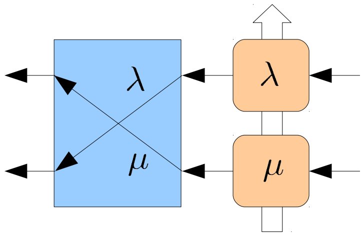
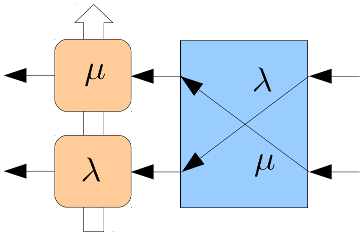


The Algebraic Bethe Ansatz And Tensor Network States



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Phys. Rev. B 86, 045125 (2012)

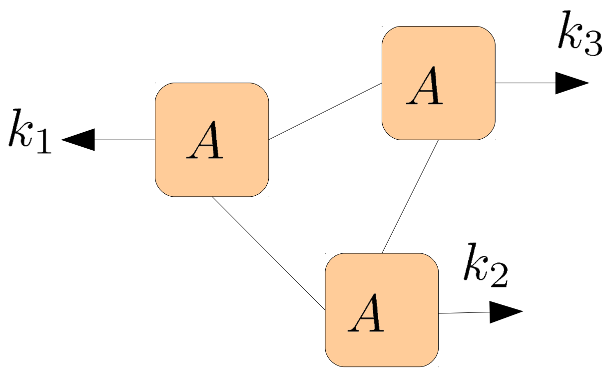
Tensor Network States ↔ Algebraic Bethe Ansatz:

Quantum Information Theory

Tensor Network States (MPS,PEPS,...)

$$|\Psi\rangle = \sum_{k_1,\ldots,k_N} \mathcal{F}(\{A_{k_j}\})|k_1,\cdots,k_n\rangle$$

$$\mathcal{F}(\{A_{k_j}\}) =$$



- Entanglement structure corresponding to tensor-network structure.
- Good variational states.
- Expectation values can be calculated approximately numerically.

Condensed Matter Theory

Integrability:

Algebraic Bethe-Ansatz

Renormalization Group Methods:

NRG, DMRG

Tensor Network States ↔ Algebraic Bethe Ansatz:

Quantum Information Theory

Condensed Matter Theory

Tensor Network States (MPS,PEPS,...)

$$|\Psi\rangle = \sum_{k_1,\ldots,k_N} \mathcal{F}(\{A_{k_j}\})|k_1,\cdots,k_n\rangle$$

Integrability:

Algebraic Bethe-Ansatz

Aim:

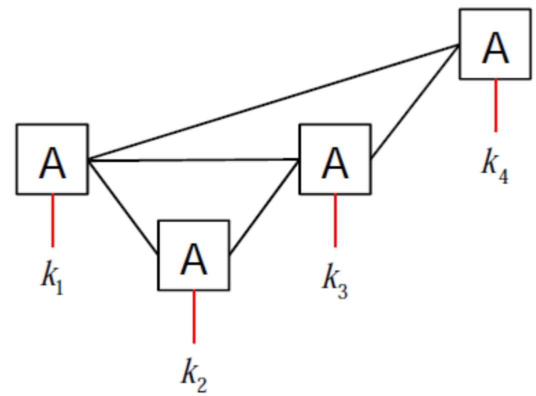
- Express Bethe-Ansatz wavefunction as Tensor Network State.
- Use Tensor Network State to calculate correlations approximately numerically.

- Entanglement structure corresponding to tensor-network structure.
- Good variational states.
- Expectation values can be calculated approximately numerically.

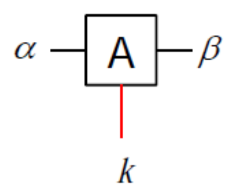
Tensor Network States

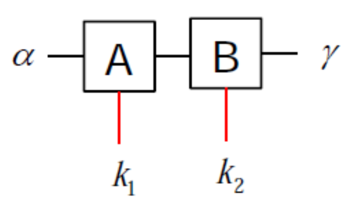
Tensor Network Ansatz: $|\Psi> \;=\; \sum_{k_1 \ldots k_M} C_{k_1 \ldots k_M} \,|\, k_1 \ldots k_M >$

Coefficients: $C_{k_1 \ldots k_M} \;=$



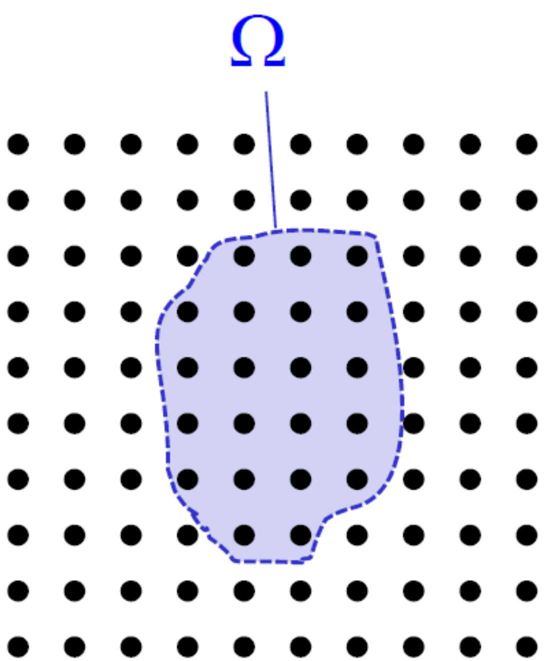
$A^k_{\alpha\beta} \;=$





$= \sum_{\beta} A^{k_1}_{\alpha\beta} B^{k_2}_{\beta\gamma}$

Local **Hamiltonians*:***



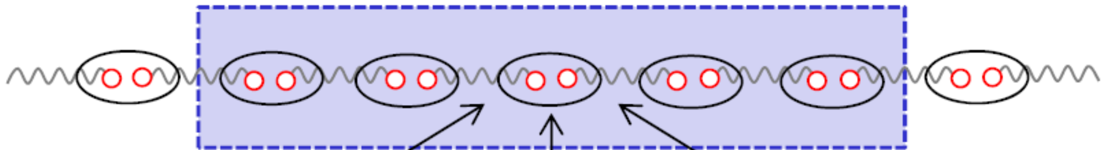
Ground State of ***local*** Hamiltonians

- Area Law: $S(\rho_\Omega) \propto \partial\Omega$
- 1D $\longleftrightarrow$ $\begin{cases} \text{Finite Correlation Length} \\ \text{Energy Gap} \end{cases}$

Tensor Network States

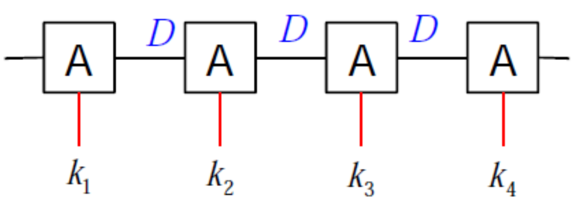
Local Hamiltonians in ***one*** dimension***:*** ***Matrix Product States (MPS)***

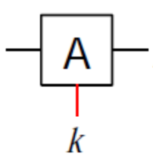
$$S(\rho_\Omega) = const$$



$$\sum_{\alpha=1}^{D} |\alpha> \otimes |\alpha> \qquad \sum_{\beta=1}^{D} |\beta> \otimes |\beta>$$

$$\sum_{k=1}^{d} \sum_{\alpha\beta=1}^{D} A^{k}_{\alpha\beta} \, |k><\alpha| \otimes <\beta|$$

$|\Psi> \;=\; \sum_{k_1 \ldots k_M}$  $|k_1 \ldots k_M>$

$A^{k}_{\alpha\beta} \;=\; \alpha$  β

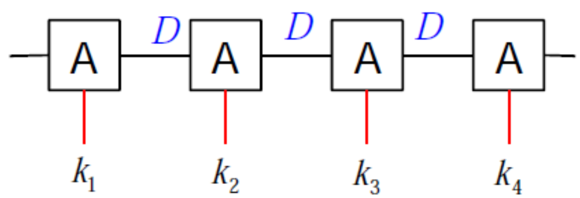
$$= \sum_{k_1 \ldots k_M} tr\left[A^{k_1} \cdots A^{k_M} \right] |k_1 \ldots k_M>$$

Tensor Network States

Local **Hamiltonians in *one* dimension:** ***Matrix Product States (MPS)***

$$|\Psi> \;=\; \sum_{k_1\ldots k_M} tr\left[A^{k_1}\cdots A^{k_M}\right]|k_1\ldots k_M>$$

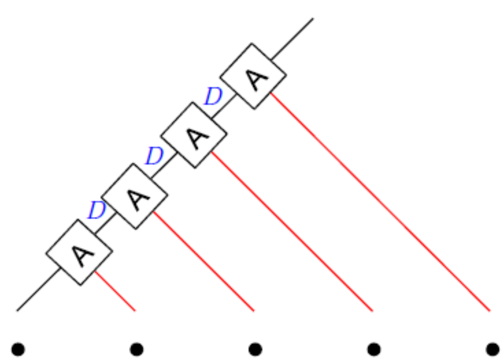
$$= \sum_{k_1\ldots k_M}$$



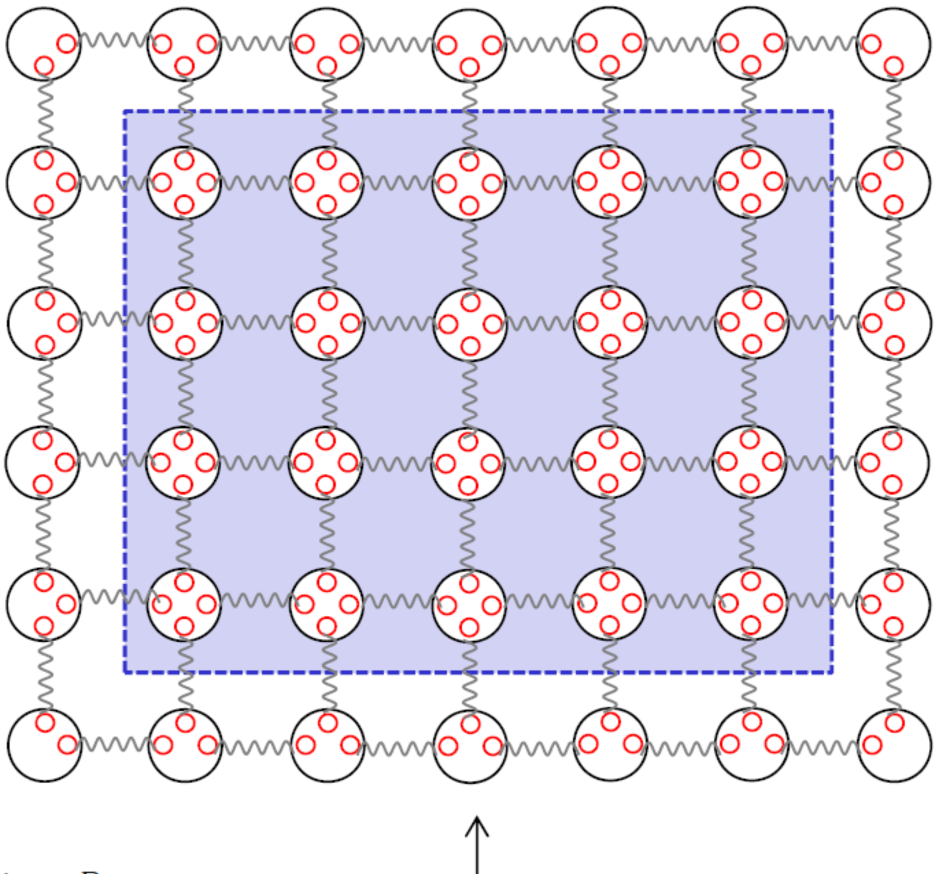
$$|k_1\ldots k_M>$$

- Every state can be represented as MPS with D ~ exp(N).
- Correlations decay exponentially: $< O_i O_{i+\Delta} > - < O_i >< O_{i+\Delta} > \propto e^{-\Delta/\zeta}$
- Ground states of gapped local Hamiltonians can be represented as MPS with D ~ poly(N).
- Renormalization:

  NRG, DMRG



Local Hamiltonians in ***two*** dimensions***:*** ***Projected Entangled Pair States (PEPS)***



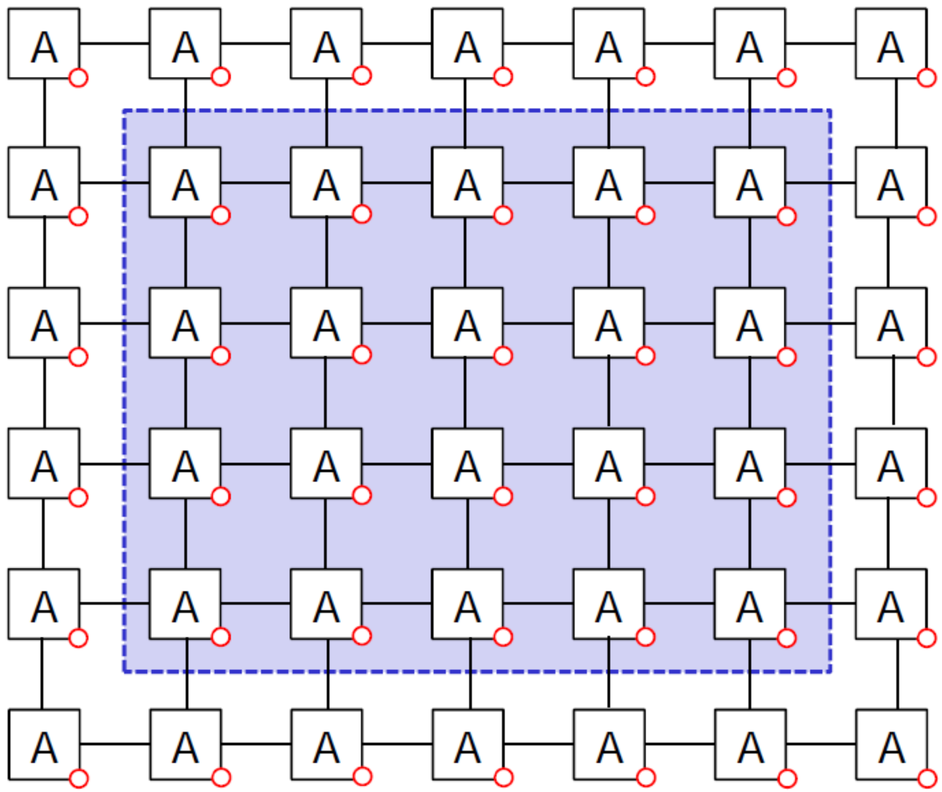
$$S(\rho_{\Omega}) = \partial\Omega$$

$$\sum_{k=1}^{d} \sum_{\alpha\beta\gamma\delta=1}^{D} A^{k}_{\alpha\beta\gamma\delta} \,|\, k >< \alpha \,|\otimes < \beta \,|\otimes < \gamma \,|\otimes < \delta \,|$$

Tensor Network States

Local **Hamiltonians in *two* dimensions*:*** ***Projected Entangled Pair States (PEPS)***

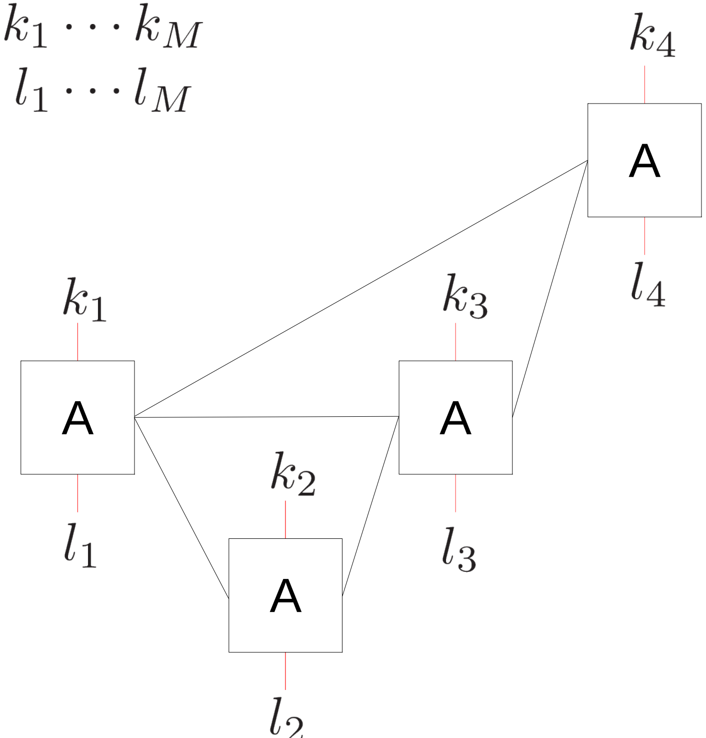
$|\Psi> \quad =$



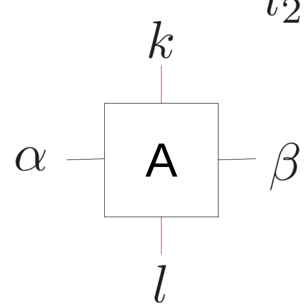
Tensor Network Operators

Tensor Network Ansatz: $$\hat{O} = \sum_{\substack{k_1 \cdots k_M \\ l_1 \cdots l_M}} C^{k_1 \cdots k_M}_{l_1 \cdots l_M} \, | \, k_1 \cdots k_M \, \rangle \langle \, l_1 \cdots l_M \, |$$

Coefficients: $$C^{k_1 \cdots k_M}_{l_1 \cdots l_M} =$$



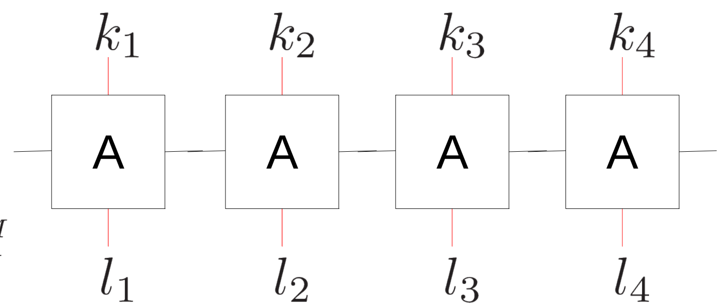
$$A^{kl}_{\alpha\beta} =$$



Operator in ***one*** dimension*:*

Matrix Product Operator (MPO)

$$\hat{O} = \sum_{\substack{k_1 \cdots k_M \\ l_1 \cdots l_M}}$$



$$|\, k_1 \cdots k_M \,\rangle\langle\, l_1 \cdots l_M \,|$$

$$= \sum_{\substack{k_1 \cdots k_M \\ l_1 \cdots l_M}} \mathrm{tr}\left[A^{k_1 l_1} \cdots A^{k_M l_M}\right] |\, k_1 \cdots k_M \,\rangle\langle\, l_1 \cdots l_M \,|$$

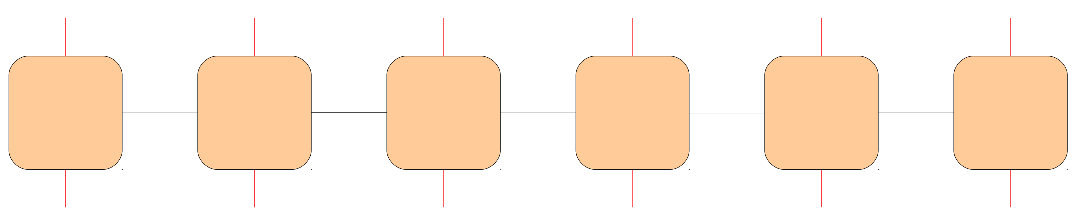
Time evolution in ***one*** dimension*:*

$$H = \sum_{j=1}^{N-1} h^{(j,j+1)} = H_{\mathrm{even}} + H_{\mathrm{odd}}$$

$$\begin{cases} H_{\mathrm{even}} = \sum_{j \text{ even}} h^{(j,j+1)} \\ H_{\mathrm{odd}} = \sum_{j \text{ odd}} h^{(j,j+1)} \end{cases}$$

$$e^{-i\delta t H} \approx e^{-i\delta t H_{\mathrm{odd}}} e^{-i\delta t H_{\mathrm{even}}}$$

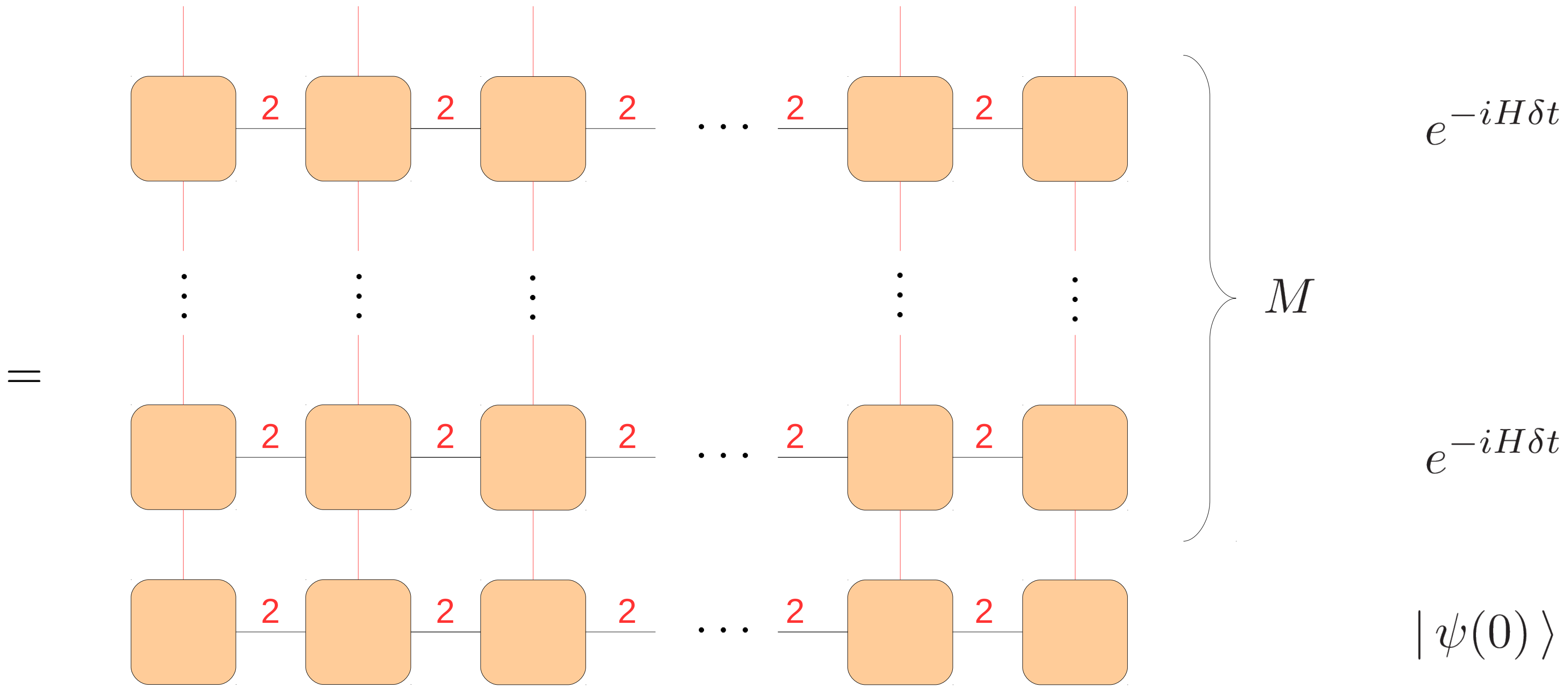
$$=$$



Time Evolution: Numerical Approximation

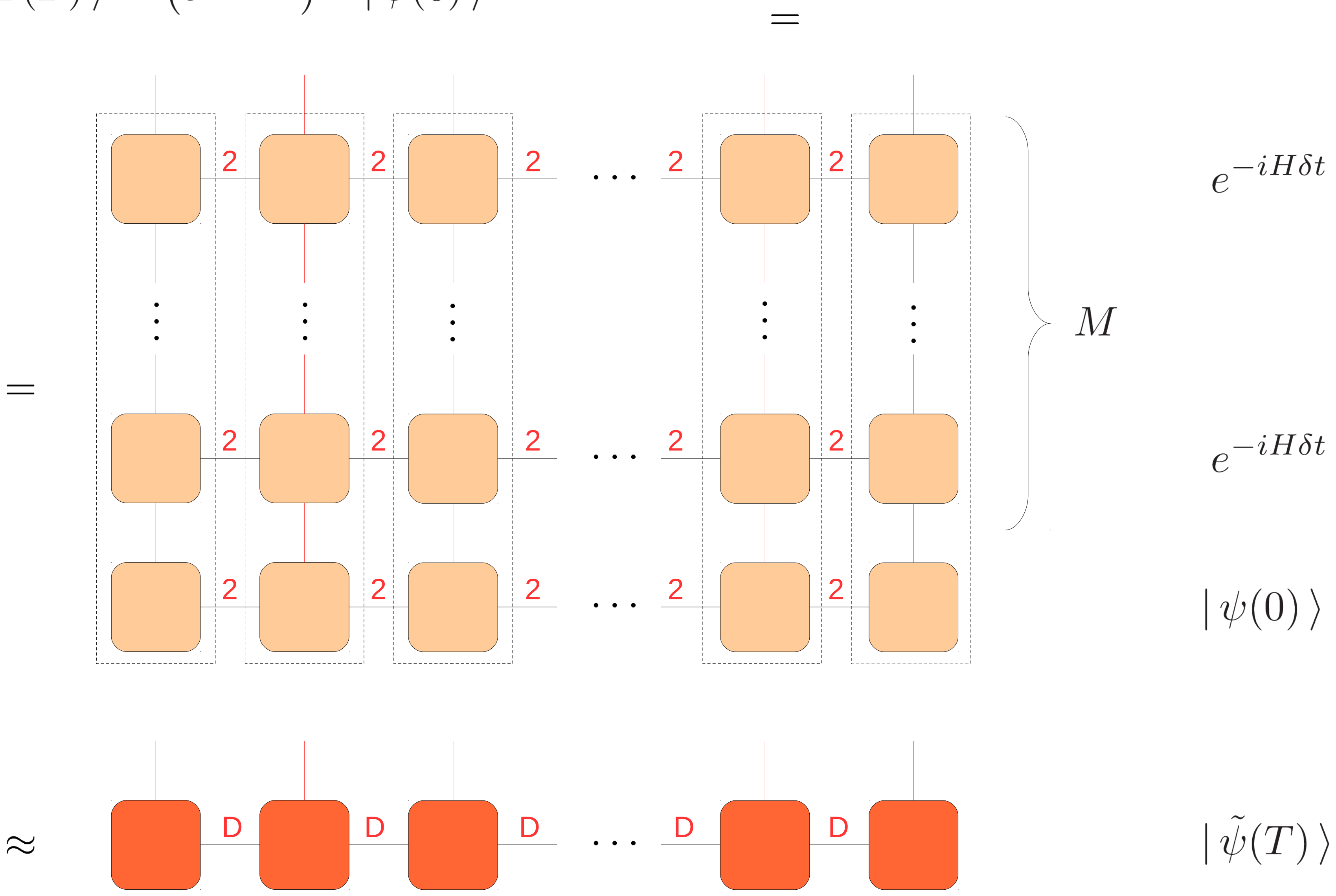
$$|\,\Psi(T)\,\rangle = \left(e^{-iH\delta t}\right)^{M}|\,\psi(0)\,\rangle$$

