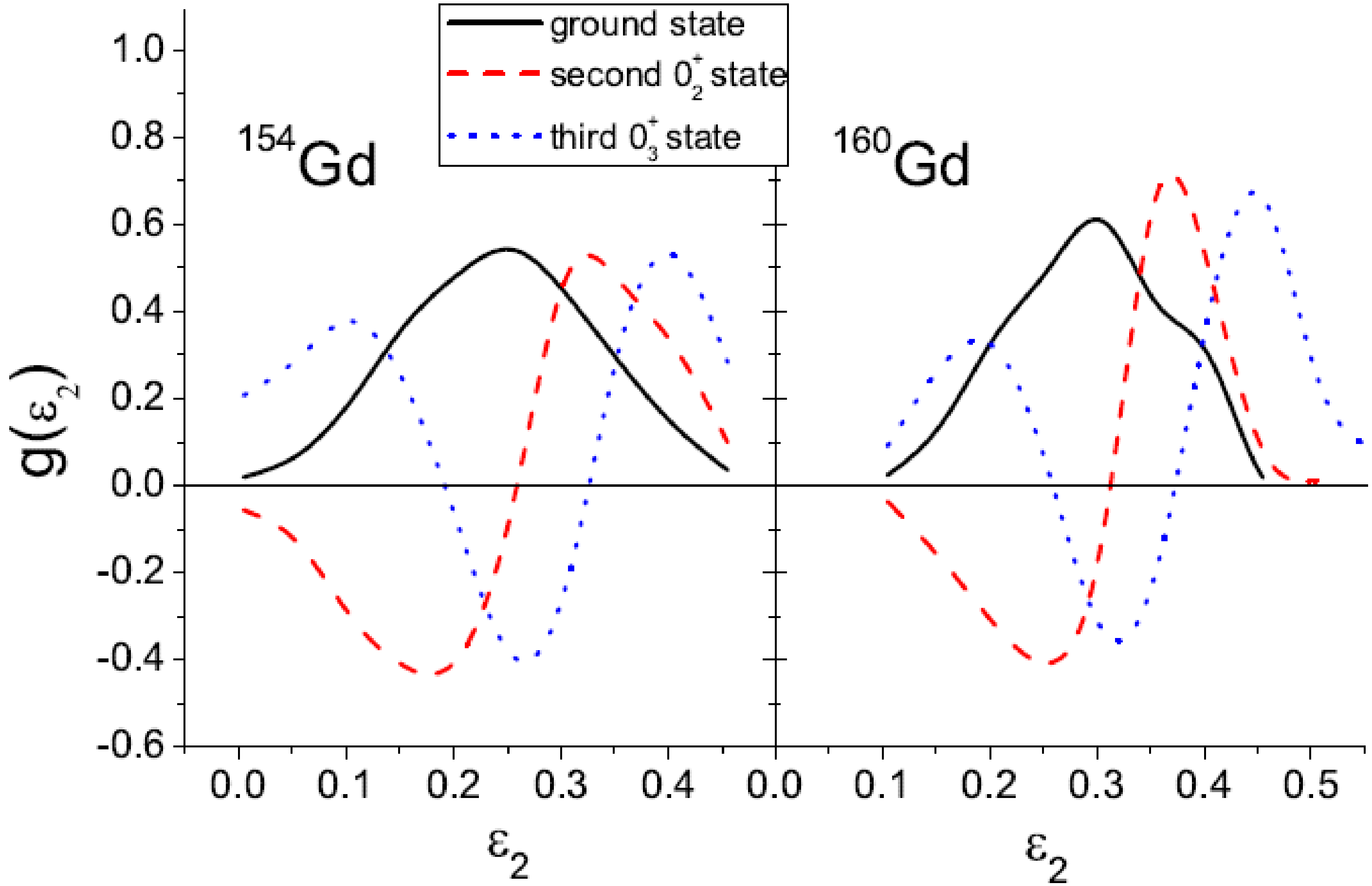
- Transformation of $f(\varepsilon_2)$ to an orthogonal basis gives

$$g(\varepsilon_2) = \int \mathcal{N}^{1/2}(\varepsilon_2, \varepsilon_2') f(\varepsilon_2') d\varepsilon_2'$$

  which can be used to present the distribution of the GCM wave functions.

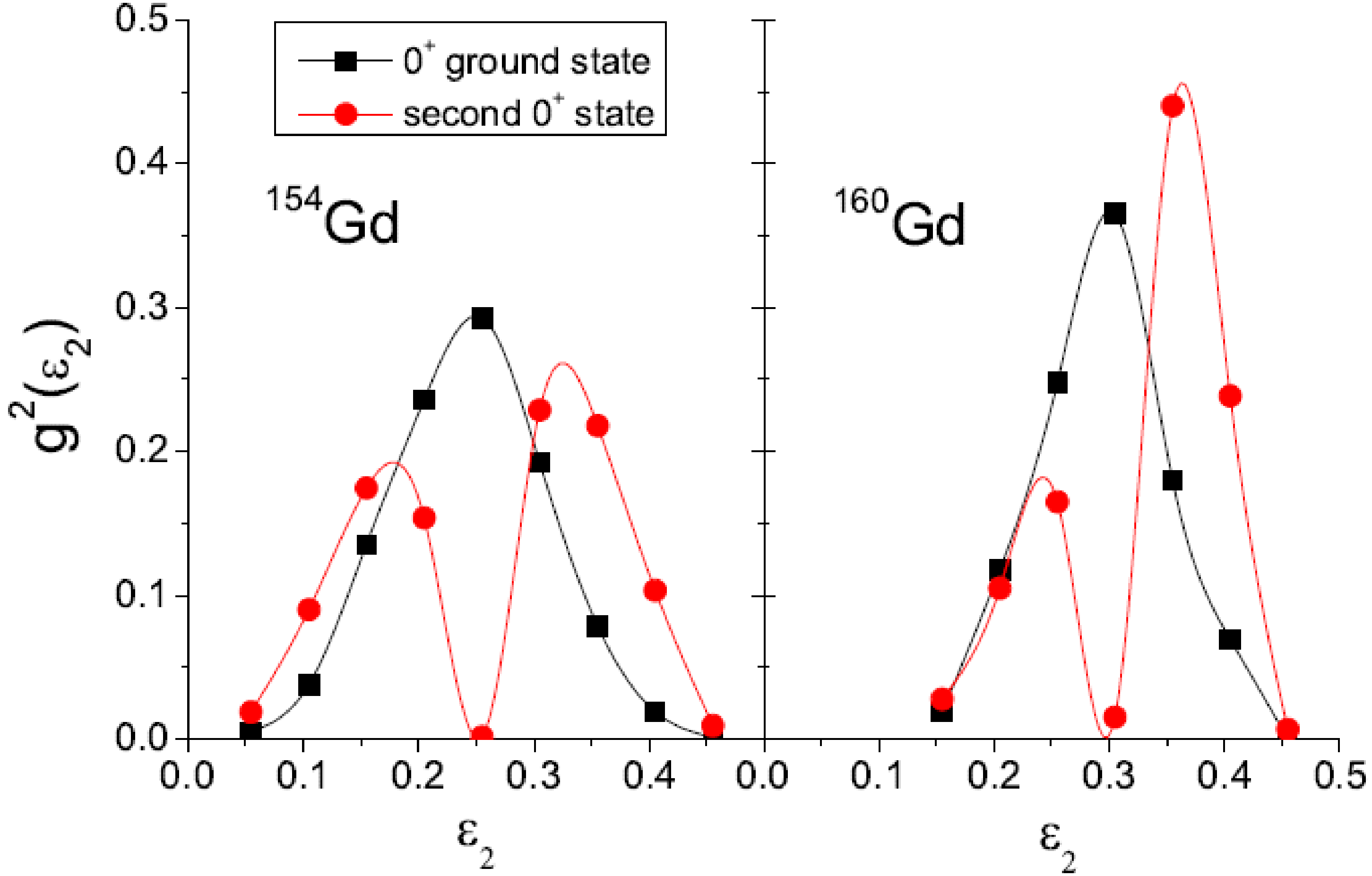
- $g^2(\varepsilon_2)$ represent the probability function.

Distribution function of deformation



Calculated distribution function of deformation for the first three 0^+ states in ^{154}Gd and ^{160}Gd

Probability function of deformation



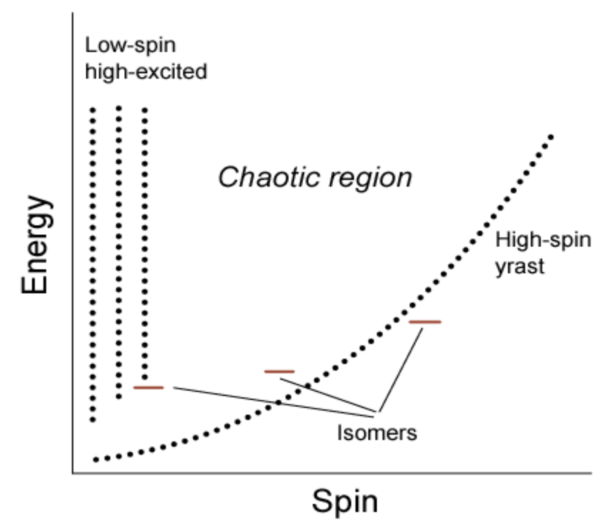
Calculated probability function of deformation for ground state 0_1^+ and excited 0_2^+ state in ^{154}Gd and ^{160}Gd.