$$=$$



Time Evolution: Numerical Approximation

$$|\,\Psi(T)\,\rangle = \left(e^{-iH\delta t}\right)^{M}|\,\psi(0)\,\rangle$$



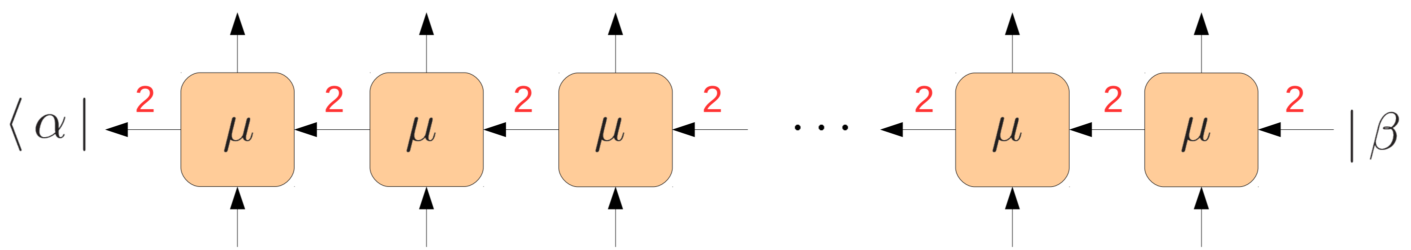
Algebraic Bethe Ansatz – XXZ Model

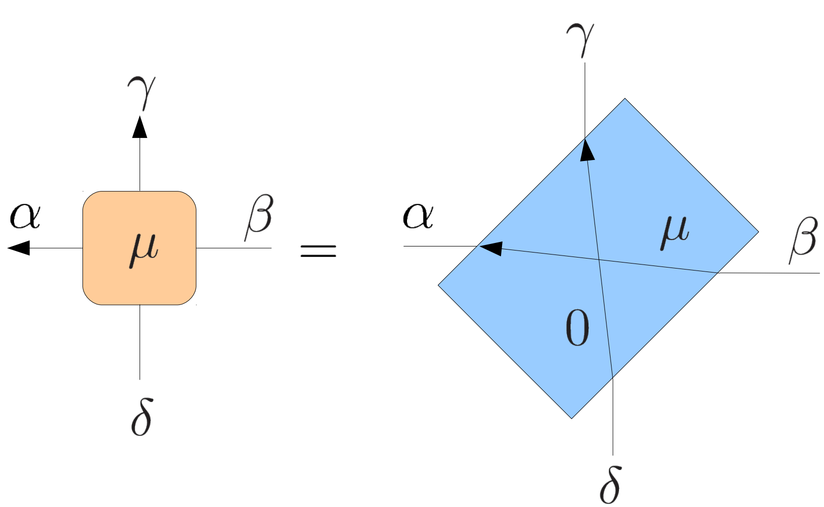
$$H = \sum_{j=1}^{N} \left(s_x^{(j)} s_x^{(j+1)} + s_y^{(j)} s_y^{(j+1)} + \Delta s_z^{(j)} s_z^{(j+1)} \right)$$

Monodromy Matrix

$$T(\lambda) = L_L(\lambda) L_{L-1}(\lambda) \cdots L_1(\lambda) \qquad T(\lambda) = \begin{pmatrix} A(\lambda) & C(\lambda) \\ B(\lambda) & D(\lambda) \end{pmatrix}$$

$T(\mu)^{\alpha}_{\beta} =$





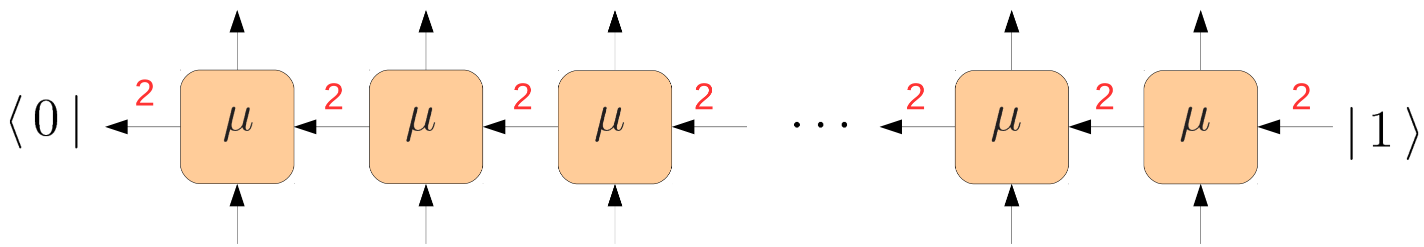
$$R(\lambda, \mu) = \begin{pmatrix} 1 & & & \\ & b(\lambda-\mu) & c(\lambda-\mu) & \\ & c(\lambda-\mu) & b(\lambda-\mu) & \\ & & & 1 \end{pmatrix}$$

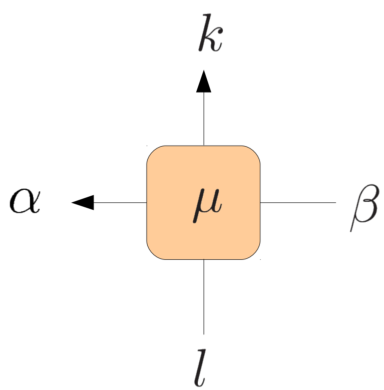
Algebraic Bethe Ansatz – XXZ Model

$$H = \sum_{j=1}^{N} \left(s_x^{(j)} s_x^{(j+1)} + s_y^{(j)} s_y^{(j+1)} + \Delta s_z^{(j)} s_z^{(j+1)} \right)$$

Creation operator for one down-spin

$B(\mu) =$





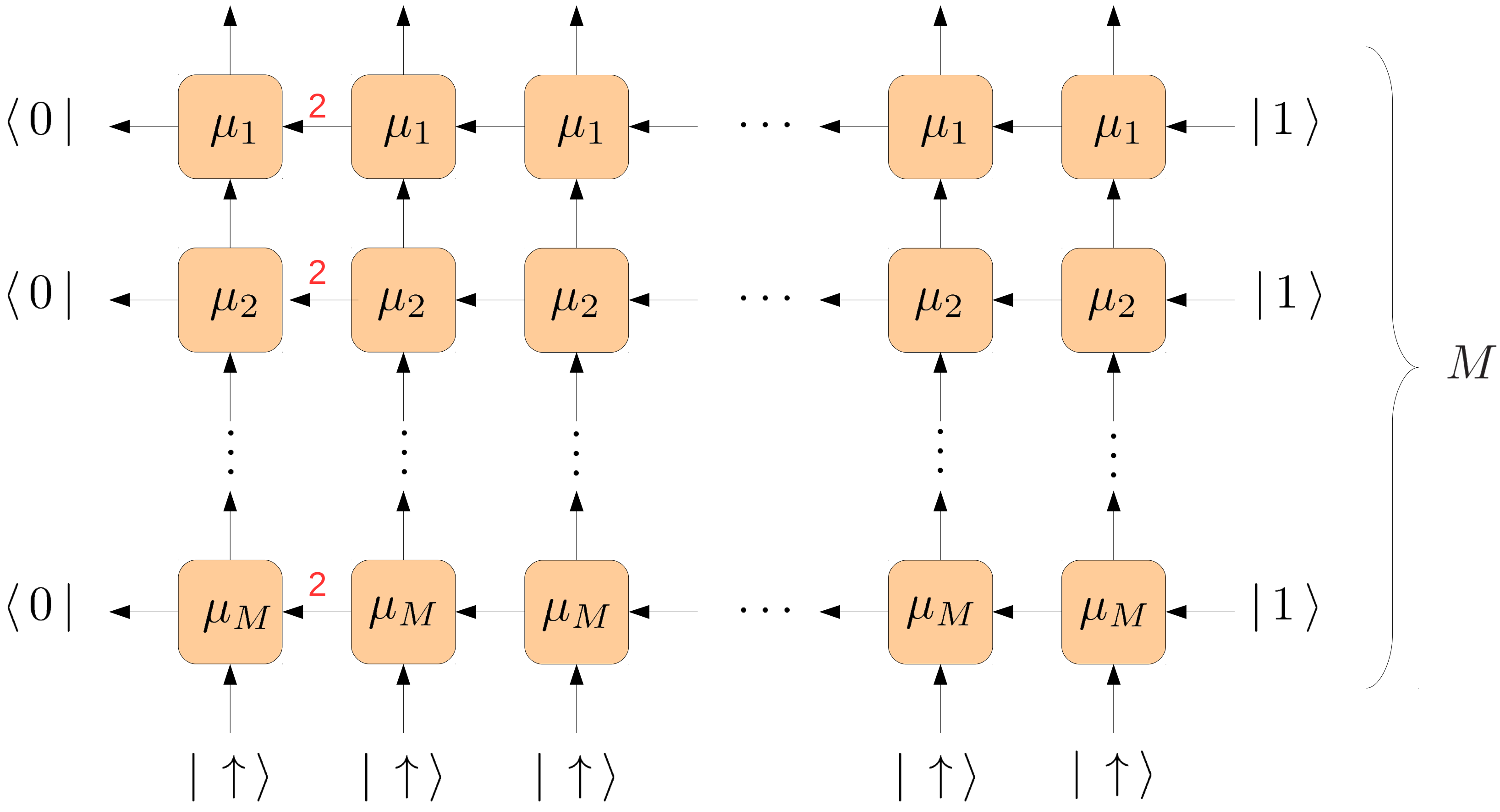
$= 0$, except $\beta = \alpha + (k - l)$

$\Longrightarrow$ The virtual bond counts the number of created down-spins!

Algebraic Bethe Ansatz – XXZ Model

$$|\,\Psi(\mu_1,\ldots,\mu_M)\,\rangle = B(\mu_1)\cdots B(\mu_M)|\uparrow,\ldots,\uparrow\rangle \qquad (S_z = \frac{1}{2}N - M)$$

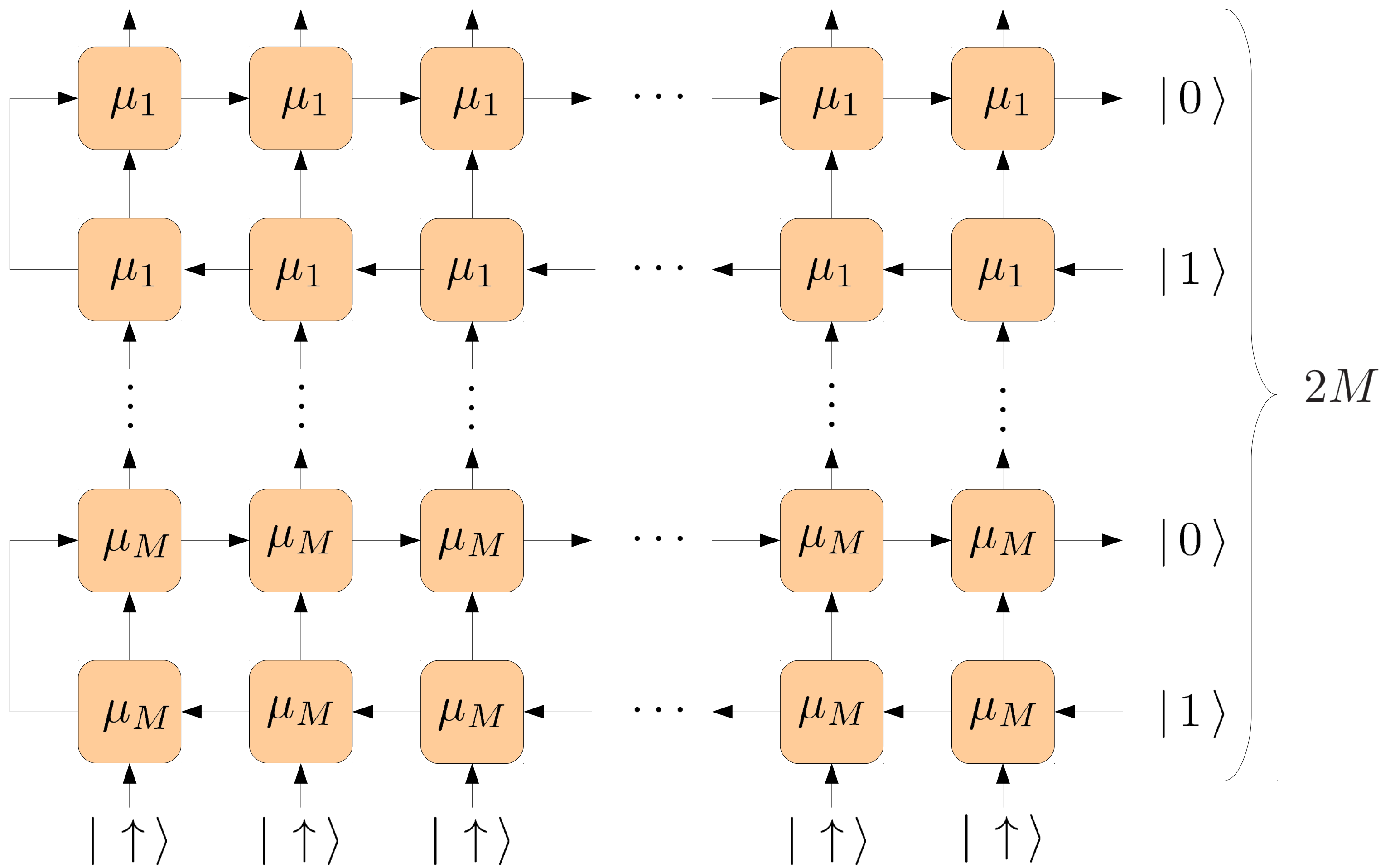
$=$



Algebraic Bethe Ansatz – XXZ Model with open boundary conditions

$$|\,\Psi(\mu_1,\ldots,\mu_M)\,\rangle = \mathcal{B}(\mu_1)\cdots\mathcal{B}(\mu_M)|\,vac\,\rangle \quad = \qquad (S_z = \frac{1}{2}N - M)$$

$=$



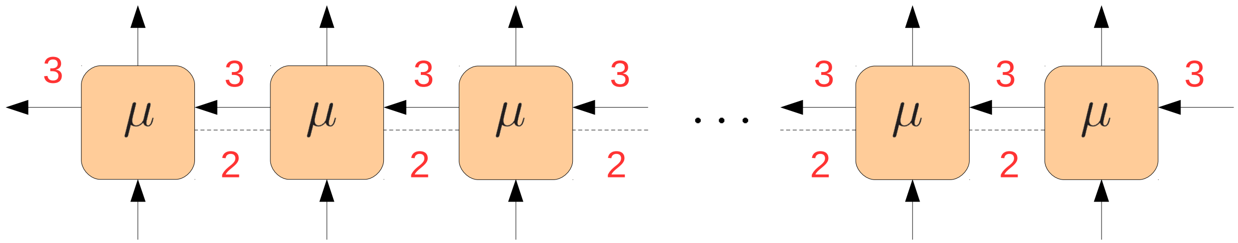
Algebraic Bethe Ansatz – supersymmetric tJ Model

$$H = -t\sum_{j\sigma}(c^{\dagger}_{j,\sigma}c_{j+1,\sigma} + h.c.) + J(\vec{S}_j \cdot \vec{S}_{j+1} - \frac{1}{4}n_j n_{j+1}) \qquad J = 2t$$

Monodromy Matrix

$$T(\lambda) = L_L(\lambda)L_{L-1}(\lambda)\cdots L_1(\lambda) \qquad T(\lambda) = \begin{pmatrix} A_{11}(\lambda) & A_{12}(\lambda) & C_1(\lambda) \\ A_{21}(\lambda) & A_{22}(\lambda) & C_2(\lambda) \\ B_1(\lambda) & B_2(\lambda) & D(\lambda) \end{pmatrix}$$

$$T(\mu)^{\alpha}_{\beta} = \quad \langle\,\alpha\,|$$



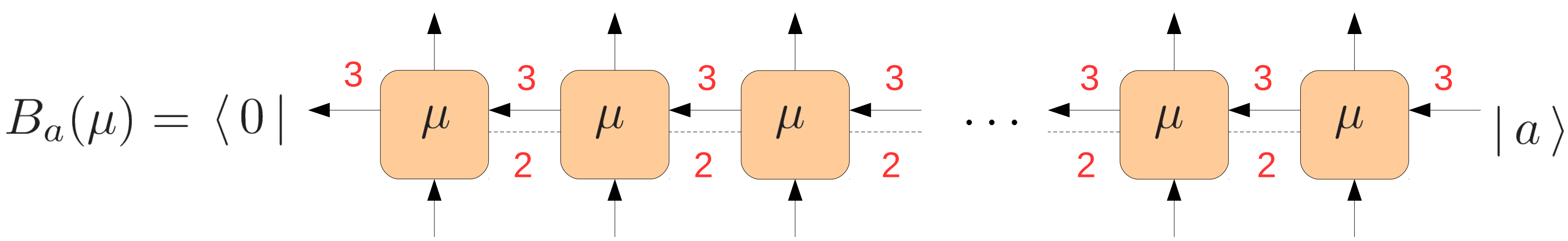
$$|\,\beta\,\rangle$$

$$|\downarrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |\uparrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |\,0\,\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Algebraic Bethe Ansatz – supersymmetric tJ Model

$$H = -t \sum_{j\sigma} (c^{\dagger}_{j,\sigma} c_{j+1,\sigma} + h.c.) + J(\vec{S}_j \cdot \vec{S}_{j+1} - \frac{1}{4} n_j n_{j+1}) \qquad J = 2t$$

Creation operator for one down-spin

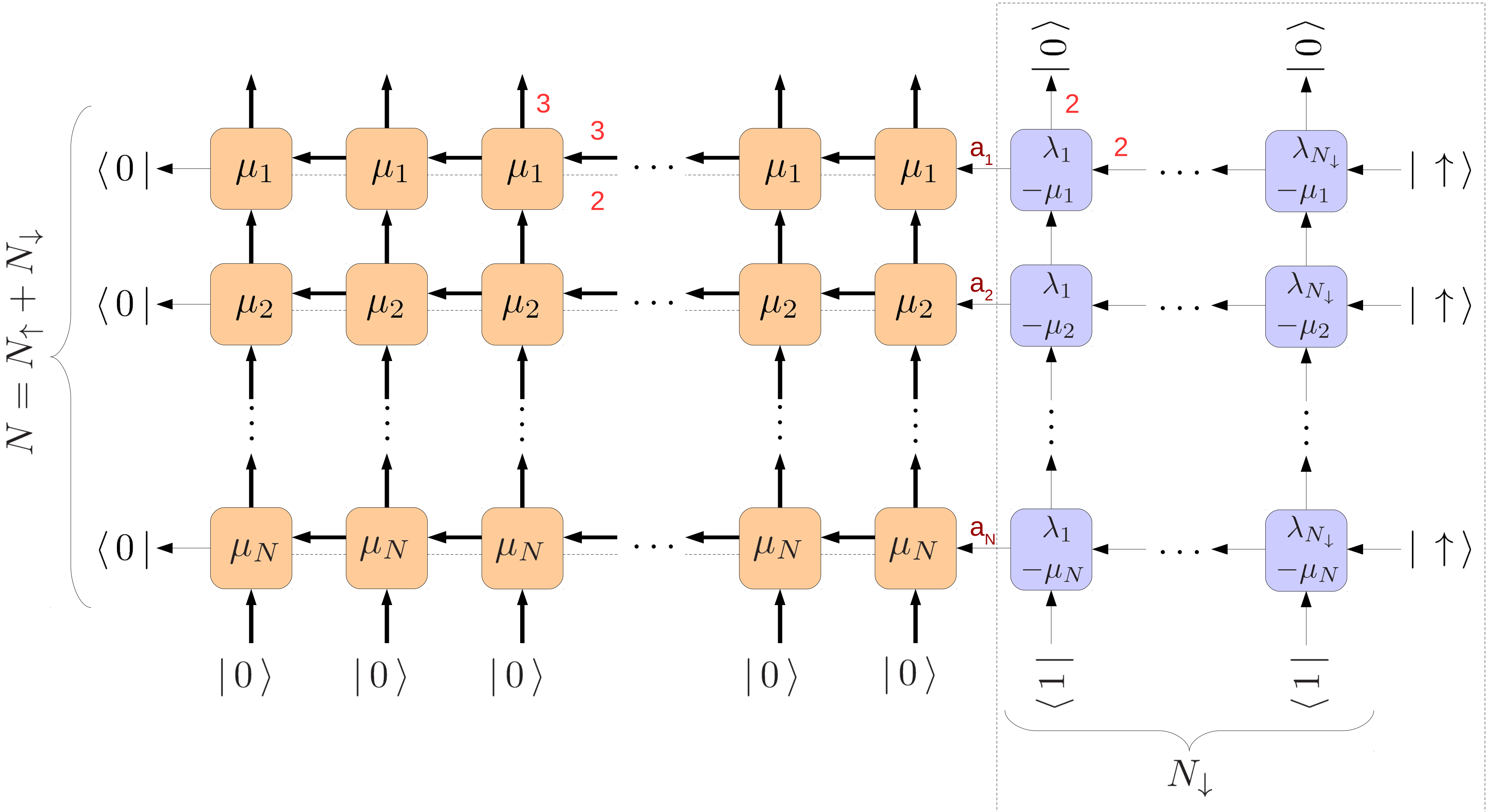


$$| a \rangle \in \{ | \downarrow \rangle, | \uparrow \rangle \}$$

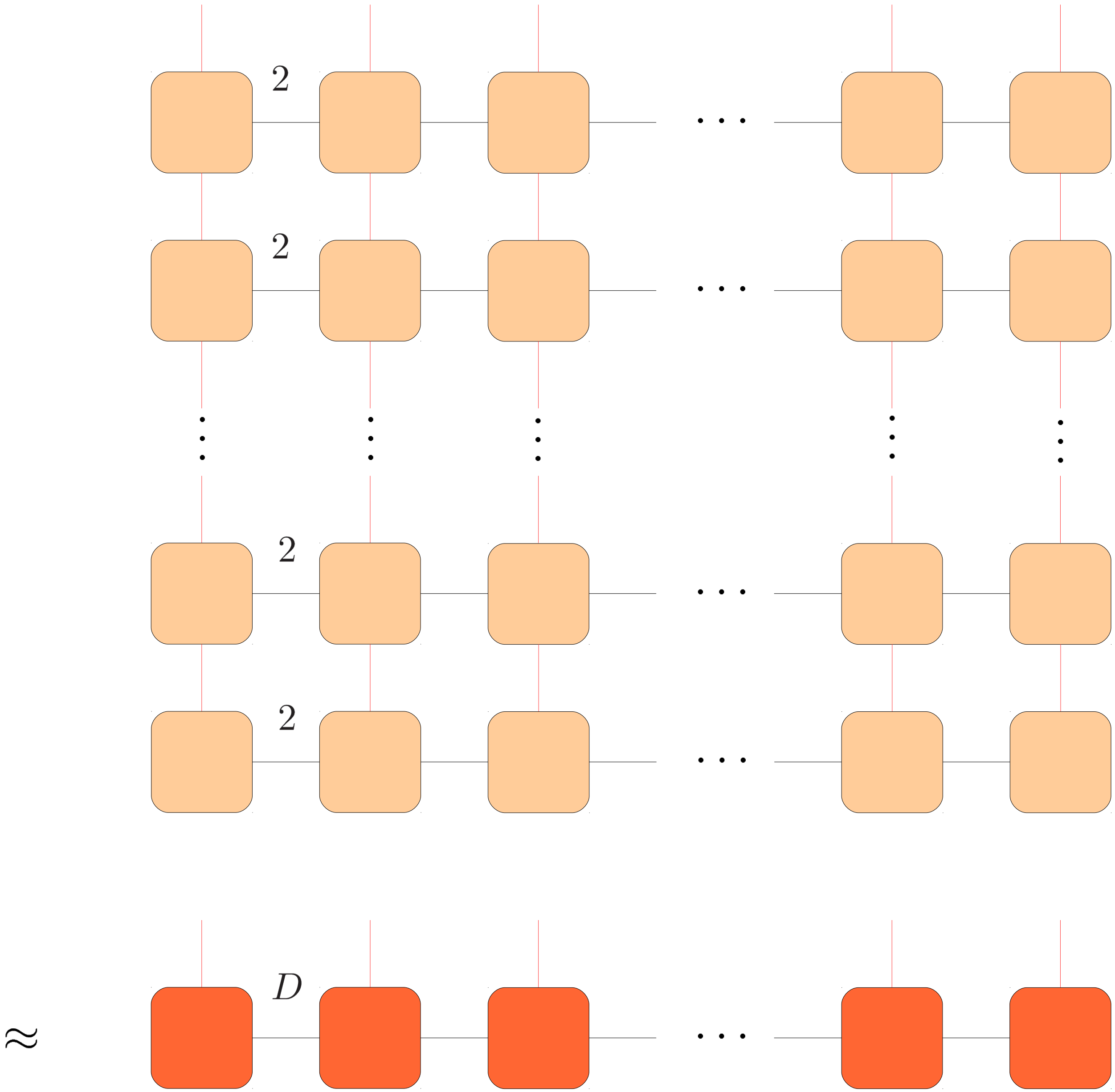
The virtual bond counts the number of created spins
and distinguished up- and down-spins!

Algebraic Bethe Ansatz – supersymmetric tJ Model

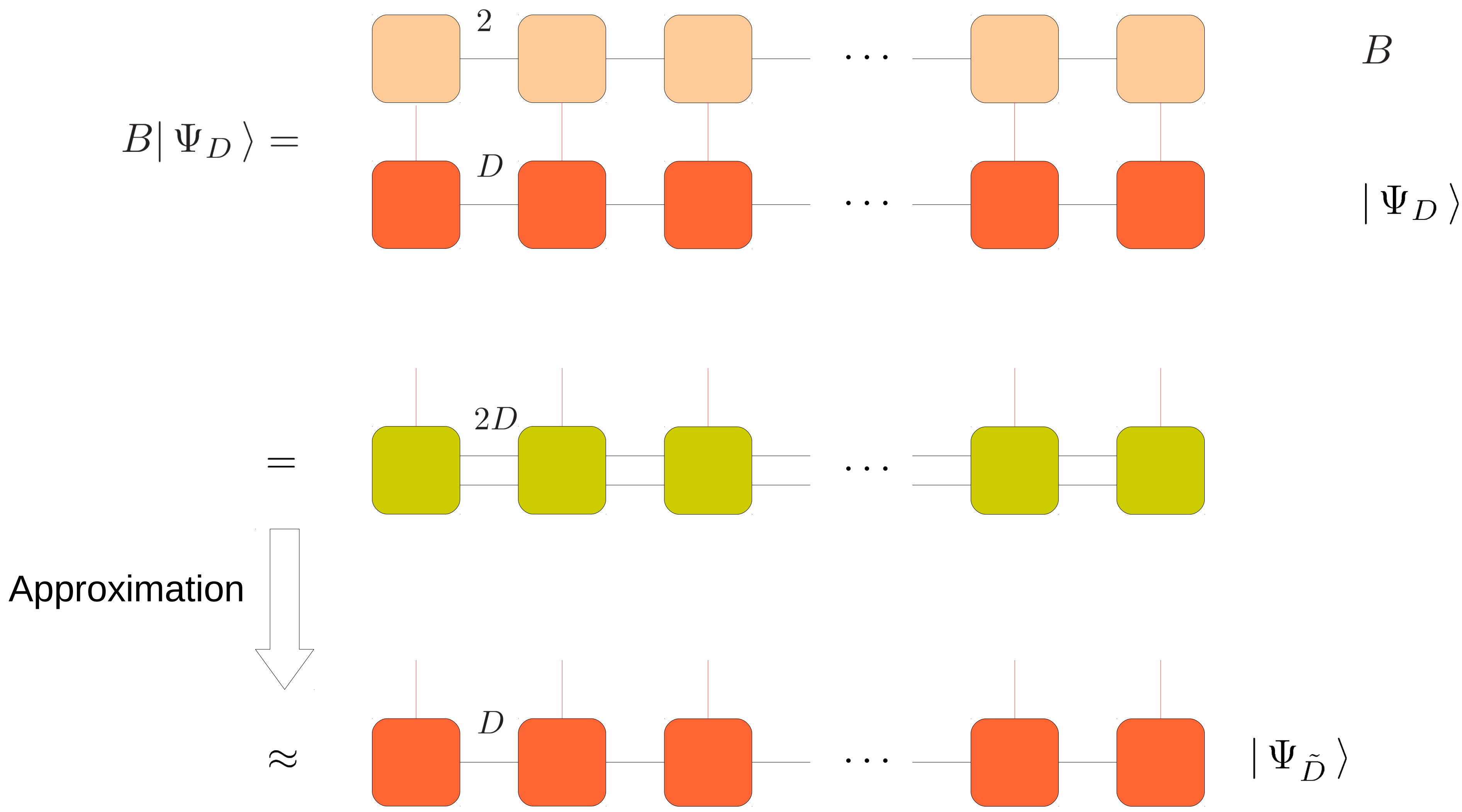
$$|\,\Psi\,\rangle = \sum_{a_1,\ldots,a_M=0}^{1} C_{a_1,\ldots,a_M} B_{a_1}(\mu_1)\cdots B_{a_N}(\mu_N)|\,vac\,\rangle =$$



Contraction of the Bethe-Network



Contraction of the Bethe-Network



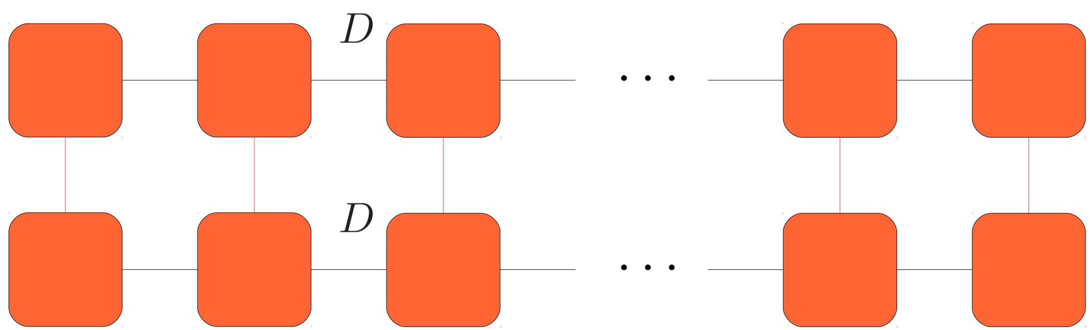
Contraction of the Bethe-Network

Approximation

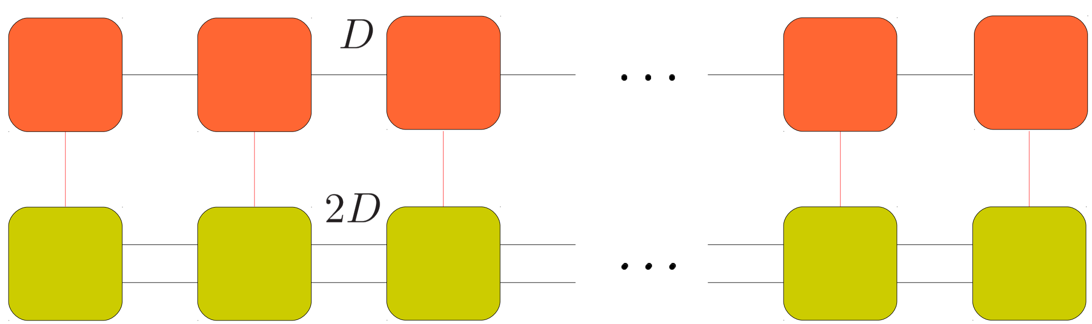
$$K = \Big\| \, |\, \Psi_{2D} \rangle - |\, \tilde{\Psi}_D \rangle \Big\|^2 \rightarrow \text{Min.}$$

$$K = \langle\, \tilde{\Psi}_D \,|\, \tilde{\Psi}_D \,\rangle + 2\text{Re}\, \langle\, \Psi_{2D} \,|\, \tilde{\Psi}_D \,\rangle + \text{const}$$

$$\langle\, \tilde{\Psi}_D \,|\, \tilde{\Psi}_D \,\rangle =$$



$$\langle\, \tilde{\Psi}_D \,|\, \Psi_{2D} \,\rangle =$$



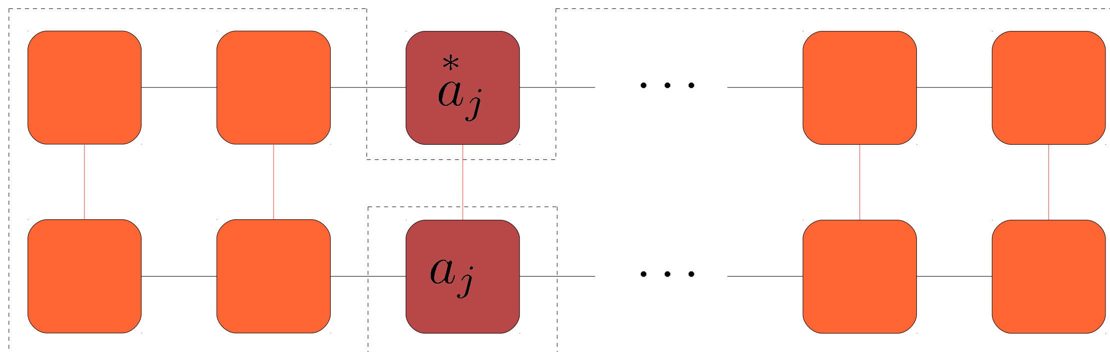
Contraction of the Bethe-Network

Approximation

$$K = \left\| |\,\Psi_{2D}\,\rangle - |\,\tilde{\Psi}_D\,\rangle \right\|^2 \rightarrow \mathrm{Min.}$$

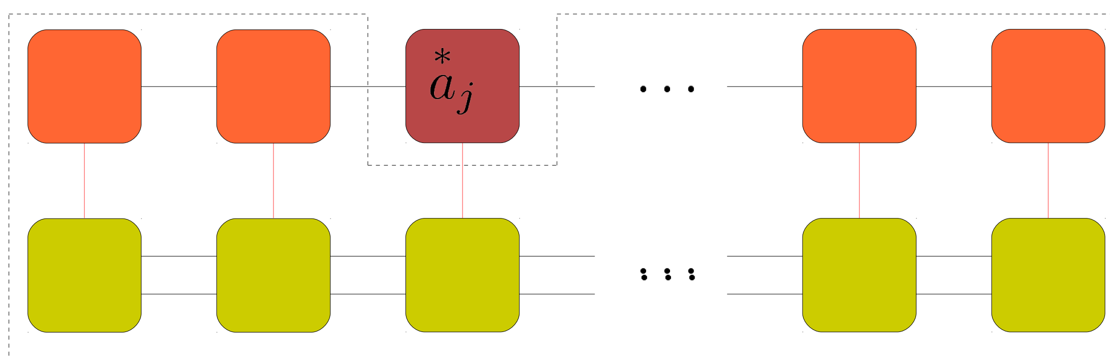
$$K = \langle\,\tilde{\Psi}_D\,|\,\tilde{\Psi}_D\,\rangle + 2\mathrm{Re}\,\langle\,\Psi_{2D}\,|\,\tilde{\Psi}_D\,\rangle + \mathrm{const}$$

$$\langle\,\tilde{\Psi}_D\,|\,\tilde{\Psi}_D\,\rangle =$$



$$= \vec{a}_j^{\dagger}\,\mathbb{N}_j\,\vec{a}_j$$

$$\langle\,\tilde{\Psi}_D\,|\,\Psi_{2D}\,\rangle =$$



j

$$= \vec{a}_j^{\dagger}\,\vec{w}$$

Approximation

$$K = \Big\| \,|\,\Psi_{2D}\,\rangle - |\,\tilde{\Psi}_D\,\rangle \Big\|^2 \to \mathrm{Min.}$$

$$K = \langle\,\tilde{\Psi}_D\,|\,\tilde{\Psi}_D\,\rangle + 2\mathrm{Re}\,\langle\,\Psi_{2D}\,|\,\tilde{\Psi}_D\,\rangle + \mathrm{const}$$

$$K = \vec{a}_j^{\dagger}\mathbb{N}_j\vec{a}_j + \vec{a}_j^{\dagger}\vec{w} + \mathrm{const}$$

$$\frac{\partial}{\partial \vec{a}_j^{*}} K = 0 \qquad \Rightarrow \qquad \boxed{\mathbb{N}_j\vec{a}_j = \vec{w}}$$

Contraction of the Bethe-Network

Order Optimization

$$|\,\Psi(\mu_1,\ldots,\mu_M)\,\rangle = B(\mu_1)\cdots B(\mu_M)|\,vac\,\rangle$$

$$B(\mu_M)|\,vac\,\rangle$$

$$B(\mu_{M-1})B(\mu_M)|\,vac\,\rangle$$

$$B(\mu_{M-2})B(\mu_{M-1})B(\mu_M)|\,vac\,\rangle$$

$$\vdots$$

$$B(\mu_1)\cdots B(\mu_M)|\,vac\,\rangle$$

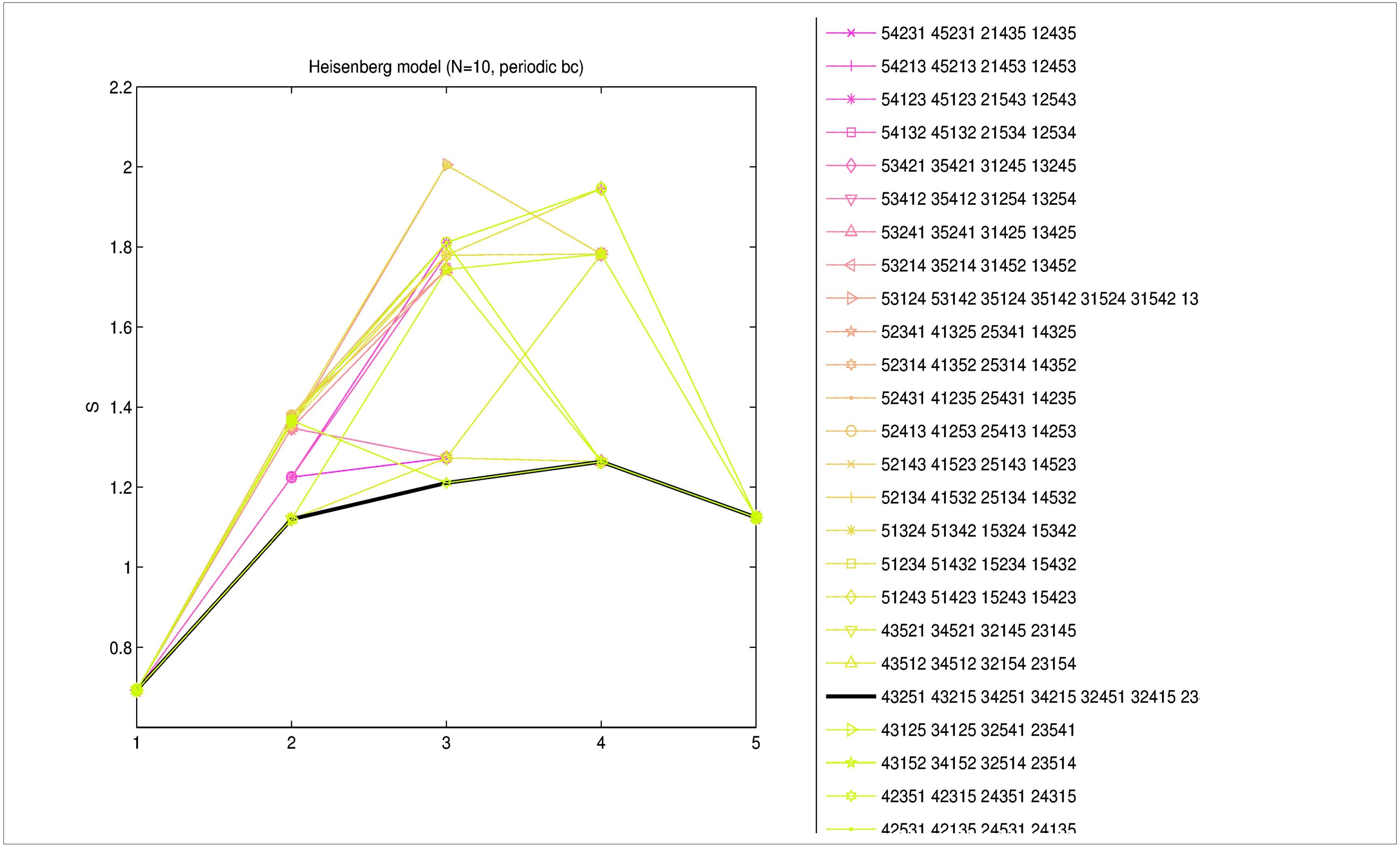
$$[B(\lambda),B(\mu)] = 0$$ $\Rightarrow$ Order of the B's is arbitrary!

Example: Heisenberg model with periodic boundary conditions

Order Optimization

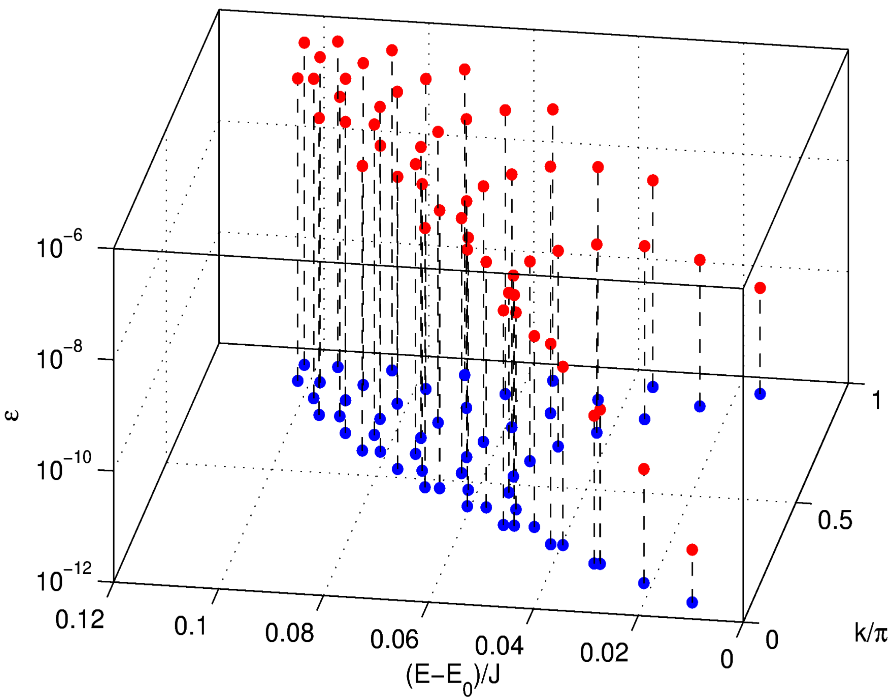
$$|\,\Psi(\mu_1,\ldots,\mu_M)\,\rangle = B(\mu_1)\cdots B(\mu_M)|\,vac\,\rangle$$

$$[B(\lambda),B(\mu)]=0$$

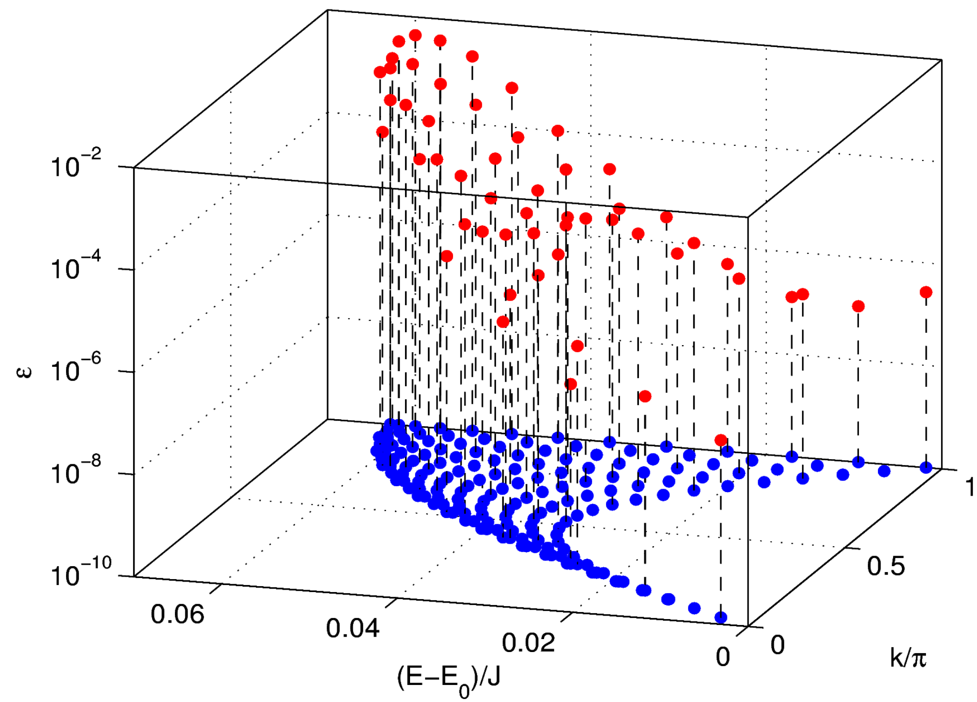


Example: Heisenberg model with periodic boundary conditions

Two-spinon excited states *Relative Error in the Energy*



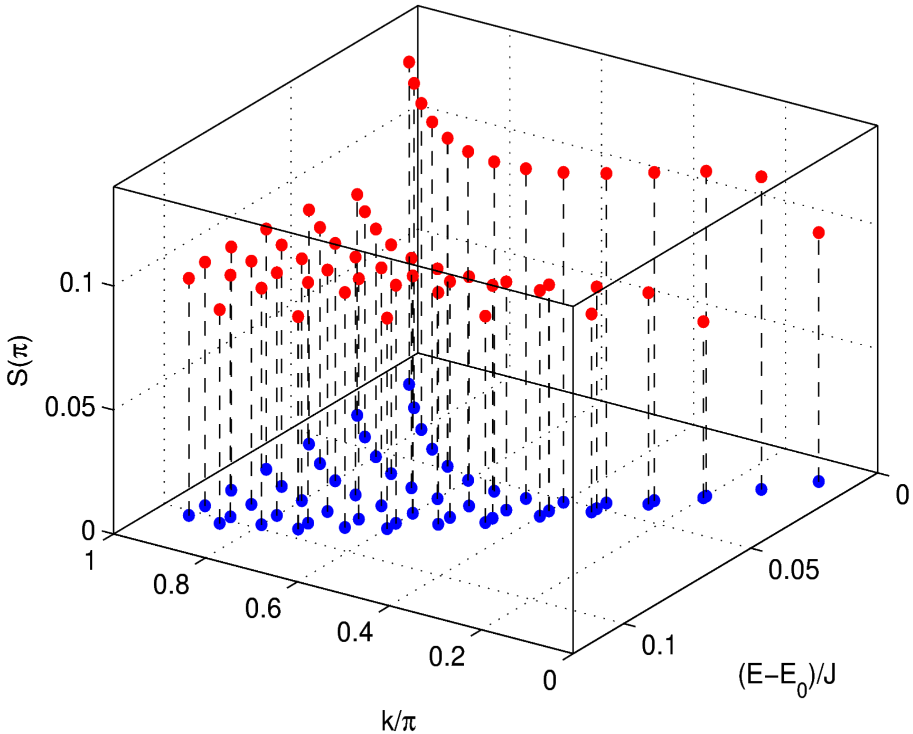
N=30, D=500



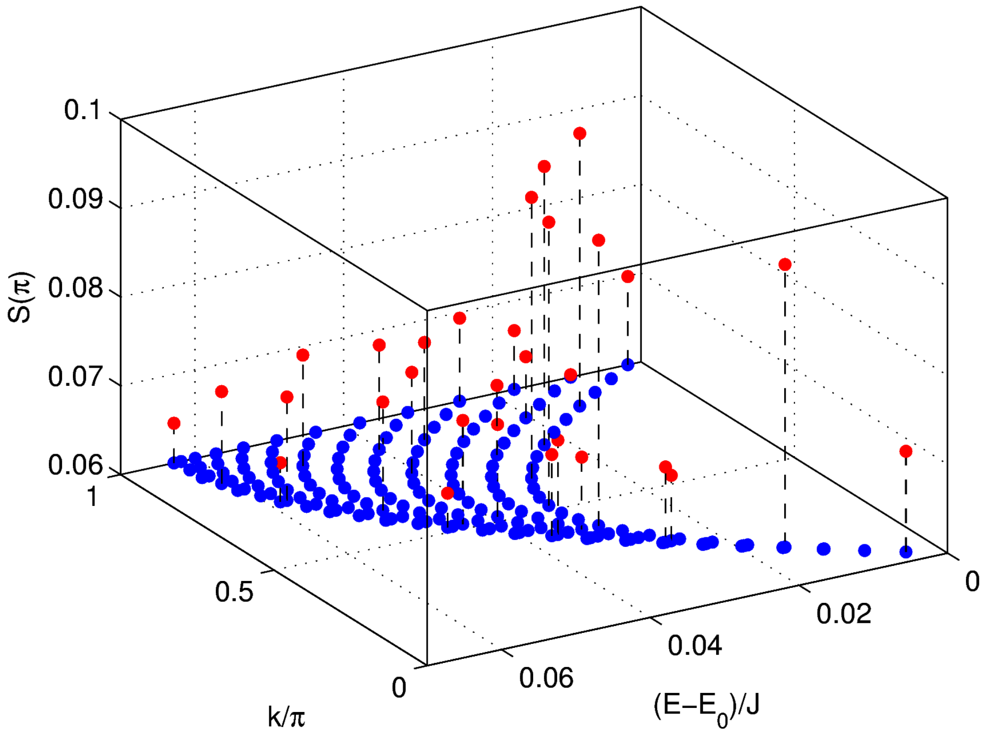
N=50, D=1000

Example: Heisenberg model with periodic boundary conditions

Two-spinon excited states *Structure Factor* $S(\pi)$



N=30, D=500

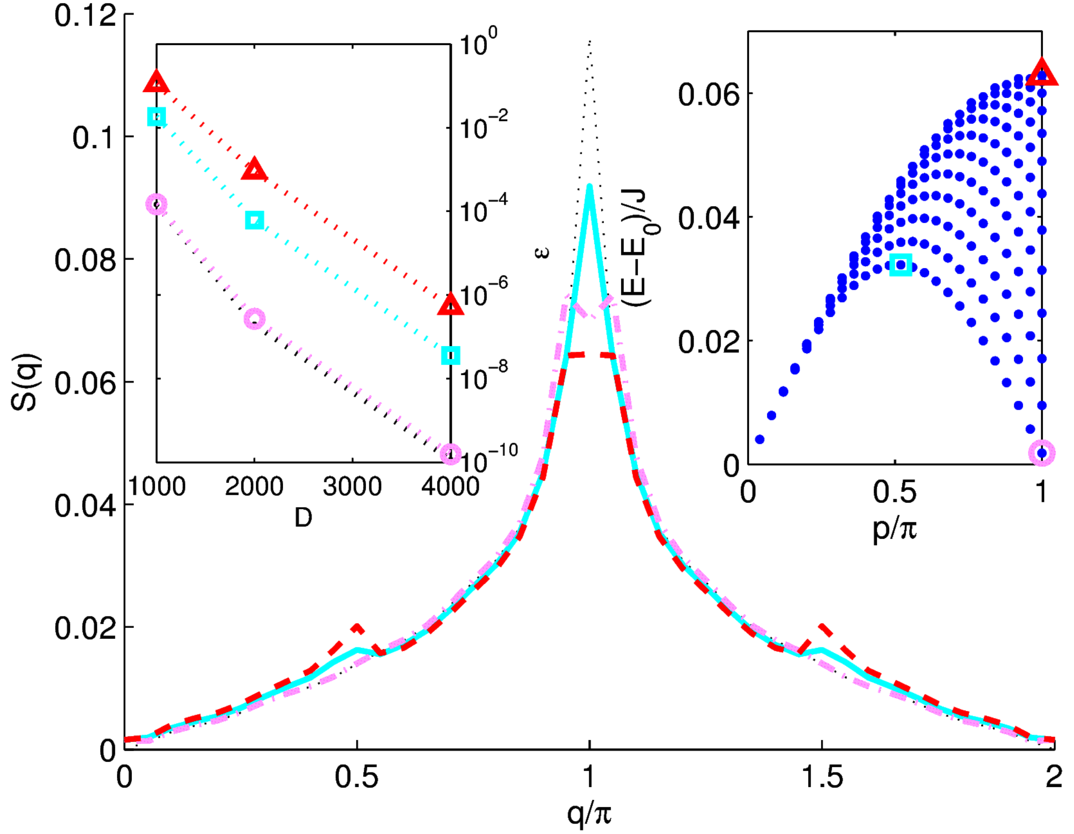


N=50, D=1000

$$S(q) = 1/N^2 \sum_{r,s} e^{iq(r-s)} \langle \sigma_z^r \sigma_z^s \rangle$$

Example: Heisenberg model with periodic boundary conditions

Two-spinon excited states   *Structure Factor* $S(q)$



Structure factor for ground state and 3 two-spinon excited states (N=50, D=1000, pbc)