Probability function of deformation

- Peak of the Gaussian defines deformation
  - ^{160}Gd being more deformed than ^{154}Gd
- The distribution is wider for ^{154}Gd
  - reflecting the softness of this nucleus
- The distribution for 0_2^+ is much more fragmented
  - reflecting a vibrational nature of these states
- For 0_1^+ , system stays mainly at system's deformation with the largest probability
- For 0_2^+ , system shows two peaks having different heights lying separately at both sides of the equilibrium
  - indicating an anharmonic oscillation
  - prefering to have a larger probability in the site of larger deformation

β-decay & electron-capture in stars (with temperature)

- Stellar weak-interaction rates are important for resolving astrophysical problems
  - for nucleosynthesis calculations
  - for core collapse supernova modeling
- Calculation of transition matrix element
  - essentially a nuclear structure problem
  - necessary to connect thermally excited parent states with many daughter states
  - for both allowed and forbidden GT transitions



Stellar enhancement of decay rate

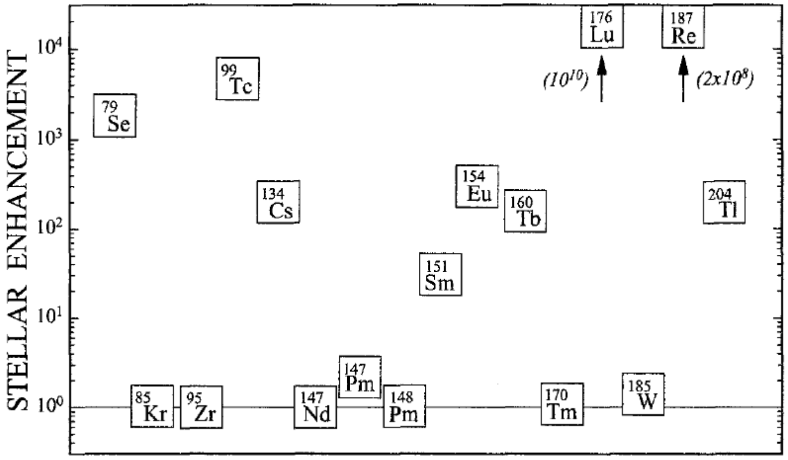
- A stellar enhancement can result from the thermal population of excited states

$$\lambda_\beta = \sum_i \left(p_i \times \sum_j \lambda_{\beta ij} \right)$$

$$p_i = \frac{(2I_i + 1) \times \exp(-E_i / kT)}{\sum_m (2I_m + 1) \times \exp(-E_m / kT)}$$

- Examples in the s-process

F. Kaeppeler,
Prog. Part. Nucl. Phys. 43 (1999) 419



Transition matrix elements in the projected basis

- Gamow-Teller rate $$B(GT)=\frac{2I_f+1}{2I_i+1}\left\langle \psi_{I_f}\middle|\hat{\beta}^{\pm}\middle|\psi_{I_i}\right\rangle^2$$
- Wavefunction $$\psi_M^I=\sum_\kappa f_\kappa \hat{P}^I_{MK_\kappa}\left|\phi_\kappa\right\rangle$$
- e-e system $$\left|\phi_e(\varepsilon_e)\right\rangle=\left\{\left|\varepsilon_e\right\rangle, b_\nu^+b_\nu^+\left|\varepsilon_e\right\rangle, b_\pi^+b_\pi^+\left|\varepsilon_e\right\rangle, b_\nu^+b_\nu^+b_\pi^+b_\pi^+\left|\varepsilon_e\right\rangle,\cdots\right\}$$
- o-o system $$\left|\phi_o(\varepsilon_o)\right\rangle=\left\{a_\nu^+a_\pi^+\left|\varepsilon_o\right\rangle, a_\nu^+a_\nu^+a_\nu^+a_\pi^+\left|\varepsilon_o\right\rangle, a_\nu^+a_\pi^+a_\pi^+a_\pi^+\left|\varepsilon_o\right\rangle,\cdots\right\}$$
- Overlapping matrix element (K. Tanabe *et al*., *PRC* 59 (1999) 2494).