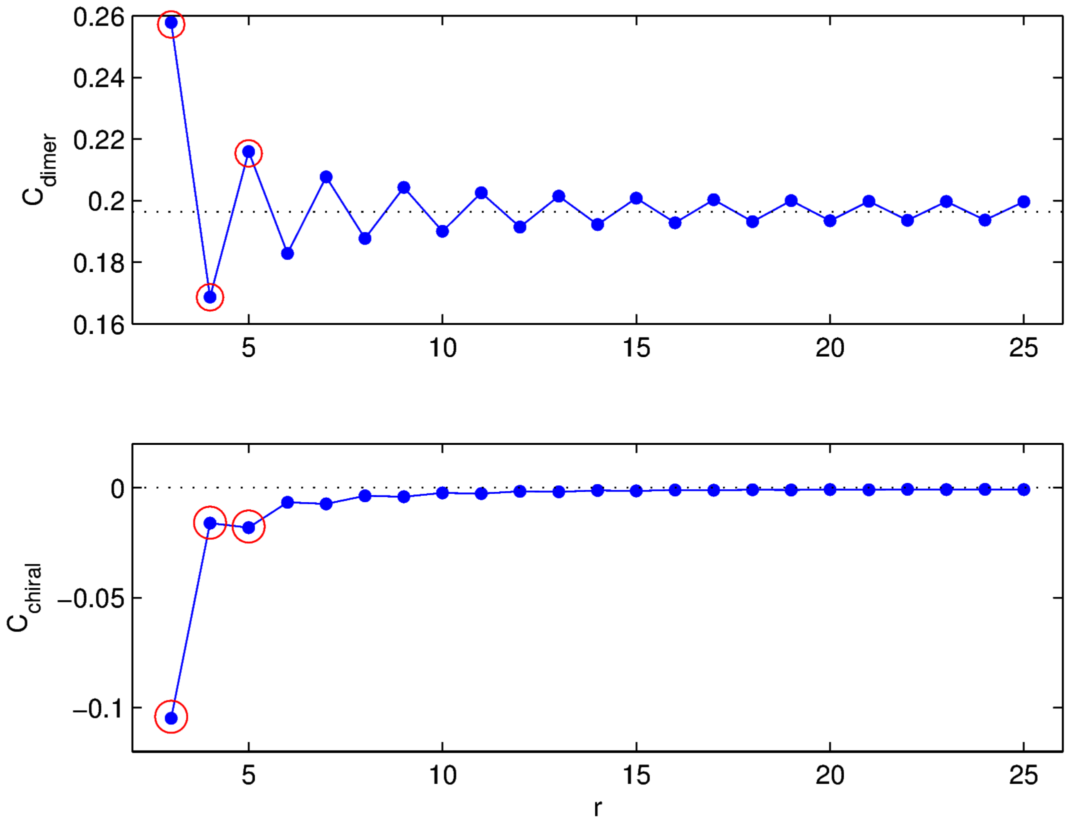
Rhs: two-spinon excited states with S=1 and Sz=1

Lhs: relative error in the energy as a function of D

Example: Heisenberg Model

Ground State

Dimer-dimer / chiral correlations



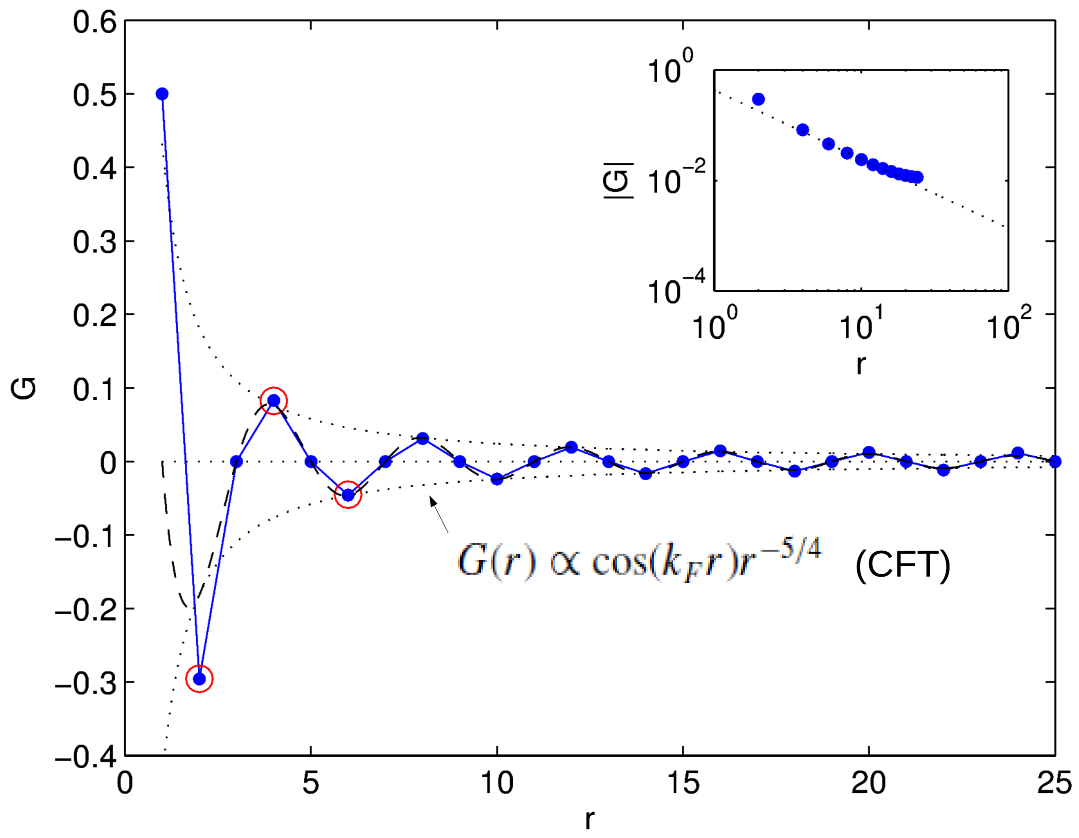
$$C_{dimer}(r) = \frac{1}{16}\langle(\vec{\sigma}_1 \cdot \vec{\sigma}_2)(\vec{\sigma}_r \cdot \vec{\sigma}_{r+1})\rangle$$

$$C_{chiral}(r) = \frac{1}{16}\langle(\vec{\sigma}_1 \times \vec{\sigma}_2) \cdot (\vec{\sigma}_r \times \vec{\sigma}_{r+1})\rangle$$

Example: Heisenberg Model

Ground State

One-particle Green's function

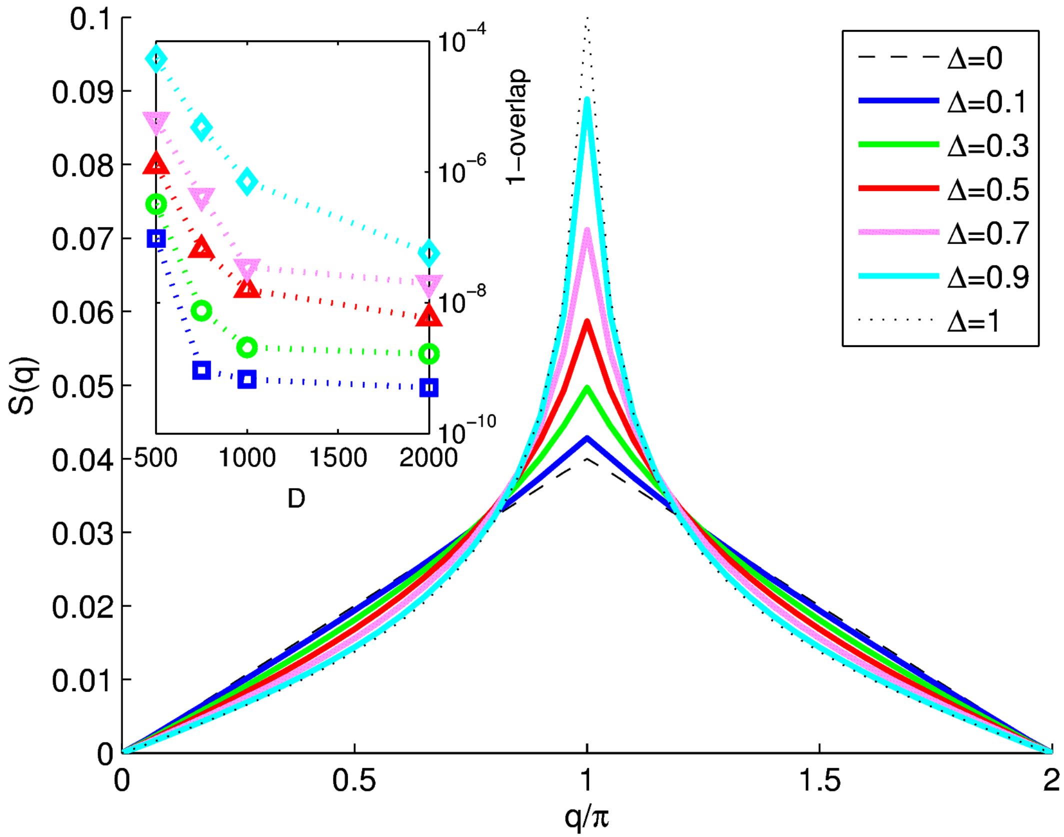


$$G(r) = \langle c_1^\dagger c_r \rangle \equiv \langle \sigma_1^+ \sigma_z^2 \cdots \sigma_z^{r-1} \sigma_r^- \rangle$$

Example: XXZ Model with open boundary conditions

Ground State

Structure Factor $S(q)$



Structure factor for ground state if the XXZ-model (N=50, D=1000, obc)

Lhs: relative error as a function of D

Algebraic Bethe Ansatz – XXZ Model

$$H = \sum_{j=1}^{N} \left(s_x^{(j)} s_x^{(j+1)} + s_y^{(j)} s_y^{(j+1)} + \Delta s_z^{(j)} s_z^{(j+1)} \right)$$

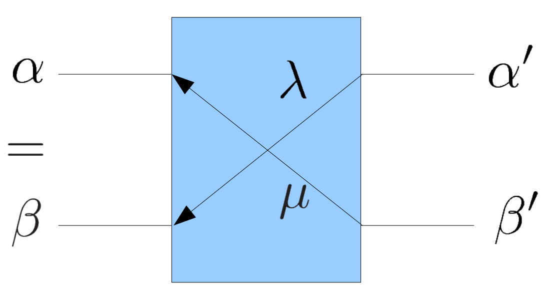
L-Operator

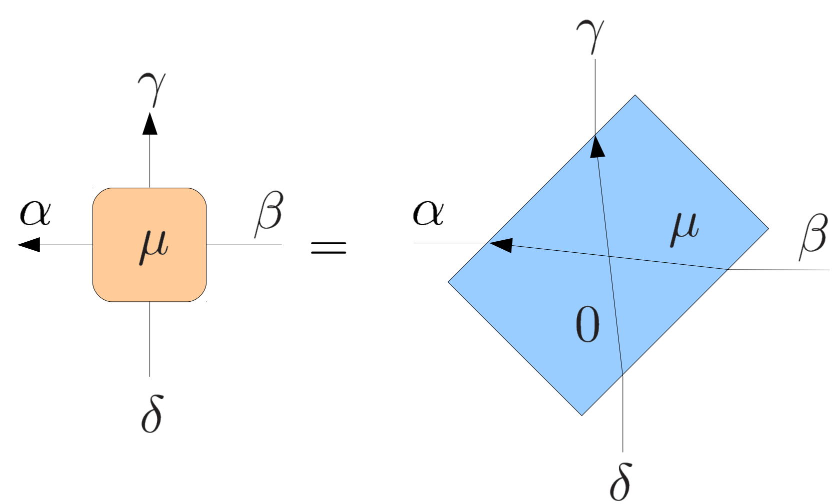
$$R(\lambda, \mu) = \begin{pmatrix} 1 & & & \\ & b(\lambda-\mu) & c(\lambda-\mu) & \\ & c(\lambda-\mu) & b(\lambda-\mu) & \\ & & & 1 \end{pmatrix}$$

$$L_{j\,\beta}^{\;\alpha}(\lambda) = R^{\gamma\alpha}_{\beta\delta}(\lambda, 0) e_{j\,\gamma}^{\;\delta}$$

$$e_{j\,\gamma}^{\;\delta} = |\, \gamma \,\rangle_j \langle \delta \,|$$

$$R^{\alpha\beta}_{\alpha'\beta'}(\lambda, \mu) =$$





Algebraic Bethe Ansatz – supersymmetric tJ Model

$$H = -t\sum_{j\sigma}(c^{\dagger}_{j,\sigma}c_{j+1,\sigma} + h.c.) + J(\vec{S}_j\cdot\vec{S}_{j+1} - \frac{1}{4}n_j n_{j+1}) \qquad J = 2t$$

L-Operator

$$L_{j\,\beta}^{\,\alpha}(\lambda) = R^{\gamma\alpha}_{\beta\delta}(\lambda,0)\,e_{j\,\gamma}^{\,\delta}(-1)^{p(\alpha)p(\gamma)}$$

$$R(\lambda,\mu) = b(\lambda-\mu)I + c(\lambda-\mu)P$$

$$b(\lambda) = \frac{i}{\lambda+i} \qquad c(\lambda) = \frac{\lambda}{\lambda+i}$$

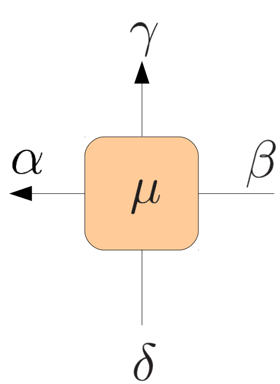
$$p(e_{j\,\gamma}^{\,\delta}) = p(\delta) + p(\gamma)$$

$$p(1) = p(2) = 1, p(3) = 0$$

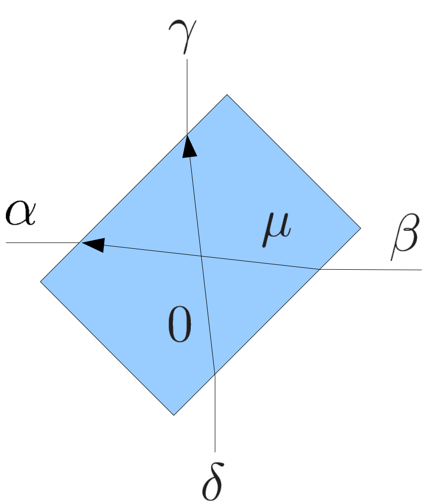
$$[e_{j\,\beta}^{\,\alpha}, e_{j\,\gamma}^{\,\delta}]_{\pm} = 0$$

$$e_{j\,\beta}^{\,\alpha}e_{j\,\gamma}^{\,\delta} = \delta^{\beta}_{\gamma}e_{j\,\alpha}^{\,\delta}$$

$L_{j\,\beta}^{\,\alpha}(\lambda) =$



$e_{j\,\gamma}^{\,\delta} \;=$



$(-1)^{p(\alpha)p(\gamma)} \;\; e_{j\,\gamma}^{\,\delta}$

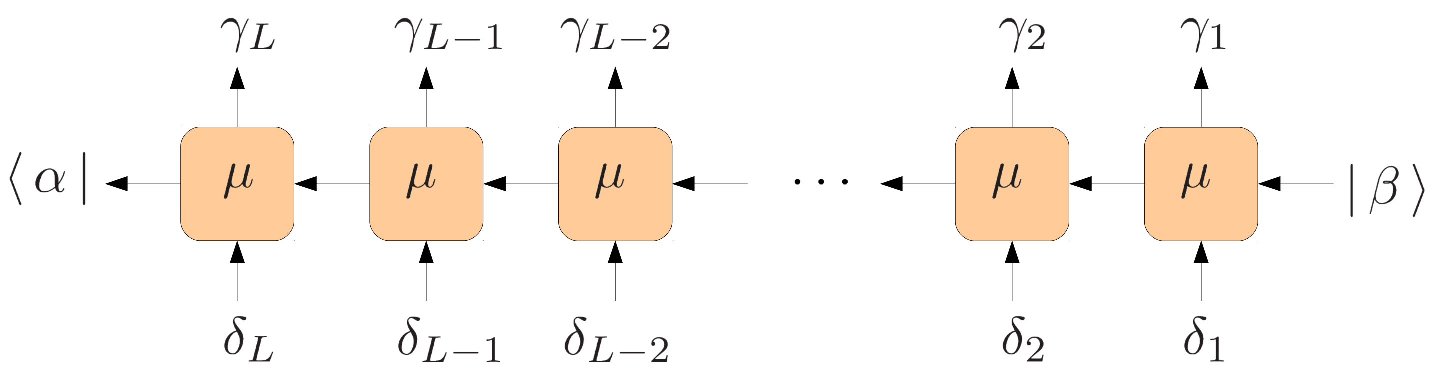
Algebraic Bethe Ansatz – supersymmetric tJ Model

$$H = -t \sum_{j\sigma} (c^\dagger_{j,\sigma} c_{j+1,\sigma} + h.c.) + J(\vec{S}_j \cdot \vec{S}_{j+1} - \frac{1}{4} n_j n_{j+1}) \qquad J = 2t$$

Transfer Matrix

$$T(\lambda) = L_L(\lambda) L_{L-1}(\lambda) \cdots L_1(\lambda) \qquad T(\lambda) = \begin{pmatrix} A_{11}(\lambda) & A_{12}(\lambda) & C_1(\lambda) \\ A_{21}(\lambda) & A_{22}(\lambda) & C_2(\lambda) \\ B_1(\lambda) & B_2(\lambda) & D(\lambda) \end{pmatrix}$$

$T(\mu)^\alpha_\beta =$



$$\times \quad {e_L}^{\delta_L}_{\gamma_L} \quad {e_{L-1}}^{\delta_{L-1}}_{\gamma_{L-1}} \, {e_{L-2}}^{\delta_{L-2}}_{\gamma_{L-2}} \quad \cdots \quad {e_2}^{\delta_2}_{\gamma_2} \quad {e_1}^{\delta_1}_{\gamma_1}$$

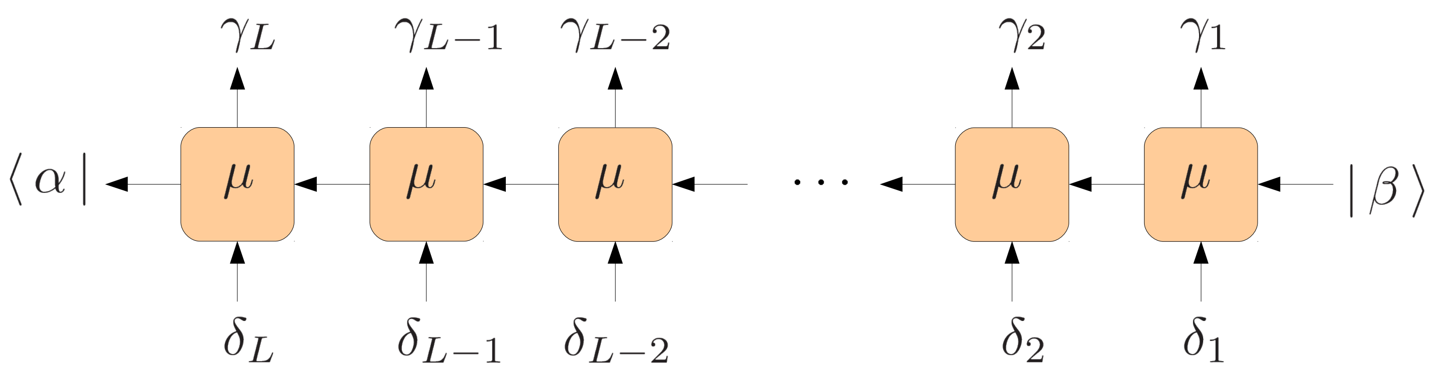
Algebraic Bethe Ansatz – supersymmetric tJ Model

$$H = -t\sum_{j\sigma}(c^{\dagger}_{j,\sigma}c_{j+1,\sigma} + h.c.) + J(\vec{S}_j\cdot\vec{S}_{j+1} - \frac{1}{4}n_j n_{j+1}) \qquad J = 2t$$

Transfer Matrix

$$T(\lambda) = L_L(\lambda)L_{L-1}(\lambda)\cdots L_1(\lambda) \qquad T(\lambda) = \begin{pmatrix} A_{11}(\lambda) & A_{12}(\lambda) & C_1(\lambda) \\ A_{21}(\lambda) & A_{22}(\lambda) & C_2(\lambda) \\ B_1(\lambda) & B_2(\lambda) & D(\lambda) \end{pmatrix}$$

$T(\mu)^{\alpha}_{\beta} =$



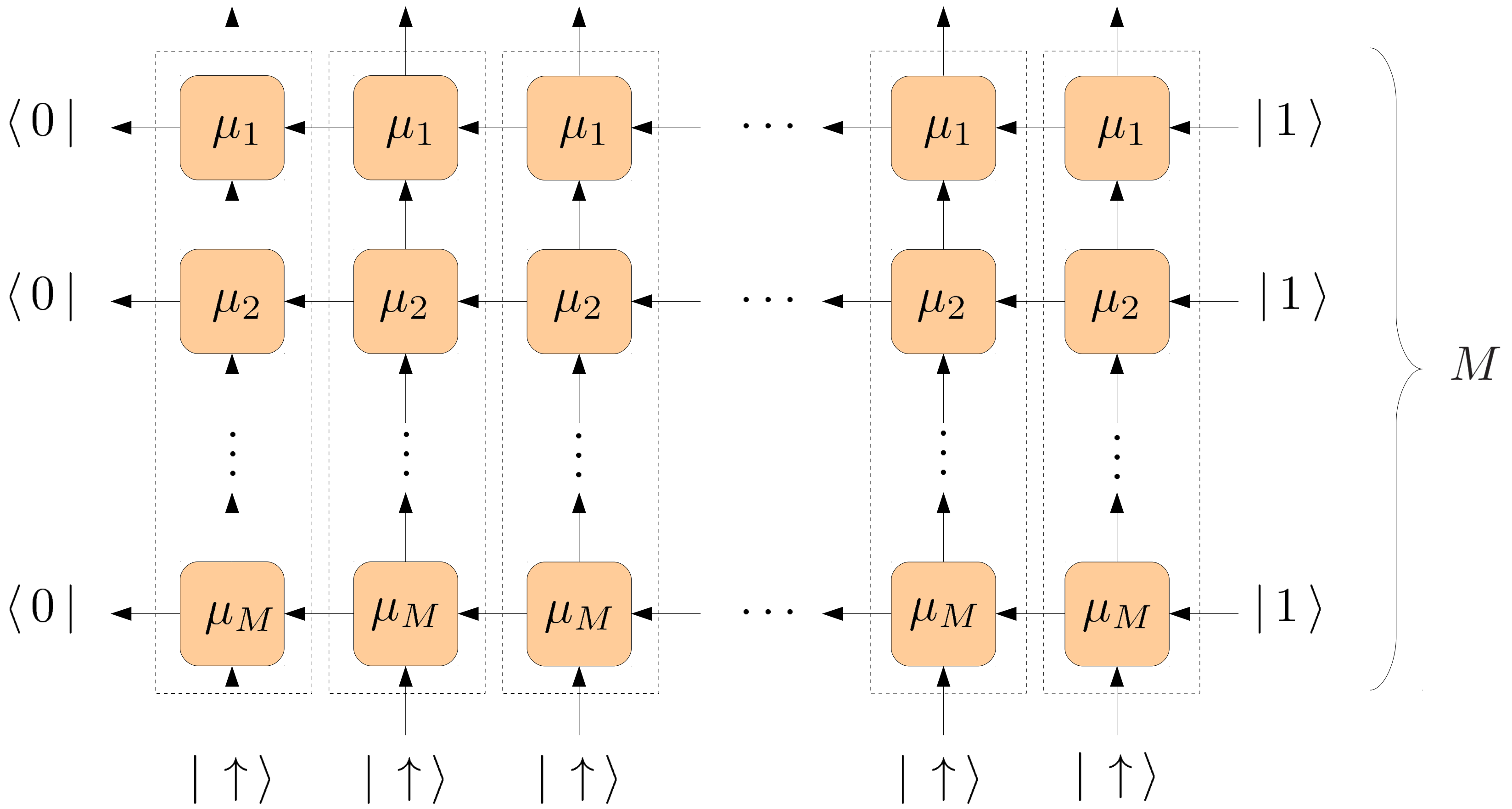
$$\times \quad (-1)^{\sum_{j=1}^{L-1}(p(\gamma_j)-p(\delta_j))\sum_{i=j+1}^{L}p(\delta_i)}\,|\,\gamma_1\cdots\gamma_L\,\rangle\langle\delta_1\cdots\delta_L\,|$$

$$H = \sum_{j=1}^{N} \left(s_x^{(j)} s_x^{(j+1)} + s_y^{(j)} s_y^{(j+1)} + \Delta s_z^{(j)} s_z^{(j+1)} \right)$$

$$|\,\Psi(\mu_1, \ldots, \mu_M)\,\rangle = B(\mu_1) \cdots B(\mu_M) |\uparrow, \ldots, \uparrow\rangle$$

(State with $S_z = \frac{1}{2} N - M$)

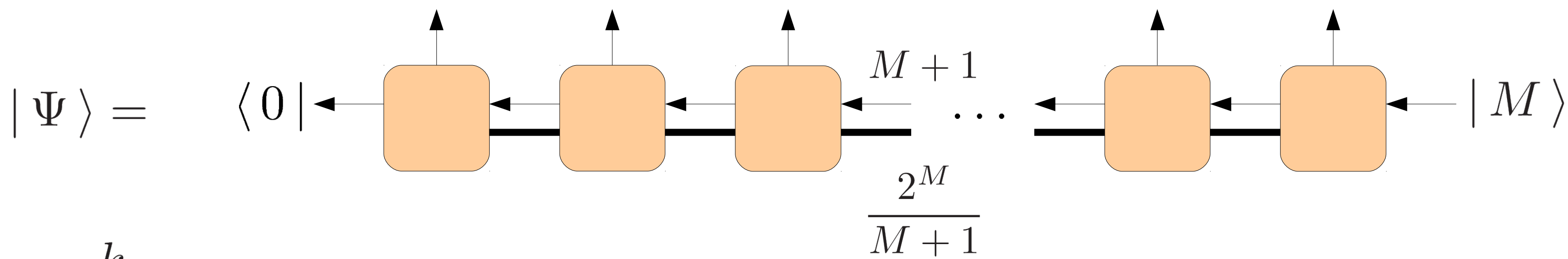
$=$

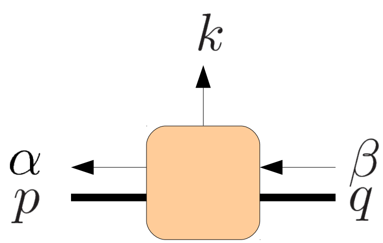


Algebraic Bethe Ansatz

$$H = \sum_{j=1}^{N} \left(s_x^{(j)} s_x^{(j+1)} + s_y^{(j)} s_y^{(j+1)} + \Delta s_z^{(j)} s_z^{(j+1)} \right)$$

Bethe Ansatz State in ***MPS Form***:



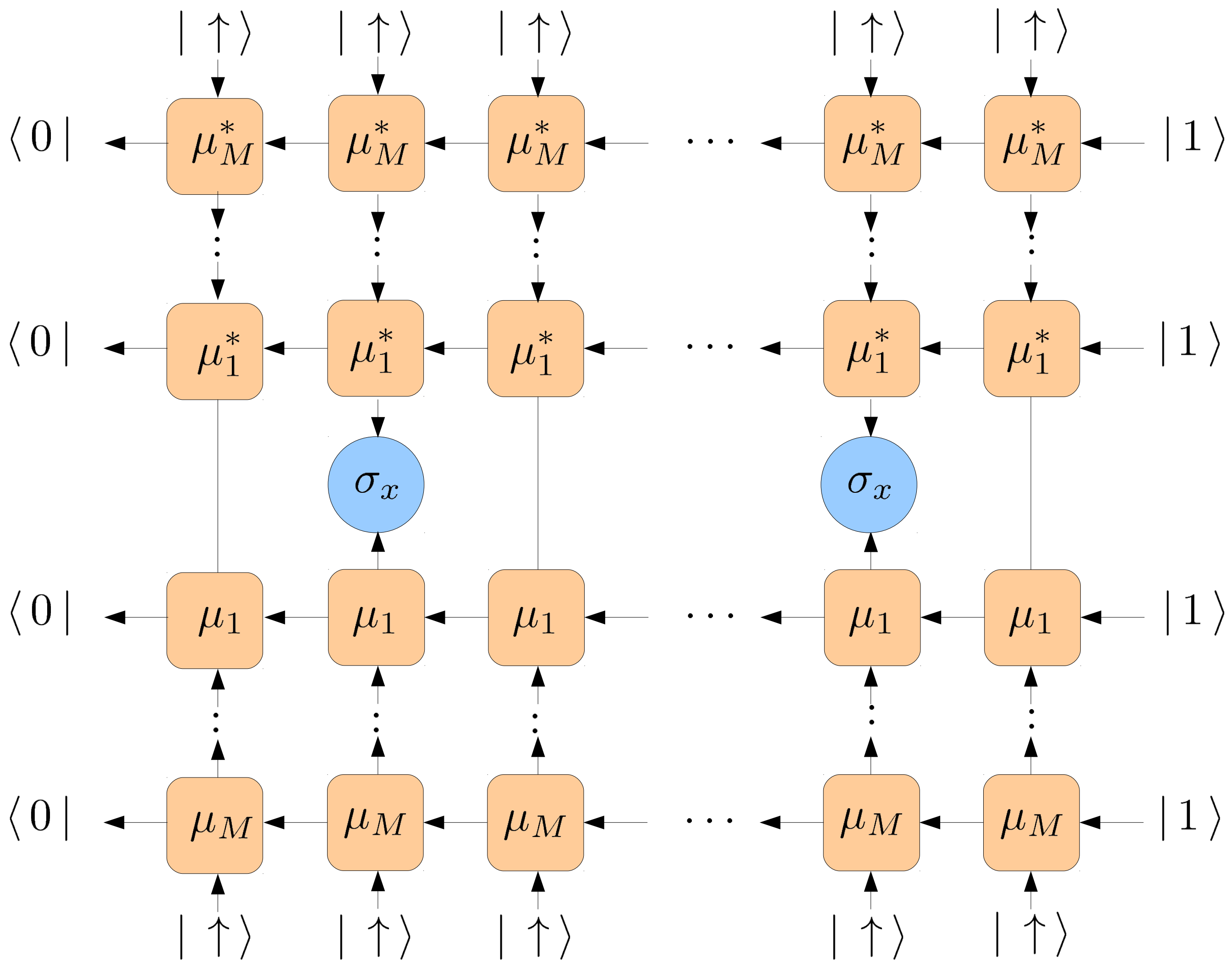


$= 0, \text{ except } \beta = \alpha + k$

$\Longrightarrow$ The virtual bond counts the number of down-spins!

$$\langle\, \Psi(\mu_1, ..., \mu_M)\, |\sigma_x^{(i)} \sigma_x^{(j)}|\, \Psi(\mu_1, ..., \mu_M)\, \rangle =$$

$$= \langle \uparrow, .., \uparrow\, |B(\mu_M)^\dagger \cdot \cdot B(\mu_1)^\dagger \sigma_x^{(i)} \sigma_x^{(j)} B(\mu_1) \cdot \cdot B(\mu_M)|\, \uparrow, .., \uparrow \rangle =$$

$=$ 

Tensor Network Formulation of the Algebraic Bethe Ansatz

Outline

Yang-Baxter Algebra

$T^{\alpha}_{\beta}(\lambda) \qquad \alpha, \beta \in \{0, 1\}$

$$t(\lambda) = \mathrm{tr}T(\lambda) = \sum_{\alpha=0}^{1} T^{\alpha}_{\alpha}(\lambda)$$

- $[t(\lambda), t(\mu)] = 0, \forall \lambda, \mu$
- $\exists \lambda_0$ s.t. $t(\lambda_0) =$ cyclic shift operator

$$H = \frac{d}{d\lambda} ln\left(t(\lambda)\right)\Big|_{\lambda=\lambda_0}$$ Local Hamiltonian

$$B(\mu) = T^{0}_{1}(\mu)$$

$$|\,\Psi(\mu_1, .., \mu_M)\,\rangle = B(\mu_1)\cdot\cdot B(\mu_M)|\,vac\,\rangle$$

$$t(\lambda)|\,\Psi\,\rangle = \tau(\lambda)|\,\Psi\,\rangle$$

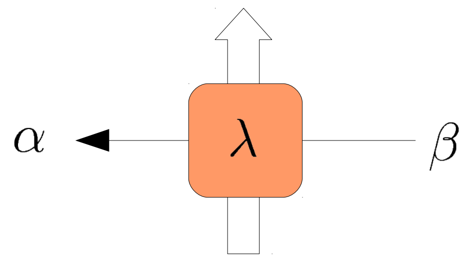
Bethe equations

$$E = \frac{d}{d\lambda} ln\left(\tau(\lambda)\right)\Big|_{\lambda=\lambda_0}$$ Energy

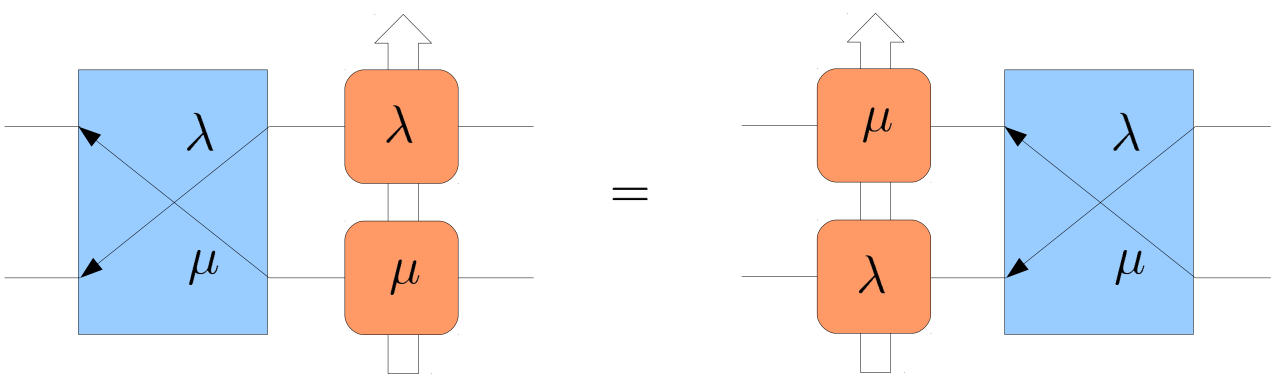
Tensor Network Formulation of the Algebraic Bethe Ansatz

Yang-Baxter Algebra:

$$T^{\alpha}_{\beta}(\lambda) \quad = $$

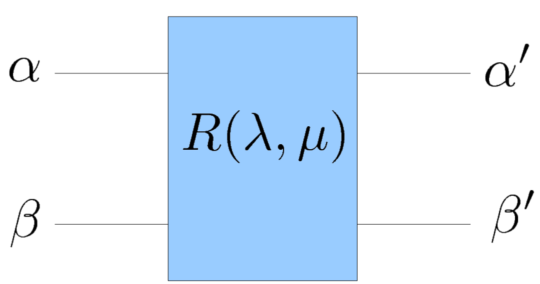


Property: $$R(\lambda,\mu)\,[T(\lambda)\otimes T(\mu)] = [T(\mu)\otimes T(\lambda)]\,R(\lambda,\mu)$$



R-Matrix:

$$R^{\alpha\beta}_{\alpha'\beta'}(\lambda,\mu) \quad = $$



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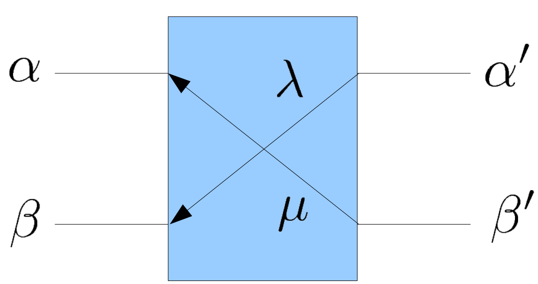
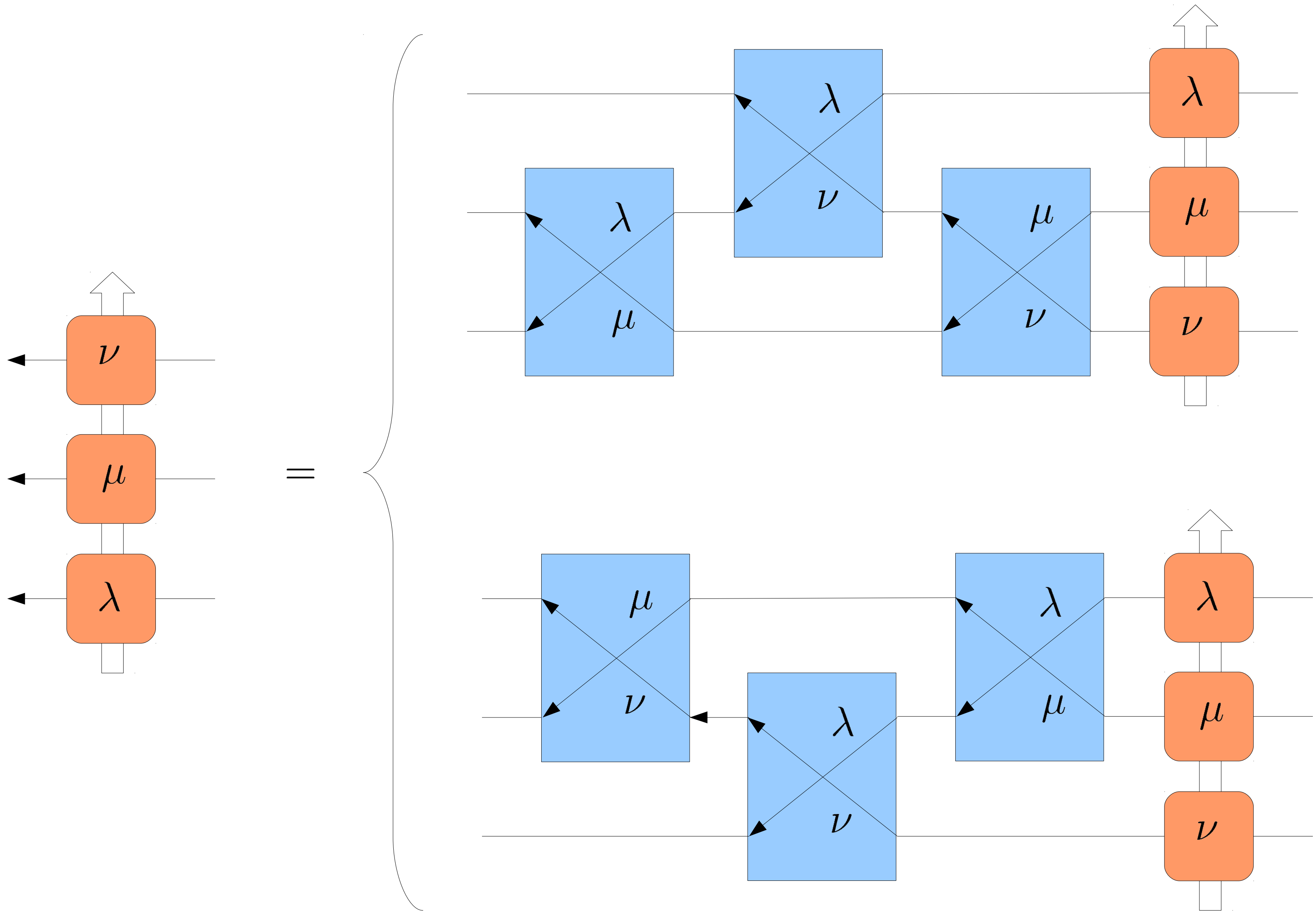
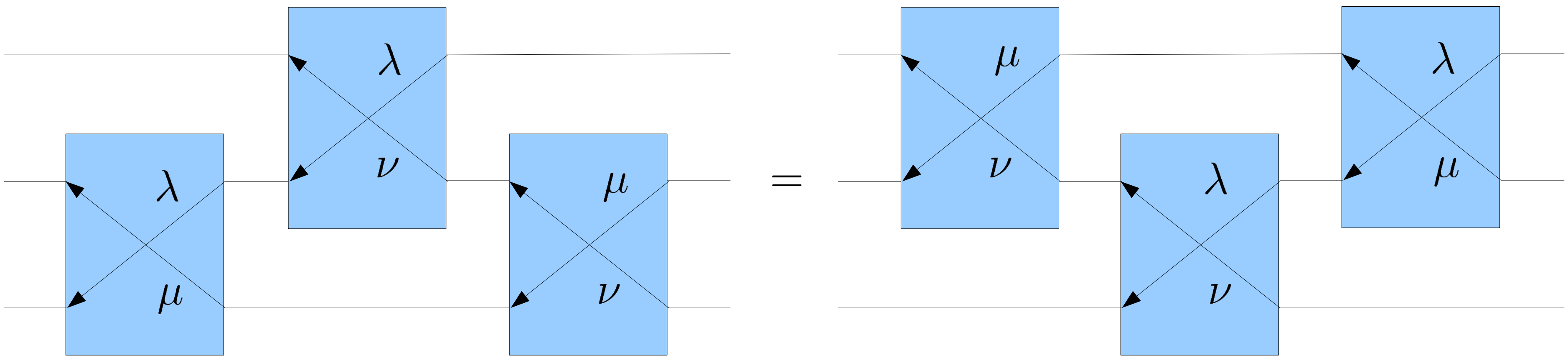


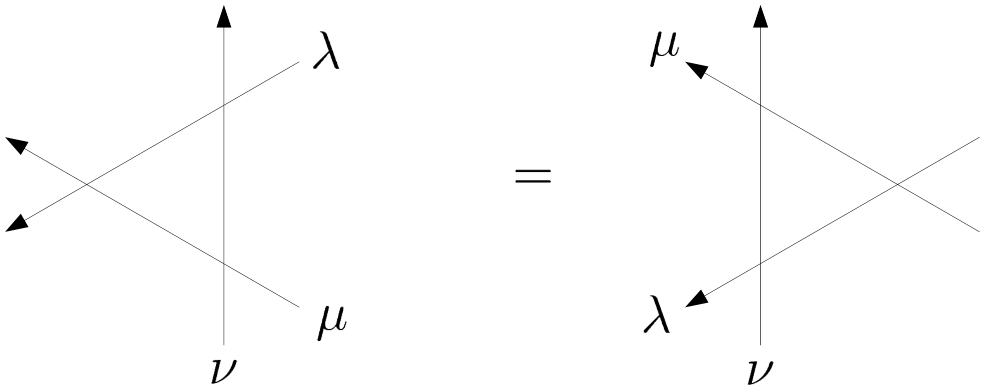
Consistency of the Yang-Baxter Algebra:



Tensor Network Formulation of the Algebraic Bethe Ansatz

Yang-Baxter Equation:

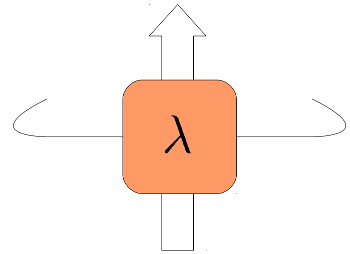




Tensor Network Formulation of the Algebraic Bethe Ansatz

Transfer Matrix:

$$t(\lambda) = \mathrm{tr}\,(T(\lambda)) \quad =$$



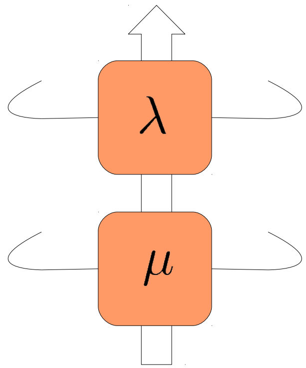
Property:

$$[t(\lambda), t(\mu)] = 0$$

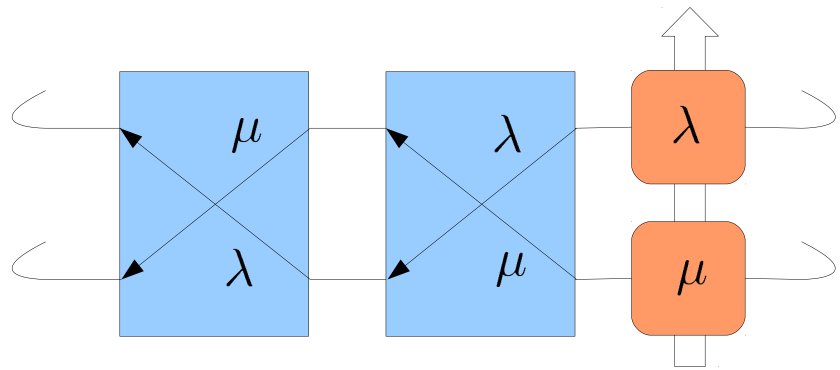
$$t(\lambda) = I_0 + \lambda I_1 + \lambda^2 I_2 + \ldots$$