$$\left\langle \phi_o(\varepsilon_o)\middle|\hat{O}\hat{P}^I_{K_oK_e}\middle|\phi_e(\varepsilon_e)\right\rangle \sim \int d\Omega D^I_{K_oK_e}(\Omega)\left\langle \phi_o(\varepsilon_o)\middle|\hat{O}\hat{R}(\Omega)\middle|\phi_e(\varepsilon_e)\right\rangle$$

The interactions

- Total Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_{QP} + \hat{H}_{GT}$
- Quadrupole + monopole-pairing + quadrupole-pairing

$$\hat{H}_{QP} = -\frac{1}{2}\chi_{QQ}\sum_{\mu}\hat{Q}^{\dagger}_{2\mu}\hat{Q}_{2\mu} - G_M\hat{P}^{\dagger}\hat{P} - G_Q\sum_{\mu}\hat{P}^{\dagger}_{2\mu}\hat{P}_{2\mu}$$

- Charge-exchange (Gamow-Teller)

$$\hat{H}_{GT} = +2\chi_{GT}\sum_{\mu}\hat{\beta}^{-}_{1\mu}(-1)^{\mu}\hat{\beta}^{+}_{1-\mu} - 2\kappa_{GT}\sum_{\mu}\hat{\Gamma}^{-}_{1\mu}(-1)^{\mu}\hat{\Gamma}^{+}_{1-\mu}$$

$$\hat{\beta}^{-}_{1\mu} = \sum_{\pi,\nu}\langle\pi|\sigma_{\mu}\tau_{-}|\nu\rangle c^{\dagger}_{\pi}c_{\nu}, \quad \hat{\beta}^{+}_{1\mu} = (-)^{\mu}(\beta^{-}_{1-\mu})^{\dagger}$$

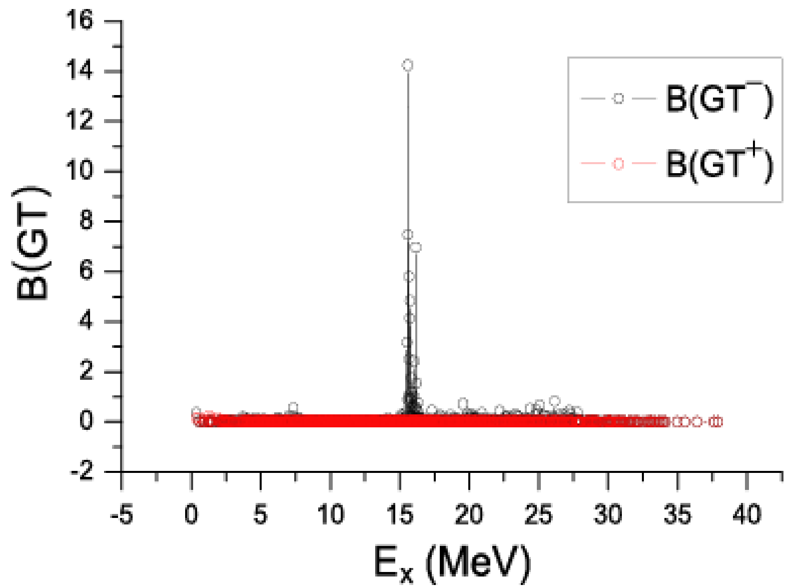
$$\hat{\Gamma}^{-}_{1\mu} = \sum_{\pi,\nu}\langle\pi|\sigma_{\mu}\tau_{-}|\nu\rangle c^{\dagger}_{\pi}c^{\dagger}_{\bar{\nu}}, \quad \hat{\Gamma}^{+}_{1\mu} = (-)^{\mu}(\Gamma^{-}_{1-\mu})^{\dagger}$$

- Kuz'min & Soloviev, *Nucl. Phys*. A 486 (1988) 118

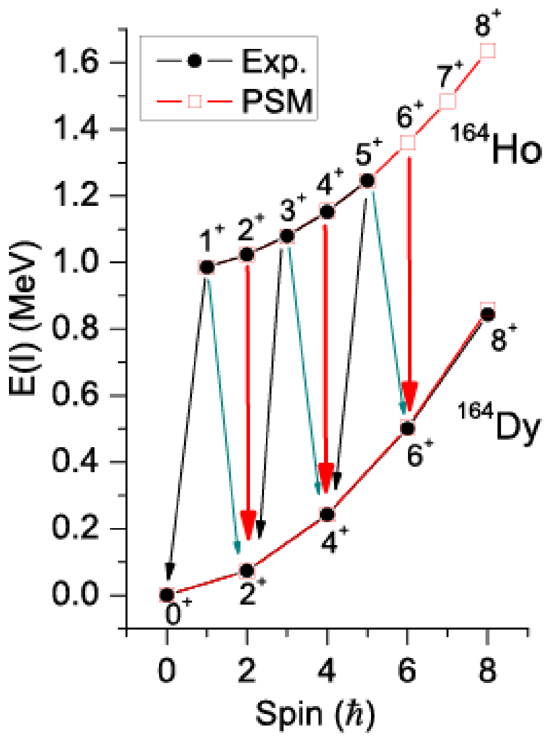
Distribution of B(GT)

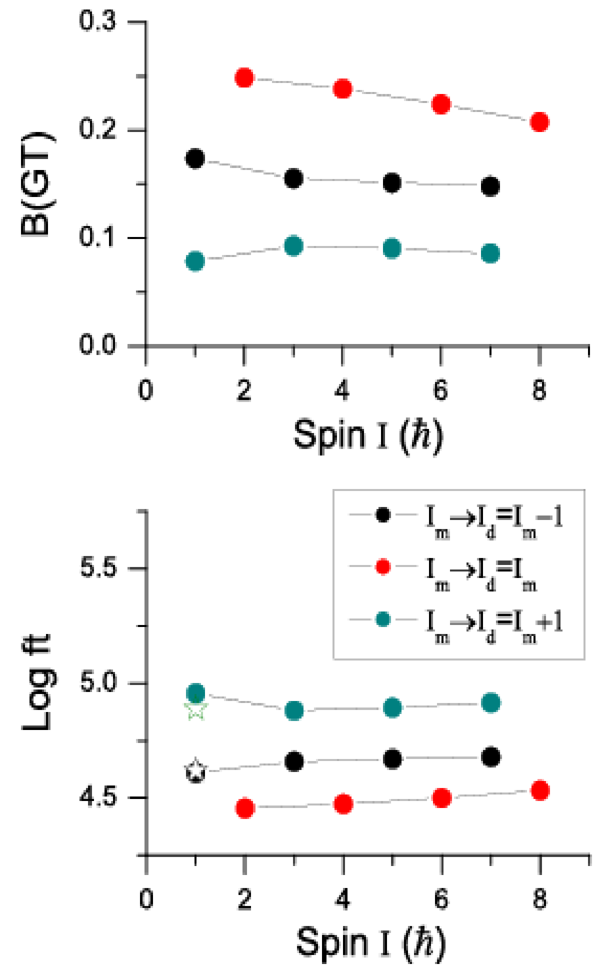
- Initial state: ground state in even-even nucleus
- Final states: all 1^+ states in odd-odd nucleus
- Ikeda sum-rule fulfilled

$$
\begin{aligned}
& S(\mathrm{GT}^-) - S(\mathrm{GT}^+) \\
= & \sum_f B(GT^-, i \rightarrow f) - \sum_f B(GT^+, i \rightarrow f) \\
= & \sum_{f,\mu} |\langle \Psi_f | \hat{\beta}^-_{1\mu} | \Psi_i \rangle|^2 - \sum_{f,\mu} |\langle \Psi_f | \hat{\beta}^+_{1\mu} | \Psi_i \rangle|^2 \\
= & 3(N - Z).
\end{aligned}
$$



B(GT) and log*ft* in $^{164}Ho \rightarrow {}^{164}Dy$





Z.-C. Gao, Y. Sun, Y.-S. Chen, *PRC 74* (2006) 054303

Summary

- Angular momentum projection is an efficient way to approach the nuclear many-body problem with the shell model concept.
- Projected Shell Model is a practical example.
  - Start from Nilsson + BCS quasiparticle states
  - Perform angular-momentum-projection on (multi-quasiparticle) states
  - Improve the PSM wave function by superimposing projected states with different deformation
  - Diagonalize the Hamiltonian in the projected basis
- Phaffian algorithm can help to simplify numerical calculations
  - Computer code can be developed when large number of quasiparticle excitations are included.

Collaboration

- (Students) Y.-C. Yang （杨迎春）， Y.-X. Liu （刘艳鑫）
  H. Jin （金华）， F.-Q. Chen （陈芳祁）
  L.-J. Wang （王龙军）， Q.-L. Hu （胡庆丽）
- (China) Z.-C. Gao （高早春）
- (Japan) T. Mizusaki （水崎高浩）
  M. Oi
- (USA) M. Guidry
- (Germany) P. Ring