$$[I_j, I_k] = 0$$

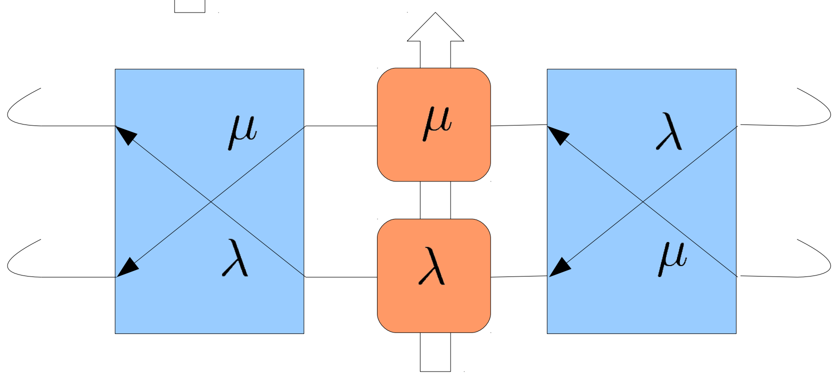
$$t(\lambda)t(\mu) \quad =$$



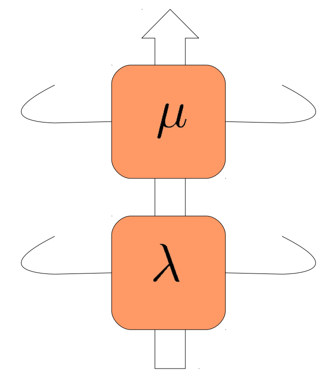
$$=$$



$$=$$



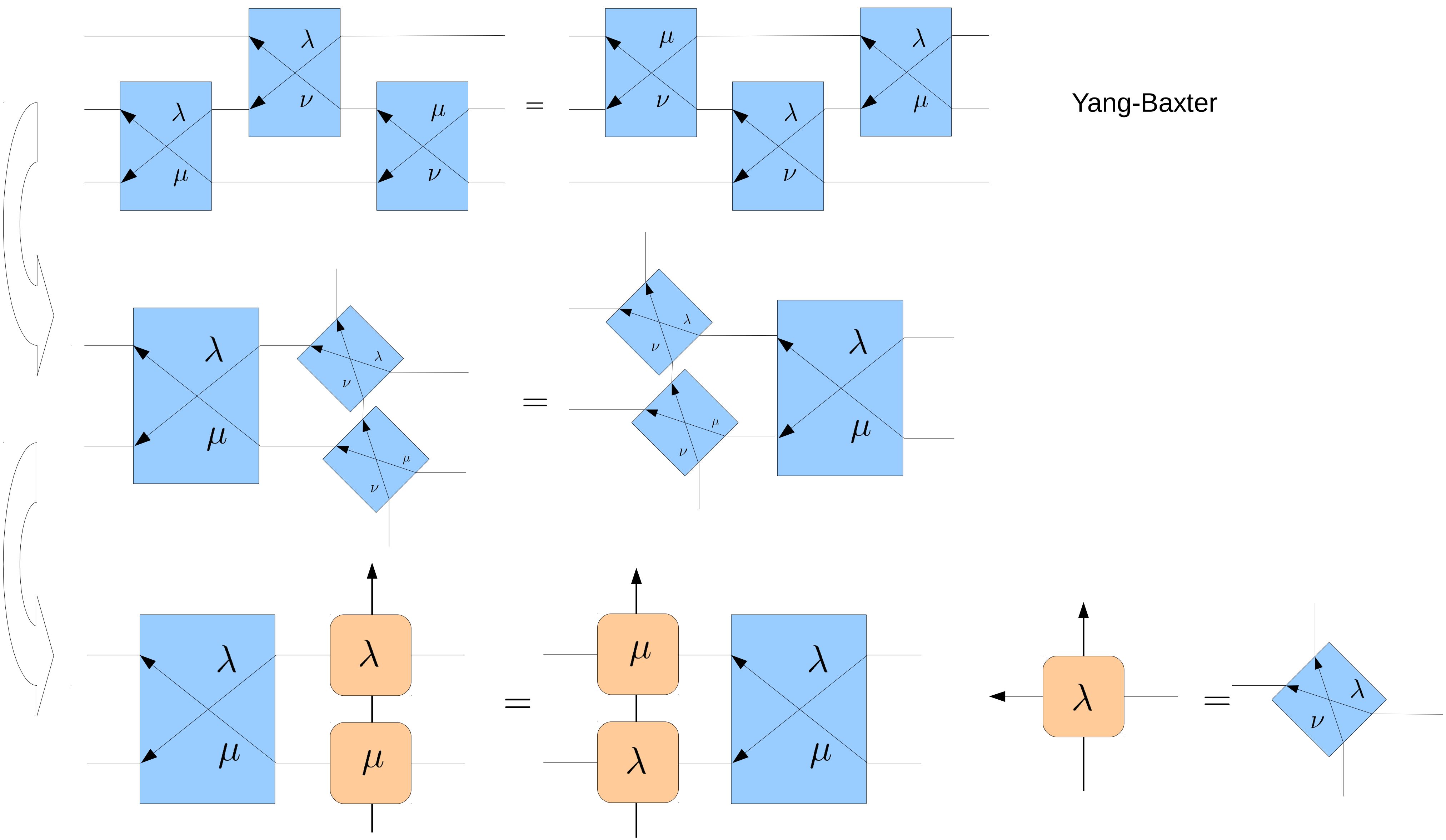
$$=$$



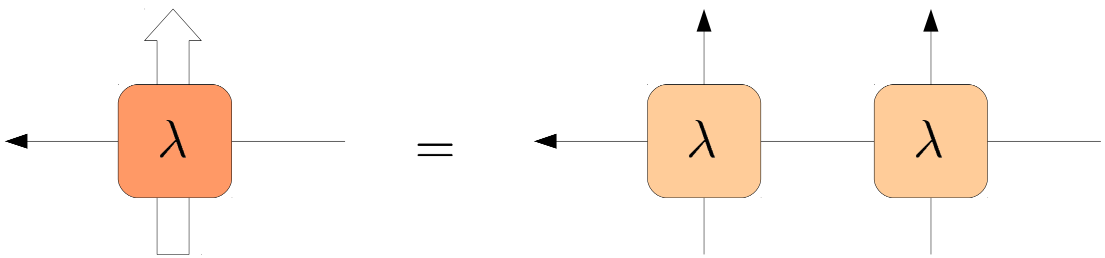
$$= \quad t(\mu)t(\lambda)$$

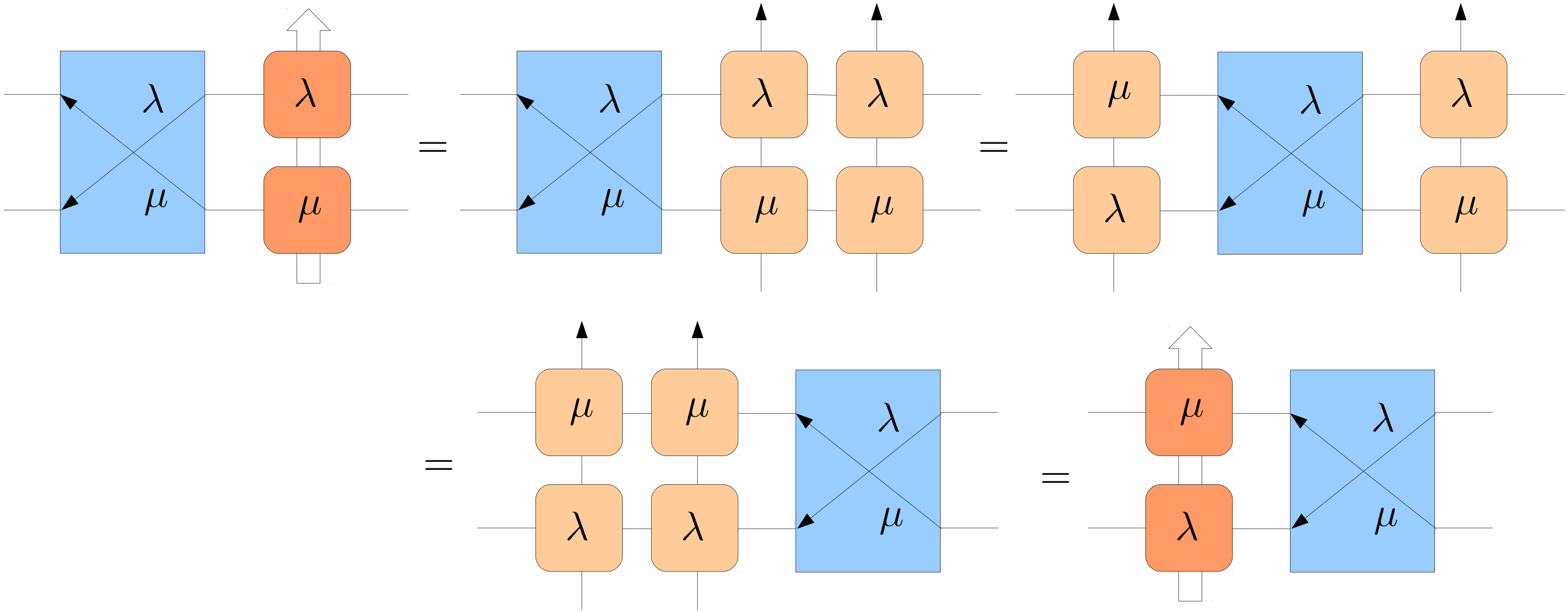
Tensor Network Formulation of the Algebraic Bethe Ansatz

Fundamental Representation:



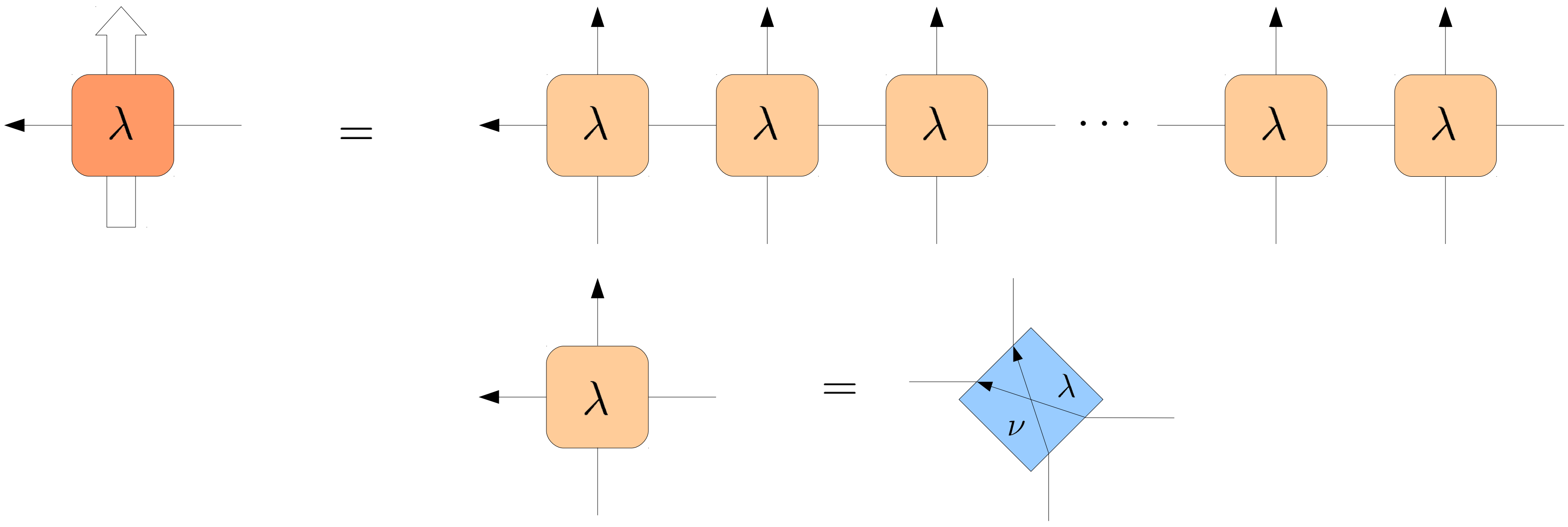
Co-Multiplication Property:





Tensor Network Formulation of the Algebraic Bethe Ansatz

Fundamental Models:

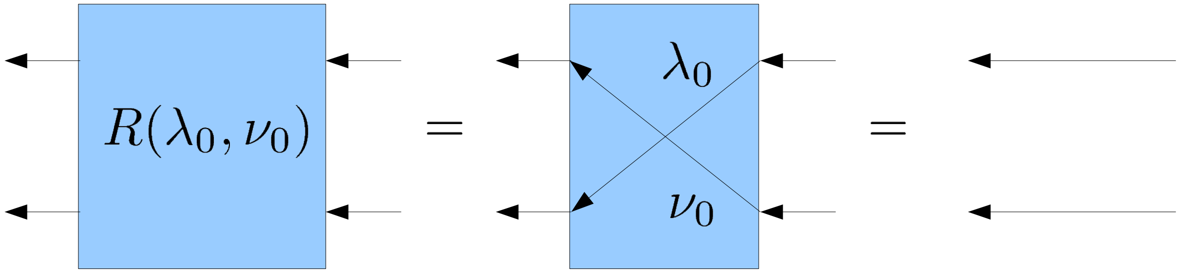


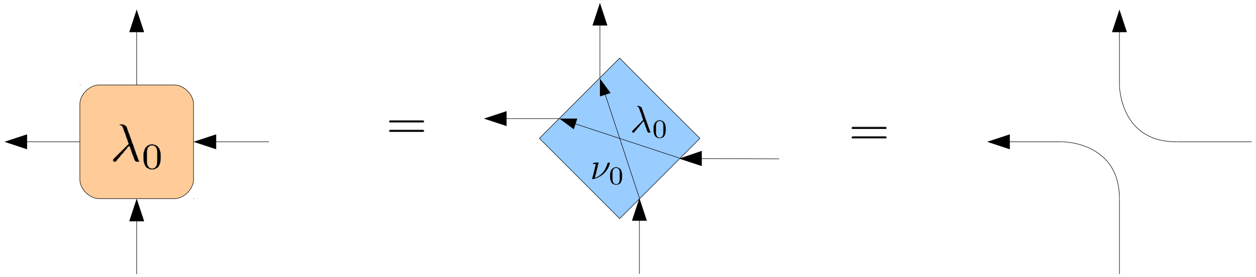
Matrix Product Operator Representation

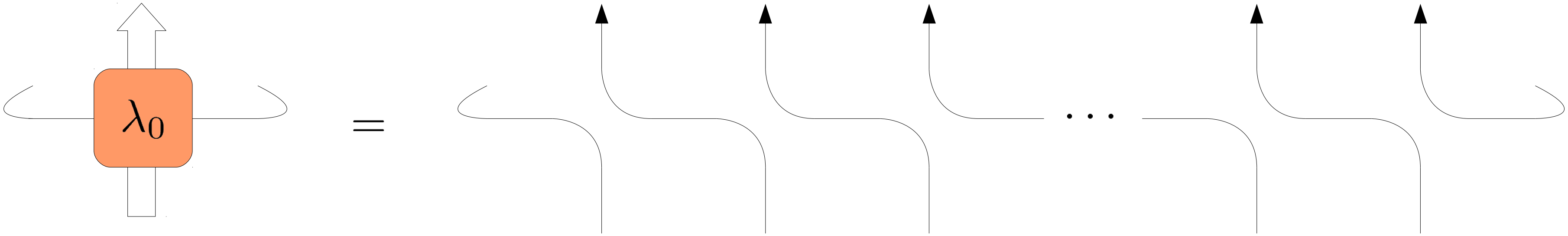
Tensor Network Formulation of the Algebraic Bethe Ansatz:

Local Hamiltonian from Transfer Matrix:

Regular Solution:

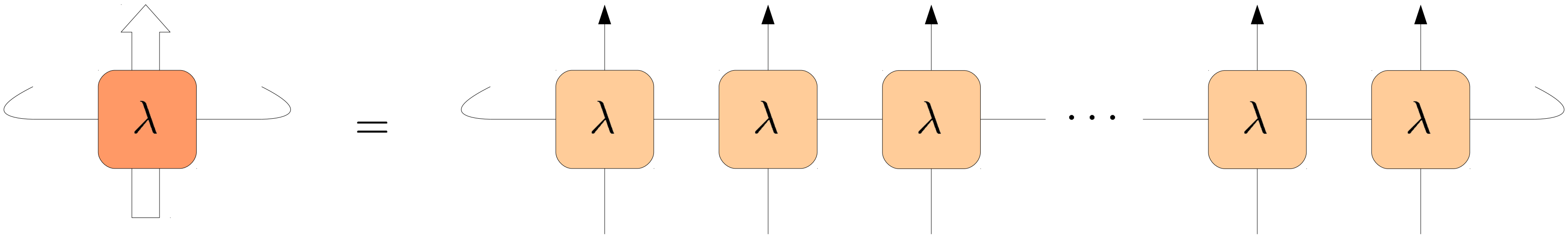






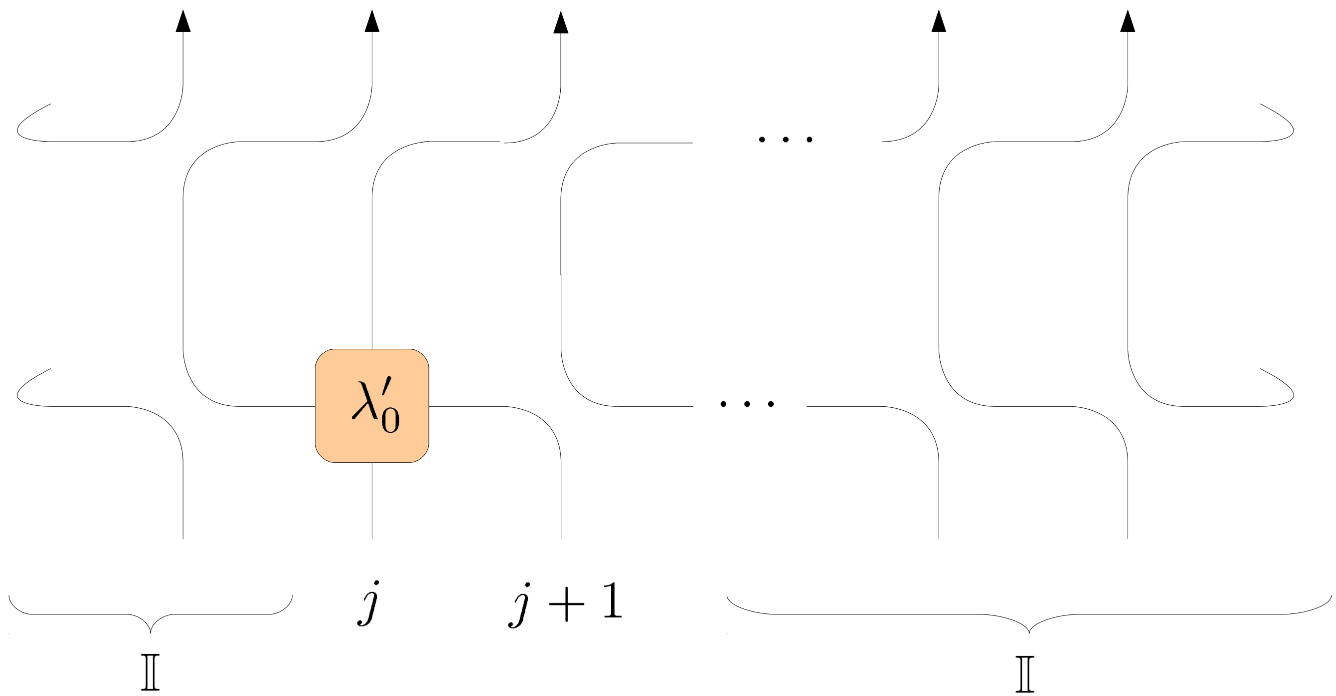
Tensor Network Formulation of the Algebraic Bethe Ansatz:

Local Hamiltonian from Transfer Matrix:



$$H = \frac{d}{d\lambda} ln\left(t(\lambda)\right)\Big|_{\lambda=\lambda_0} = t(\lambda_0)^{-1} t'(\lambda_0)$$

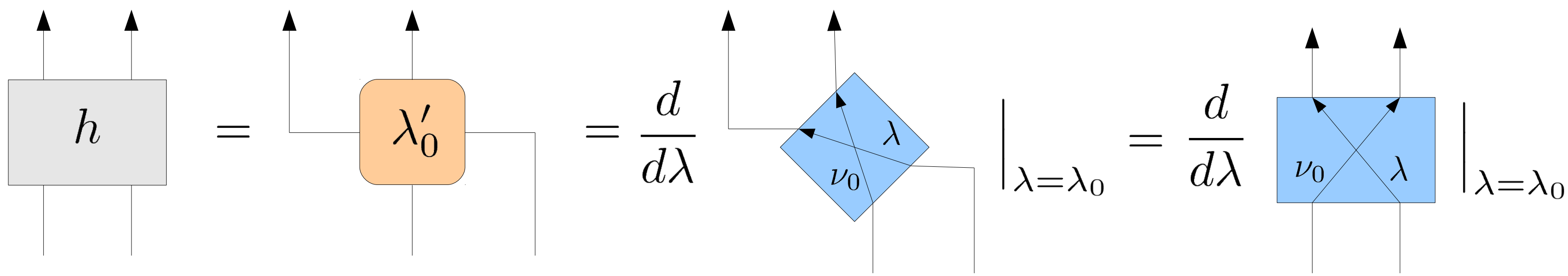
$$= \sum_j$$



Tensor Network Formulation of the Algebraic Bethe Ansatz:

Local Hamiltonian from Transfer Matrix:

$$H = \sum_{j=1}^{N} h^{(j,j+1)}$$



$$h = \frac{d}{d\lambda} R(\lambda, \nu_0)\Big|_{\lambda=\lambda_0}$$

Tensor Network Formulation of the Algebraic Bethe Ansatz:

Local Hamiltonian from Transfer Matrix:

Heisenberg Model and XXZ-Model:

$$R(\lambda,\mu) = \begin{pmatrix} 1 & & & \\ & b(\lambda,\mu) & c(\lambda,\mu) & \\ & c(\lambda,\mu) & b(\lambda,\mu) & \\ & & & 1 \end{pmatrix}$$

$$\begin{aligned} b(\lambda,\mu) &= b(\lambda-\mu) \\ c(\lambda,\mu) &= c(\lambda-\mu) \end{aligned}$$

Heisenberg Model:

$$\begin{aligned} b(\lambda) &= \frac{1}{1+\lambda} \\ c(\lambda) &= \frac{\lambda}{1+\lambda} \end{aligned}$$

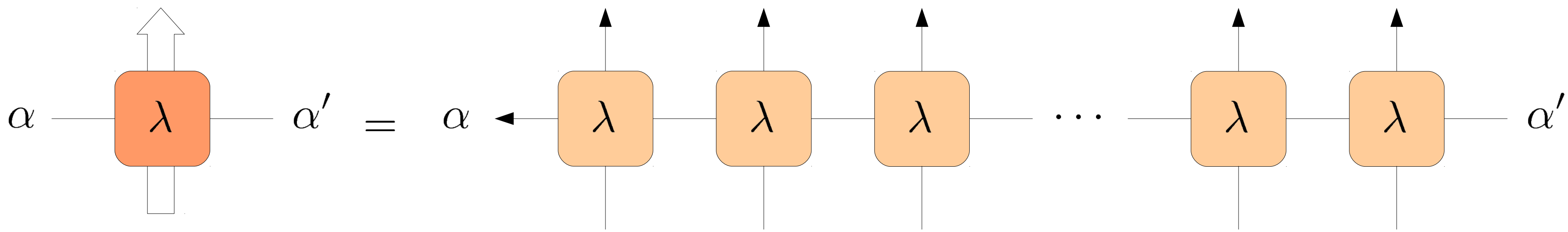
XXZ Model:

$$\begin{aligned} b(\lambda) &= \frac{\sinh(2i\eta)}{\sinh(\lambda+2i\eta)} \\ c(\lambda) &= \frac{\sinh(\lambda)}{\sinh(\lambda+2i\eta)} \end{aligned}$$

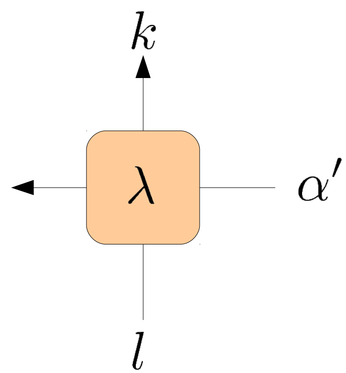
$$\Delta = cos(2\eta)$$

Tensor Network Formulation of the Algebraic Bethe Ansatz:

Symmetries:



$\langle \alpha | \mathcal{L}_l^k(\lambda) | \alpha' \rangle =$



$\neq 0 \quad \Leftrightarrow$

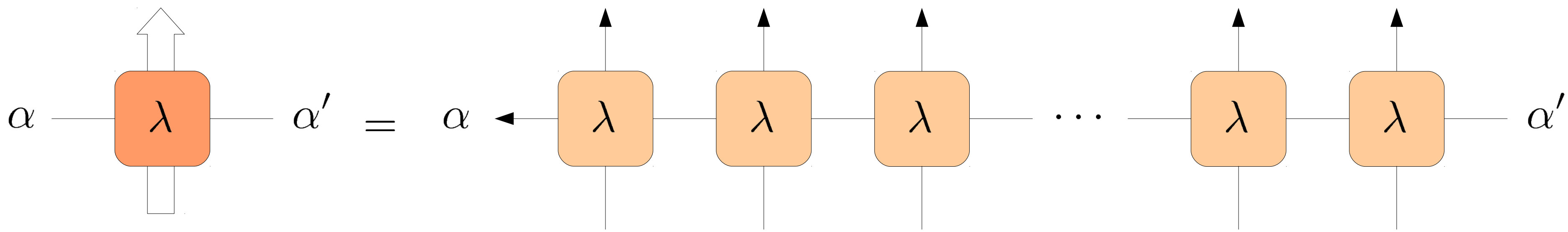
$$\alpha' = \alpha + (k - l)$$
$$(0 \leq \alpha, \alpha' \leq 1)$$

$$\mathcal{L}_0^0(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & c(\lambda) \end{pmatrix}, \quad \mathcal{L}_1^0(\lambda) = \begin{pmatrix} 0 & 0 \\ b(\lambda) & 0 \end{pmatrix}$$

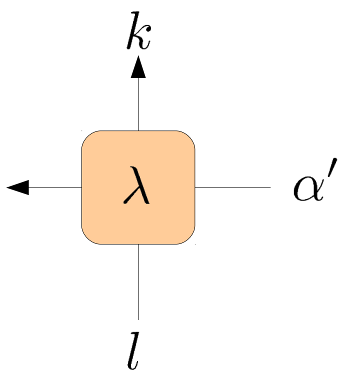
$$\mathcal{L}_0^1(\lambda) = \begin{pmatrix} 0 & b(\lambda) \\ 0 & 0 \end{pmatrix}, \quad \mathcal{L}_1^1(\lambda) = \begin{pmatrix} c(\lambda) & 0 \\ 0 & 1 \end{pmatrix}$$

Tensor Network Formulation of the Algebraic Bethe Ansatz:

Symmetries:



$$\langle \alpha | \mathcal{L}_l^k(\lambda) | \alpha' \rangle =$$



$$\neq 0 \quad \Leftrightarrow \quad \begin{array}{c} \alpha' = \alpha + (l - k) \\ (0 \leq \alpha, \alpha' \leq 1) \end{array}$$

$$A(\lambda) = T_0^0(\lambda)$$

$$D(\lambda) = T_1^1(\lambda)$$

Keeps number of down-spins constant

$$B(\lambda) = T_1^0(\lambda)$$

Creates one down-spin

$$C(\lambda) = T_0^1(\lambda)$$

Annihilates one down-spin

Tensor Network Formulation of the Algebraic Bethe Ansatz:

Bethe Ansatz:

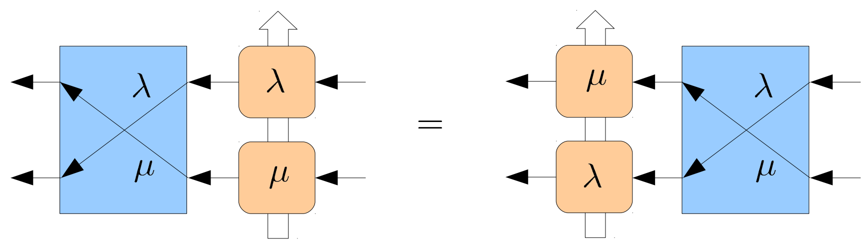
$$| \Psi(\mu_1, \ldots, \mu_M) \rangle = B(\mu_1) \cdots B(\mu_M) | \uparrow, \ldots, \uparrow \rangle$$

$$A(\lambda) = T_0^0(\lambda)$$
$$B(\lambda) = T_1^0(\lambda)$$
$$D(\lambda) = T_1^1(\lambda)$$
$$C(\lambda) = T_0^1(\lambda)$$

Eigenvalue Problem:

$$t(\lambda) | \Psi \rangle = \tau(\lambda) | \Psi \rangle \qquad t(\lambda) = A(\lambda) + D(\lambda)$$

Solvable, using only:



$$\begin{aligned} B(\lambda)B(\mu) &= B(\mu)B(\lambda) \\ A(\lambda)B(\mu) &= \tfrac{1}{c(\mu,\lambda)} B(\mu)A(\lambda) - \tfrac{b(\mu,\lambda)}{c(\mu,\lambda)} B(\lambda)A(\mu) \\ D(\lambda)B(\mu) &= \tfrac{1}{c(\lambda,\mu)} B(\mu)D(\lambda) - \tfrac{b(\lambda,\mu)}{c(\lambda,\mu)} B(\lambda)D(\mu). \end{aligned}$$

Bethe Equations:

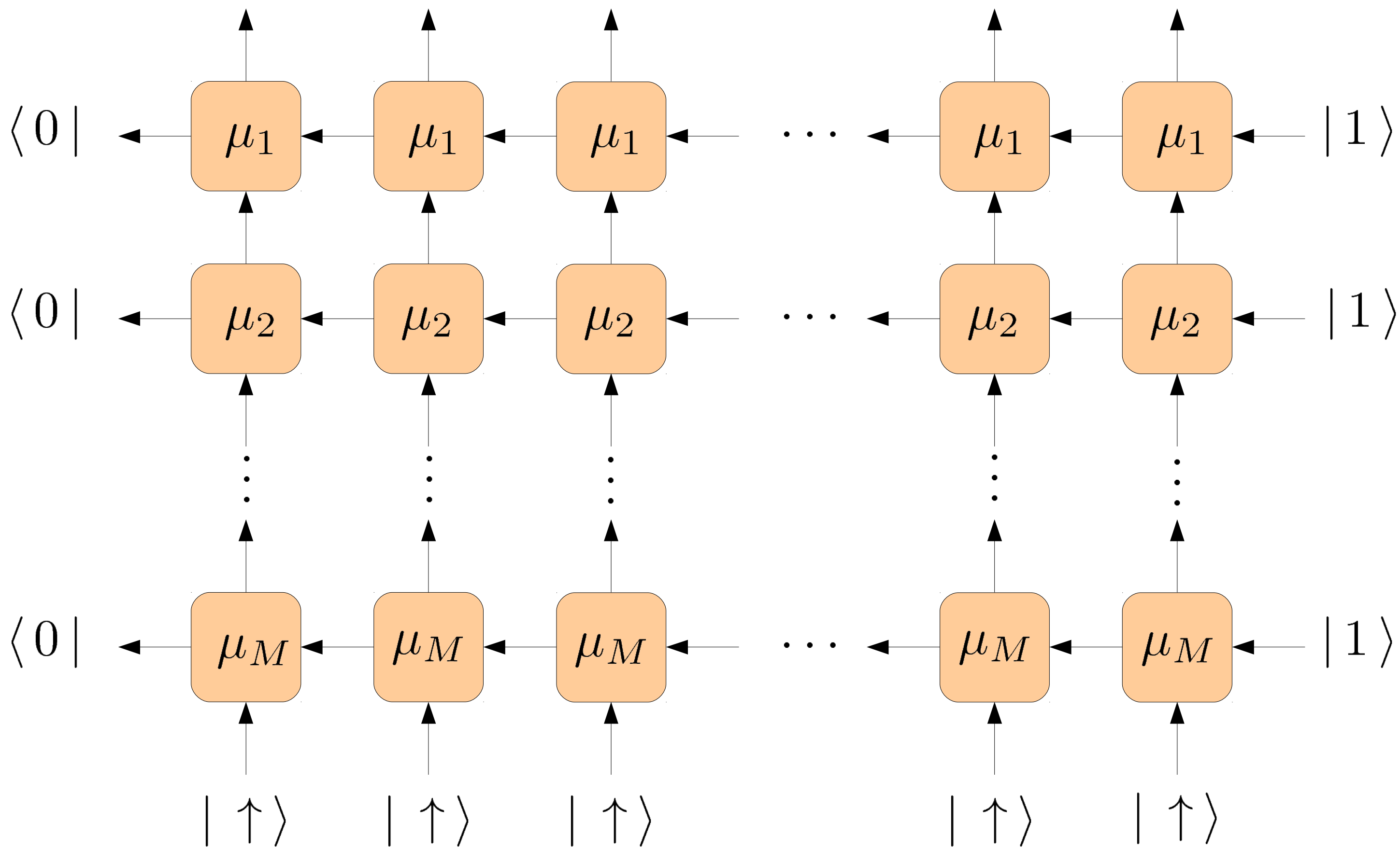
$$\frac{d(\mu_n)}{a(\mu_n)} = \prod_{\substack{j=1 \\ j \neq n}}^{M} \frac{c(\mu_n, \mu_j)}{c(\mu_j, \mu_n)}$$

Tensor Network Formulation of the Algebraic Bethe Ansatz:

Bethe-Eigenstate:

$$|\,\Psi(\mu_1,\ldots,\mu_M)\,\rangle = B(\mu_1)\cdots B(\mu_M)|\,vac\,\rangle \quad =$$

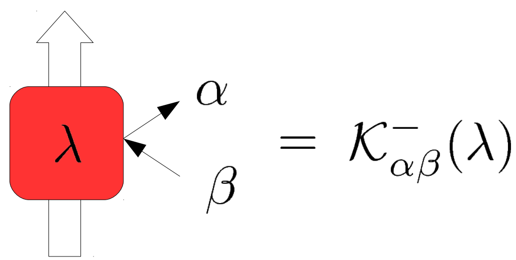
$$=$$

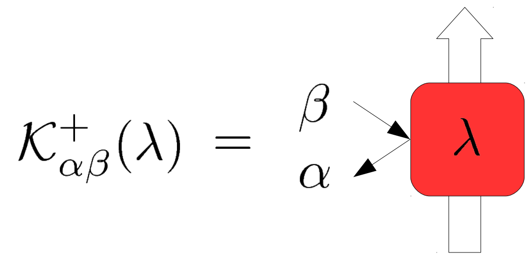


State with $\quad S_z = \frac{1}{2}N - M$

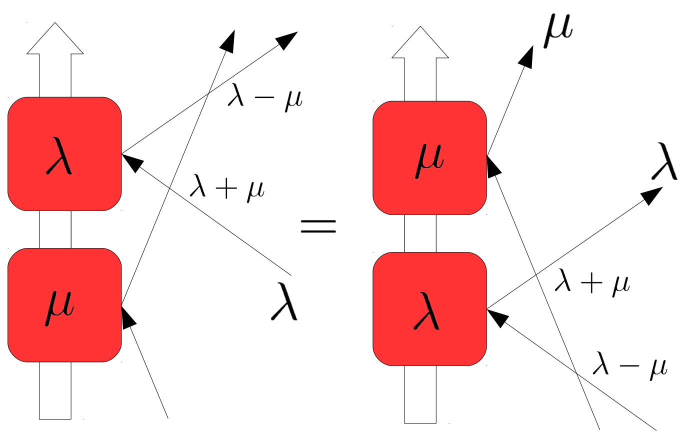
Tensor Network Formulation of the Algebraic Bethe Ansatz (open boundary conditions)

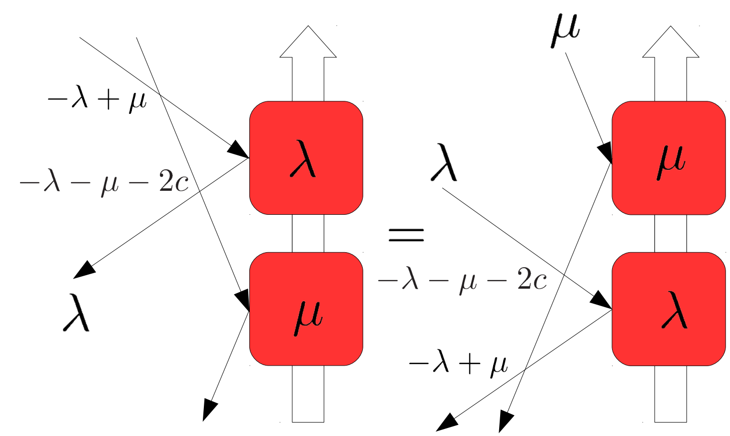
Reflection Algebra:





Reflection Equations:

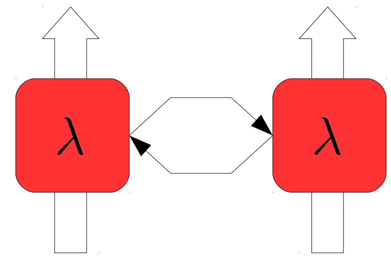




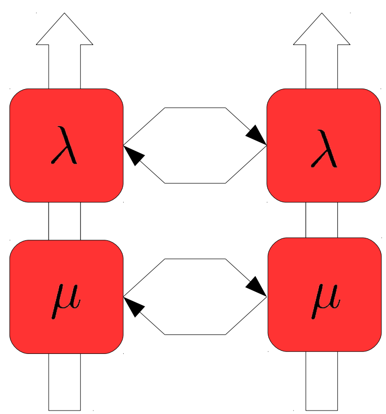
Tensor Network Formulation of the Algebraic Bethe Ansatz (open boundary conditions)

Transfer Matrix:

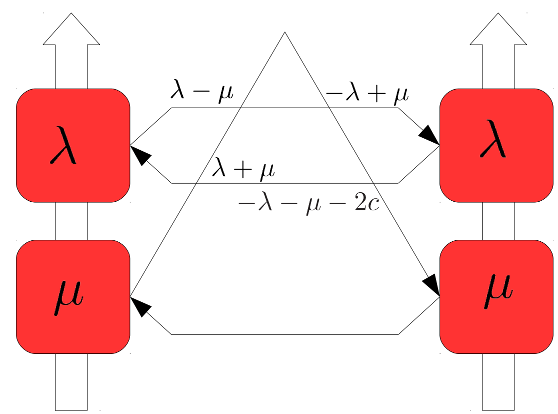
$$\tau(\lambda) = \mathrm{tr}\left(\mathcal{K}^{-}(\lambda)\mathcal{K}^{+}(\lambda)\right) =$$



$$\tau(\lambda)\tau(\mu) =$$

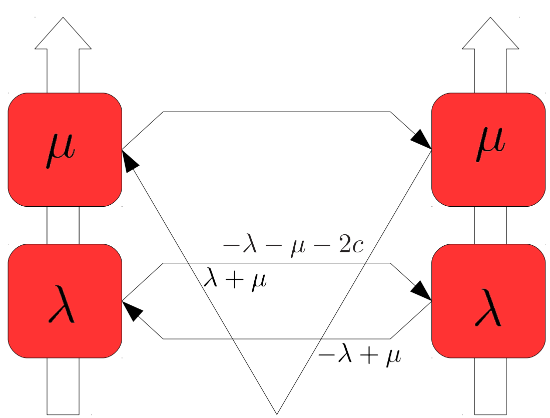


$$=$$

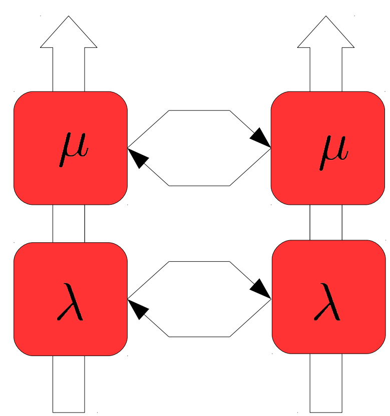


$$=$$

$$=$$



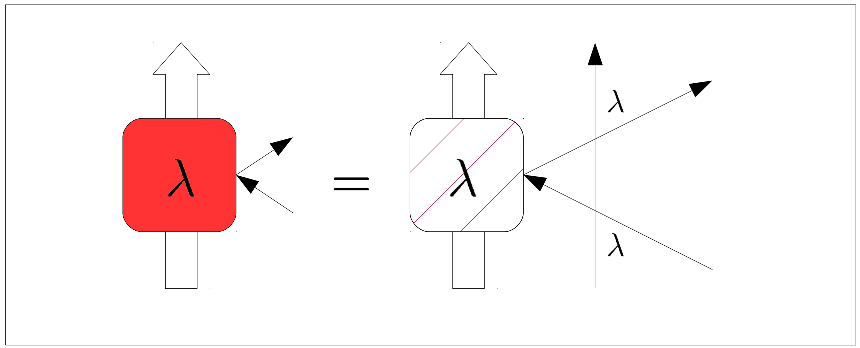
$$=$$



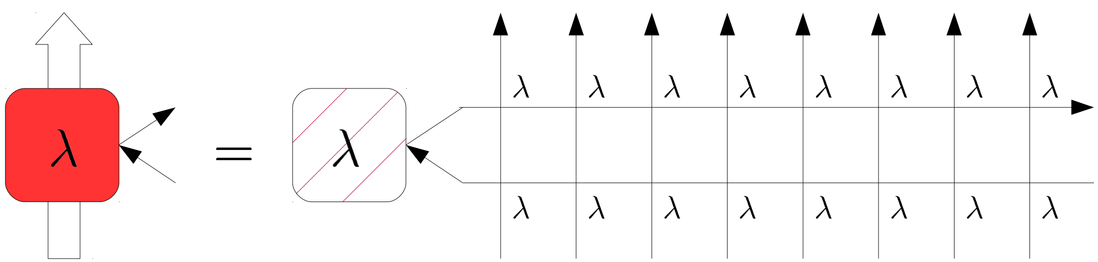
$$= \tau(\mu)\tau(\lambda)$$

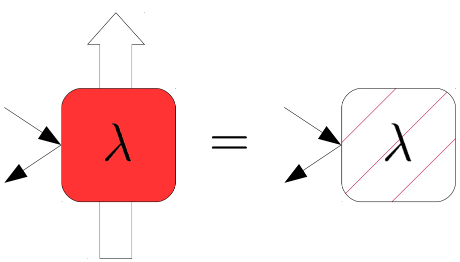
Tensor Network Formulation of the Algebraic Bethe Ansatz (open boundary conditions)

Fundamental Models:



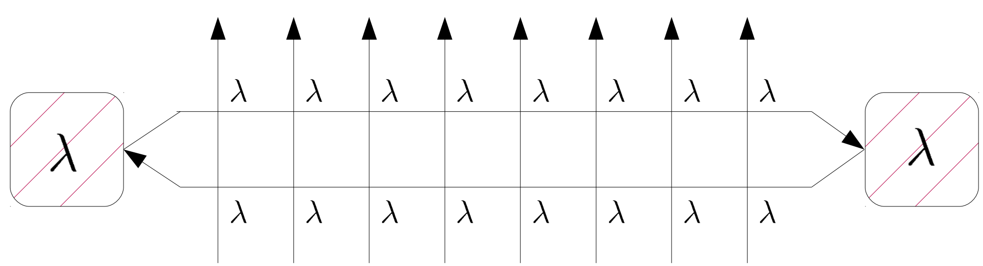
Composition of a new representation out of 2 R-matrices and a known representation.



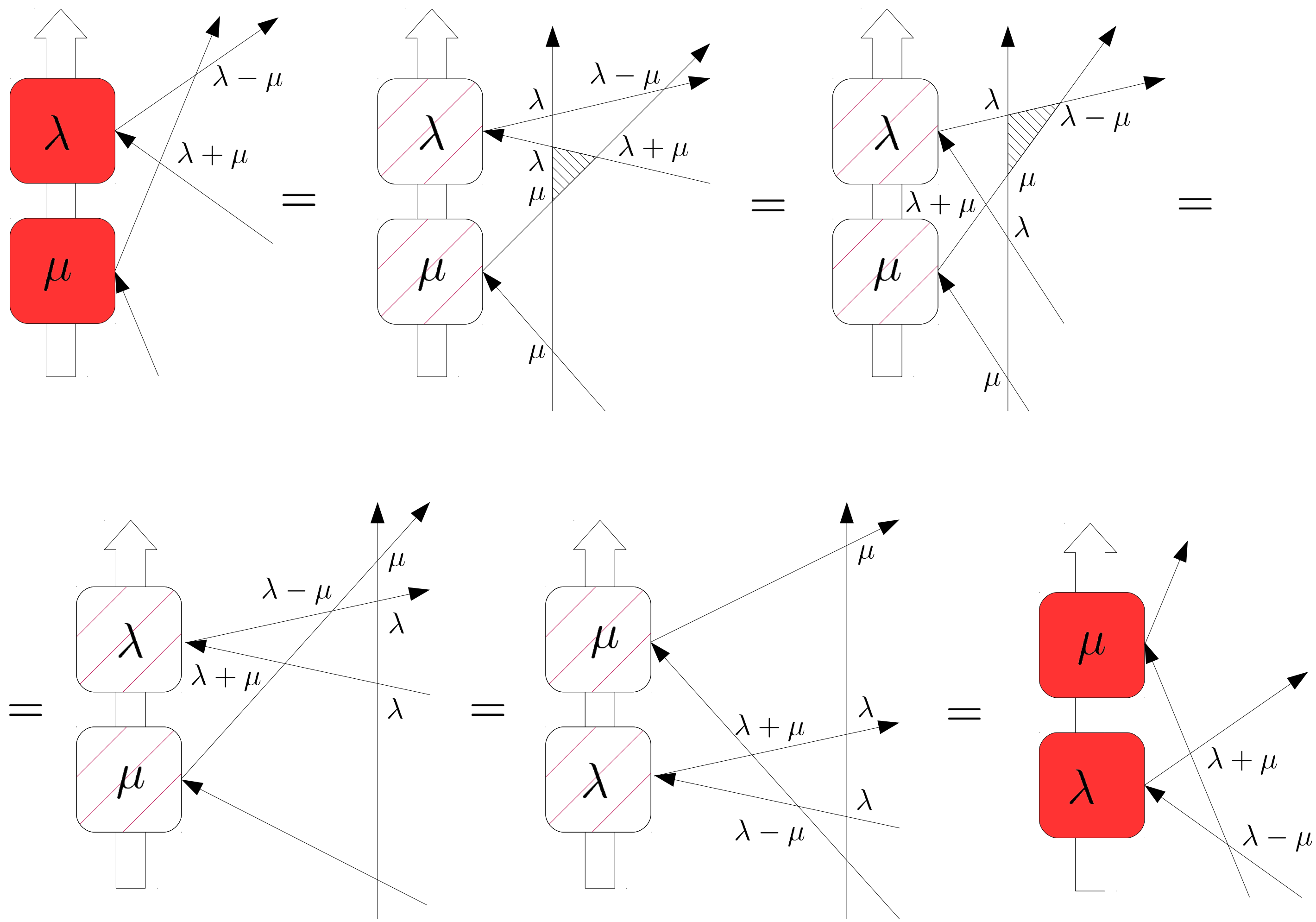


Transfer Matrix:

$$\tau(\lambda) =$$

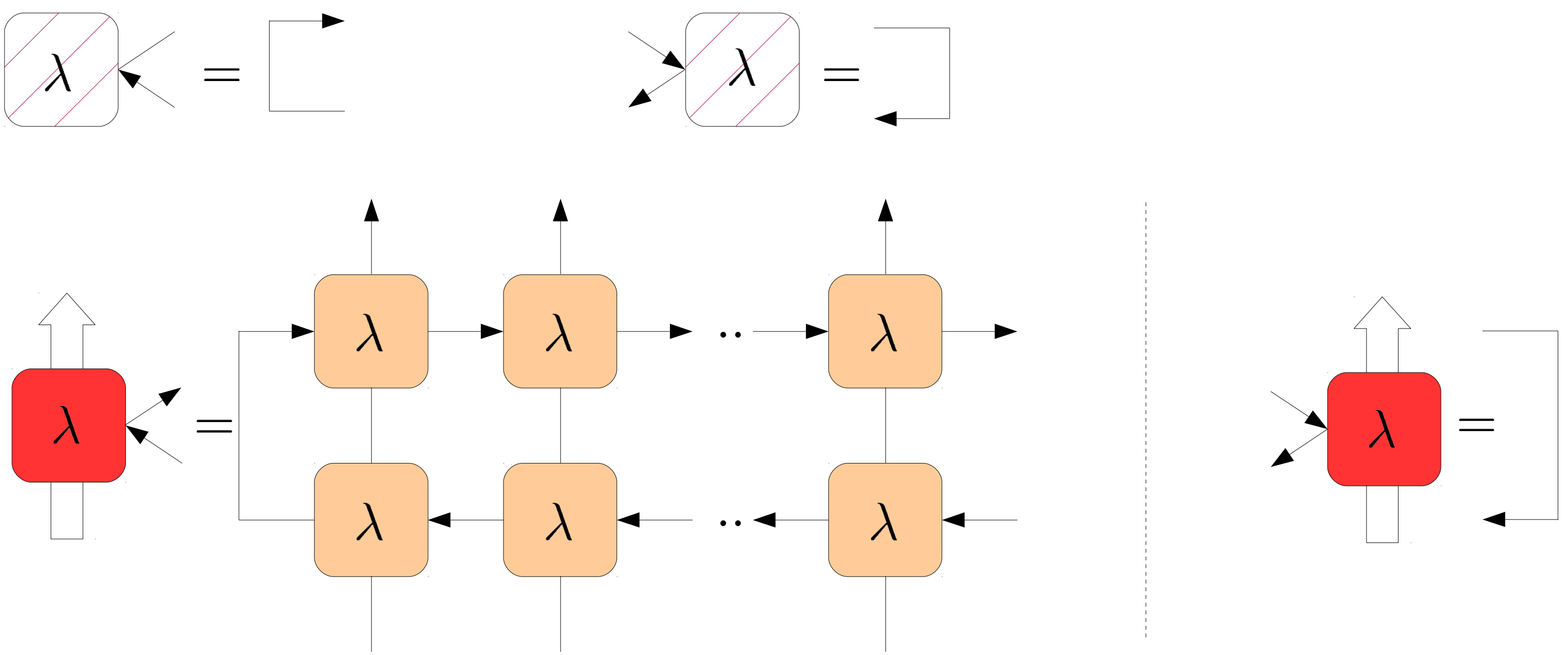


Proof of Composition-Equation:

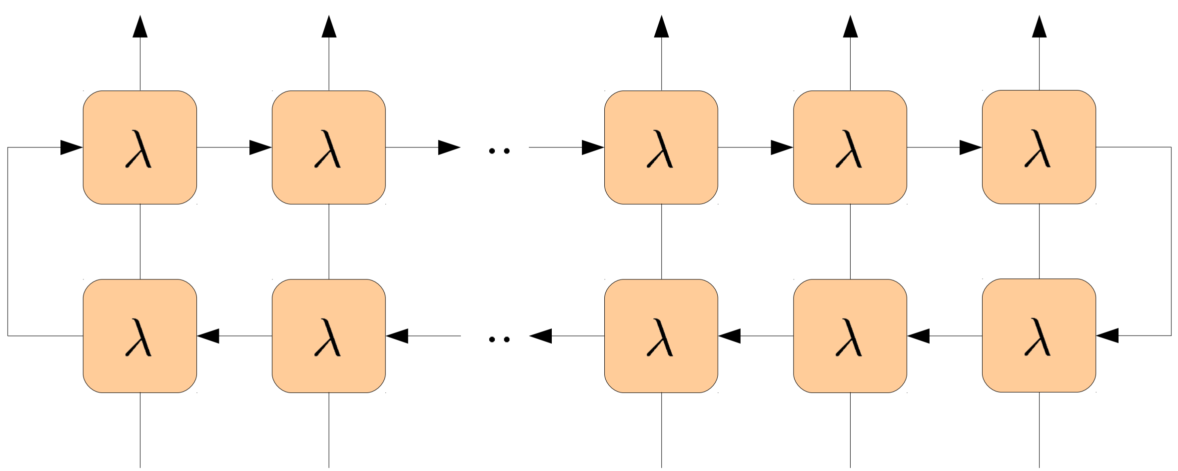


Tensor Network Formulation of the Algebraic Bethe Ansatz (open boundary conditions)

Simplest Representation:

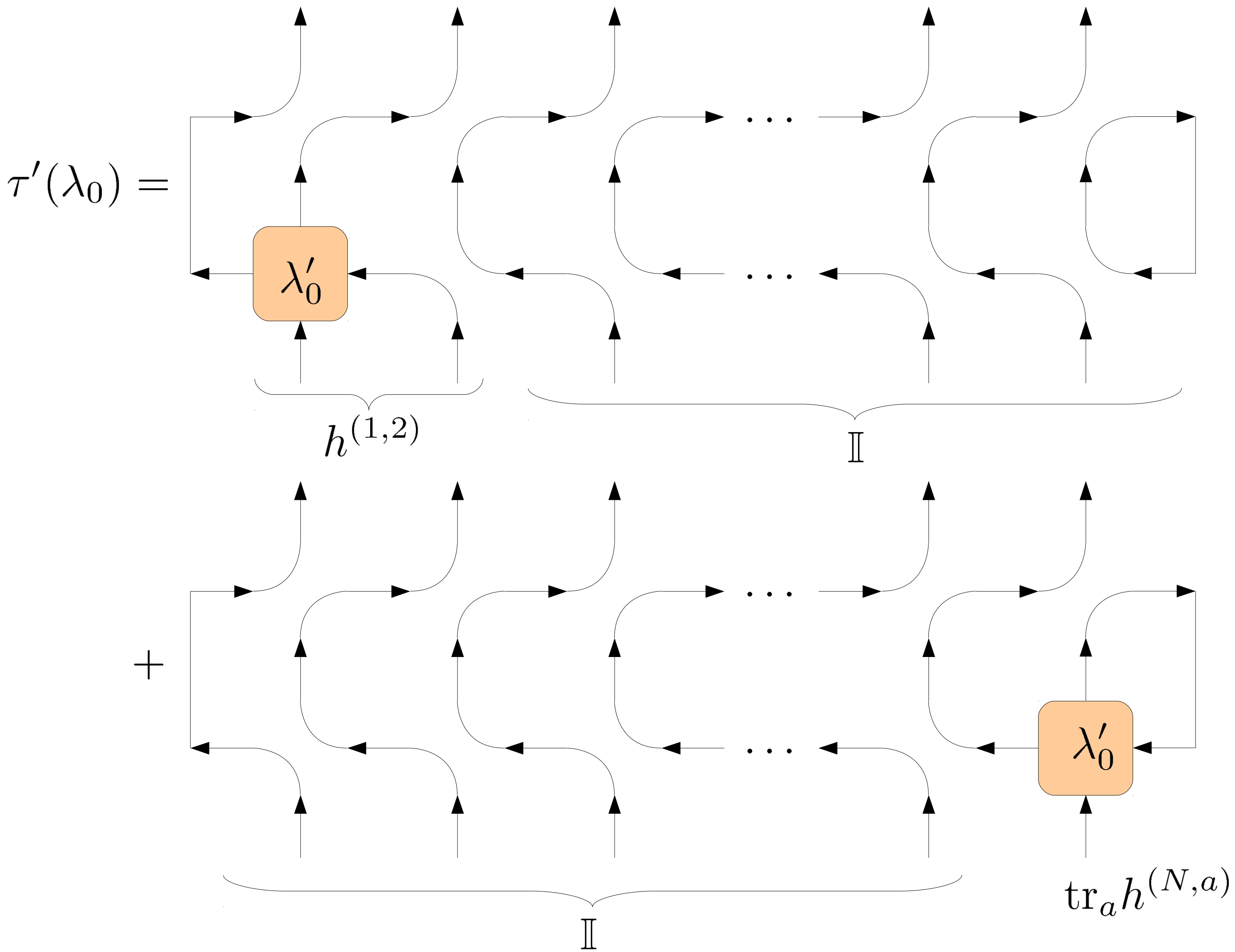


$\tau(\lambda) =$



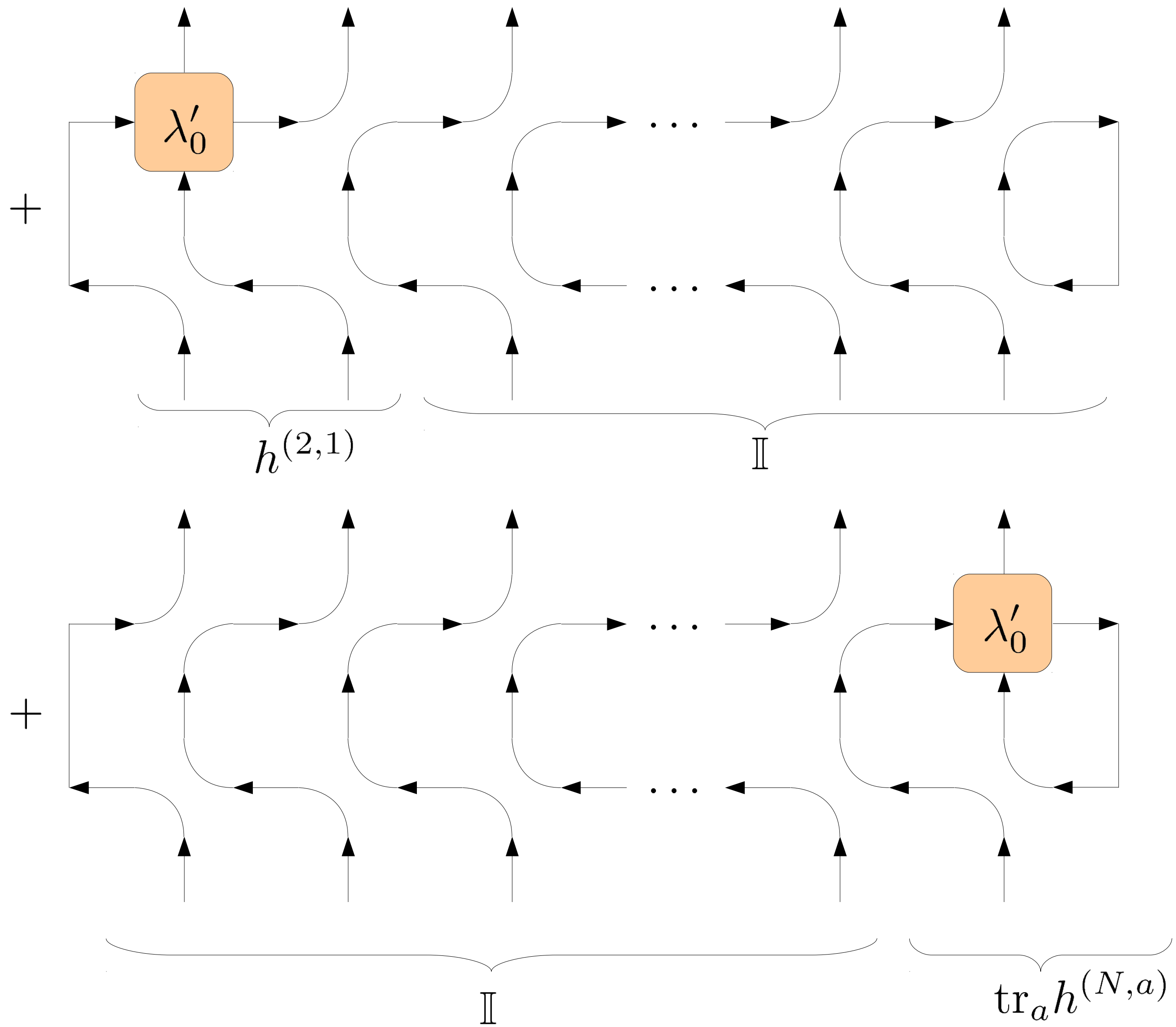
Tensor Network Formulation of the Algebraic Bethe Ansatz (open boundary conditions)

Local Hamiltonian from Transfer Matrix:



Tensor Network Formulation of the Algebraic Bethe Ansatz (open boundary conditions)

Local Hamiltonian from Transfer Matrix:

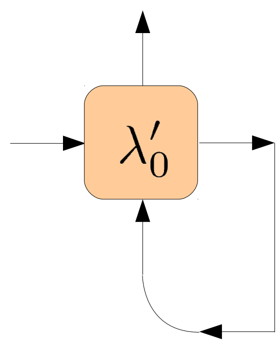


Tensor Network Formulation of the Algebraic Bethe Ansatz (open boundary conditions)

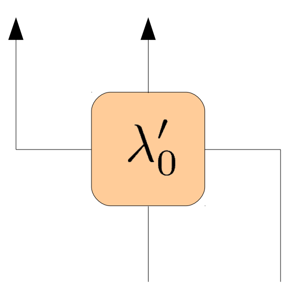
Local Hamiltonian from Transfer Matrix:

$$H = \frac{1}{2d}\tau'(\lambda_0) \equiv \sum_{i=1}^{N-1} h^{(i,i+1)} + \frac{1}{d}\mathrm{tr}_a h^{(N,a)}$$

$$\mathrm{tr}_a h^{(N,a)} =$$

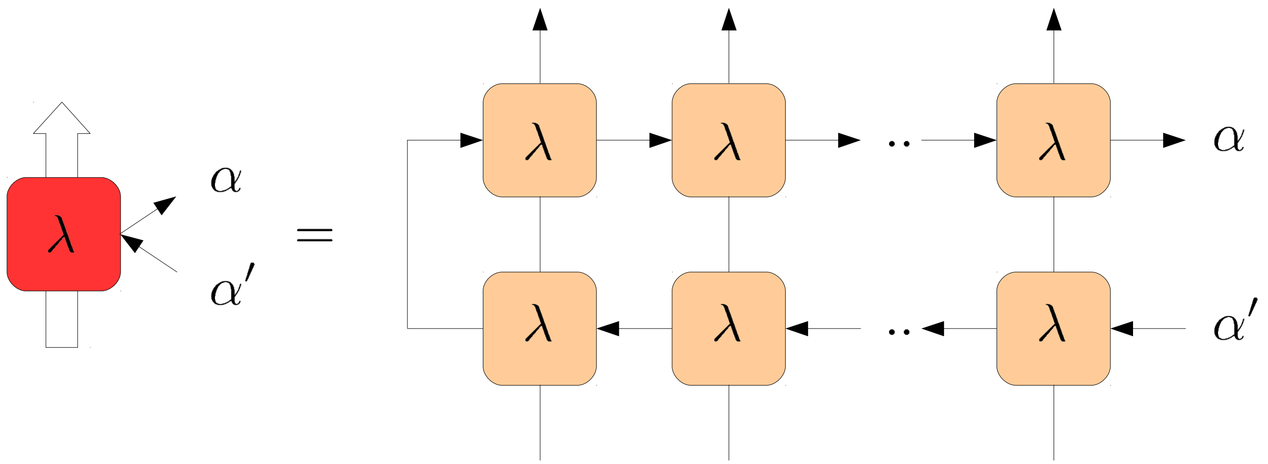


$$h =$$



Tensor Network Formulation of the Algebraic Bethe Ansatz (open boundary conditions)

Symmetries:



$$\mathcal{A}(\lambda) = \mathcal{K}^{-}_{00}(\lambda)$$
$$\mathcal{D}(\lambda) = \mathcal{K}^{-}_{11}(\lambda)$$

Keeps number of down-spins constant

$$\mathcal{B}(\lambda) = \mathcal{K}^{-}_{01}(\lambda)$$ Creates one down-spin

$$\mathcal{C}(\lambda) = \mathcal{K}^{-}_{10}(\lambda)$$ Annihilates one down-spin

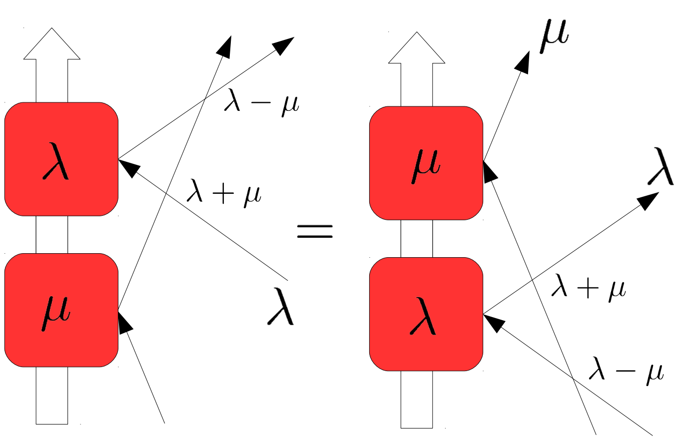
Tensor Network Formulation of the Algebraic Bethe Ansatz (open boundary conditions)

Bethe Ansatz:

$$|\,\Psi(\mu_1,\ldots,\mu_M)\,\rangle = \mathcal{B}(\mu_1)\cdots\mathcal{B}(\mu_M)|\,vac\,\rangle$$

Eigenvalue Problem: $\mathcal{T}(\lambda)|\,\Psi\,\rangle = \tau(\lambda)|\,\Psi\,\rangle \qquad \mathcal{T}(\lambda) = A(\lambda) + D(\lambda)$

Solvable, using only:



$\Rightarrow$ Bethe Equations

Time Evolution: Numerical Approximation

Order Optimization

$$|\,\Psi(\mu_1,\ldots,\mu_M)\,\rangle = B(\mu_1)\cdots B(\mu_M)|\,vac\,\rangle$$

$$B(\mu_M)|\,vac\,\rangle$$

$$B(\mu_{M-1})B(\mu_M)|\,vac\,\rangle$$

$$B(\mu_{M-2})B(\mu_{M-1})B(\mu_M)|\,vac\,\rangle$$

$$\vdots$$

$$B(\mu_1)\cdots B(\mu_M)|\,vac\,\rangle$$

$$[B(\lambda),B(\mu)]=0 \quad \Rightarrow$$ Order of the B's is arbitrary!

Time evolution in ***one*** dimension***:***

Matrix Product Operator (MPO)

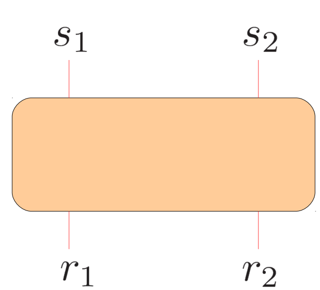
$$H = \sum_{j=1}^{N-1} h^{(j,j+1)} = H_{\text{even}} + H_{\text{odd}}$$

$$H_{\text{even}} = \sum_{j \text{ even}} h^{(j,j+1)}$$

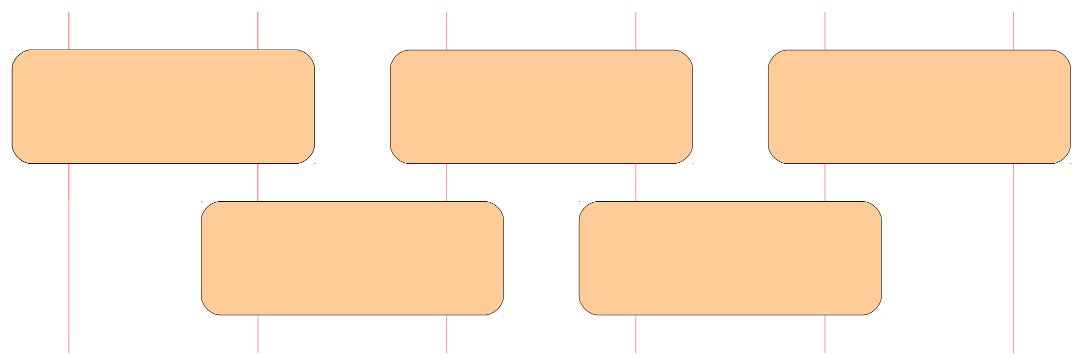
$$H_{\text{odd}} = \sum_{j \text{ odd}} h^{(j,j+1)}$$

$$e^{-i\delta t H} \approx e^{-i\delta t H_{\text{odd}}} e^{-i\delta t H_{\text{even}}}$$

$$[e^{-i\delta t h}]^{s_1 s_2}_{r_1 r_2} =$$



$$e^{-i\delta t H} \approx$$



$e^{-i\delta t H_{\text{odd}}}$

$e^{-i\delta t H_{\text{even}}}$

Tensor Network Operators

Time evolution in ***one*** dimension***:***

Matrix Product Operator (MPO)

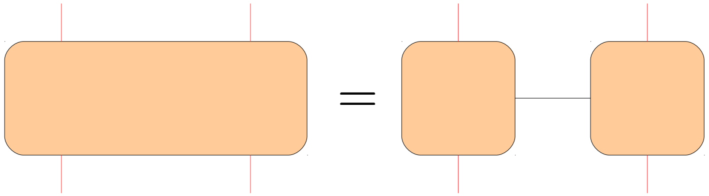
$$H = \sum_{j=1}^{N-1} h^{(j,j+1)} = H_{\text{even}} + H_{\text{odd}}$$

$$H_{\text{even}} = \sum_{j \text{ even}} h^{(j,j+1)}$$

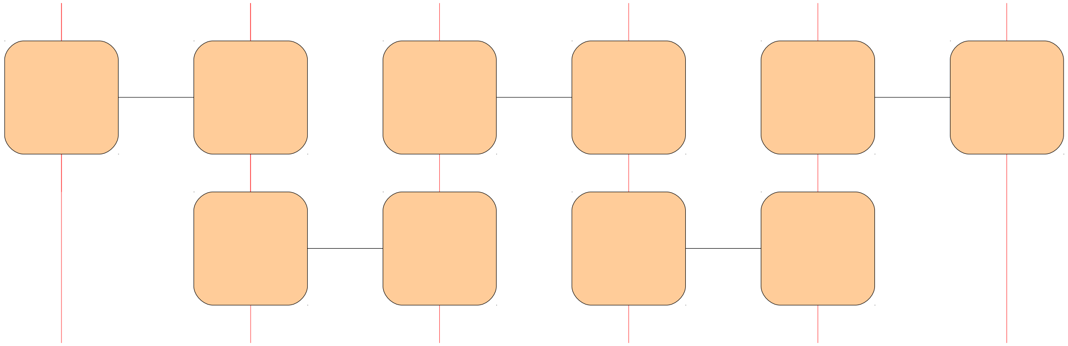
$$H_{\text{odd}} = \sum_{j \text{ odd}} h^{(j,j+1)}$$

$$e^{-i\delta t H} \approx e^{-i\delta t H_{\text{odd}}} e^{-i\delta t H_{\text{even}}}$$

$$e^{-i\delta t h} =$$



$$e^{-i\delta t H} \approx$$



$$e^{-i\delta t H_{\text{odd}}}$$

$$e^{-i\delta t H_{\text{even}}}$$

Tensor Network Operators

Time evolution in ***one*** dimension***:***

Matrix Product Operator (MPO)

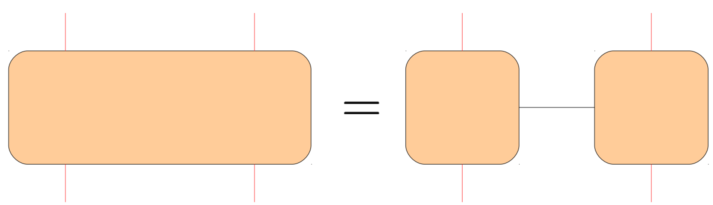
$$H = \sum_{j=1}^{N-1} h^{(j,j+1)} = H_{\text{even}} + H_{\text{odd}}$$

$$H_{\text{even}} = \sum_{j \text{ even}} h^{(j,j+1)}$$

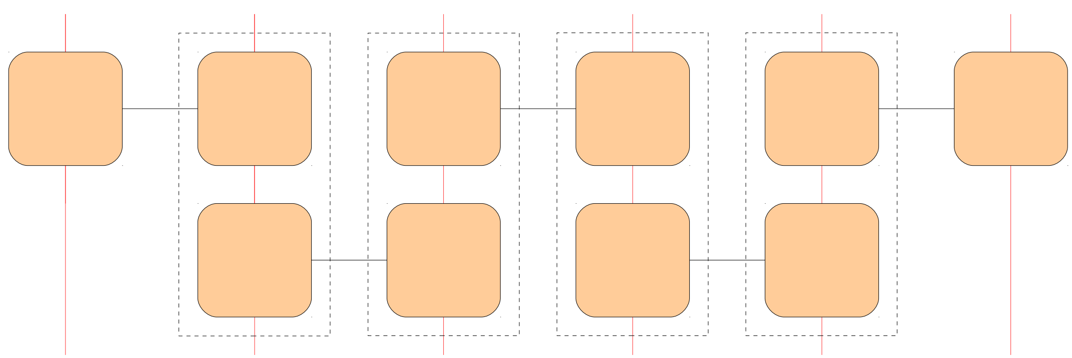
$$H_{\text{odd}} = \sum_{j \text{ odd}} h^{(j,j+1)}$$

$$e^{-i\delta t H} \approx e^{-i\delta t H_{\text{odd}}} e^{-i\delta t H_{\text{even}}}$$

$$e^{-i\delta t h} =$$



$$e^{-i\delta t H} \approx$$



$$e^{-i\delta t H_{\text{odd}}}$$

$$e^{-i\delta t H_{\text{even}}}$$

Tensor Network Operators

Time evolution in ***one*** dimension***:***

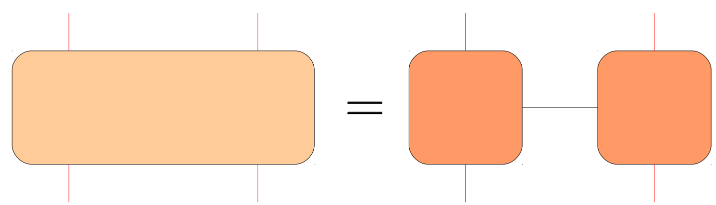
Matrix Product Operator (MPO)

$$H = \sum_{j=1}^{N-1} h^{(j,j+1)} = H_{\text{even}} + H_{\text{odd}}$$

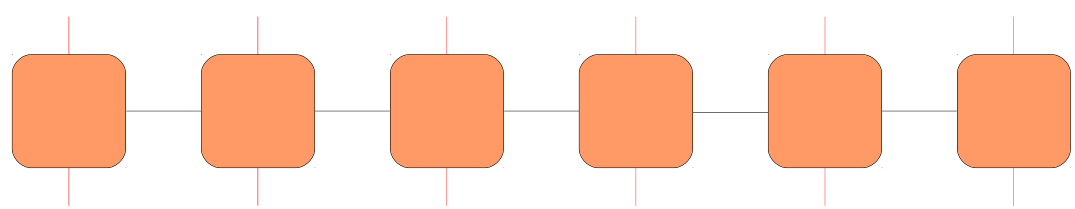
$$\begin{cases} H_{\text{even}} = \sum_{j \text{ even}} h^{(j,j+1)} \\ H_{\text{odd}} = \sum_{j \text{ odd}} h^{(j,j+1)} \end{cases}$$

$$e^{-i\delta t H} \approx e^{-i\delta t H_{\text{odd}}} e^{-i\delta t H_{\text{even}}}$$

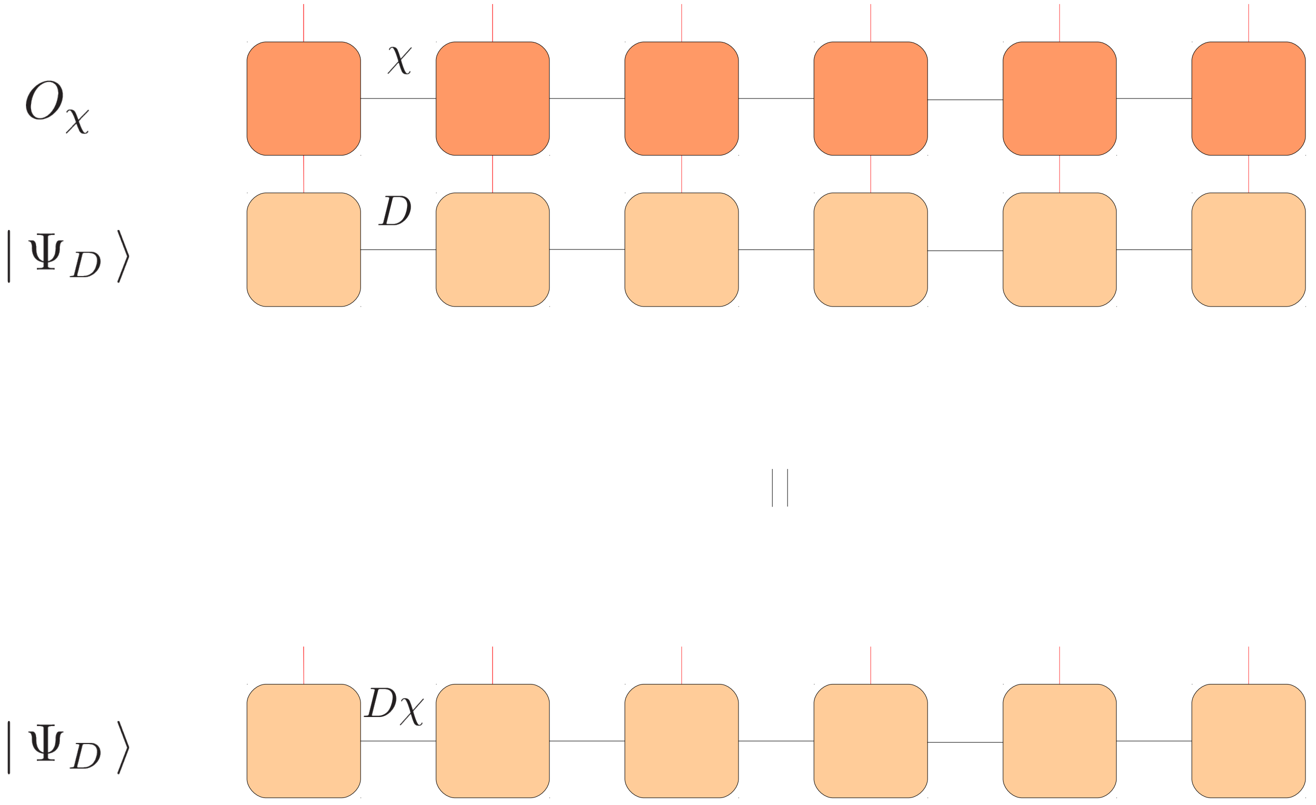
$$e^{-i\delta t h} =$$



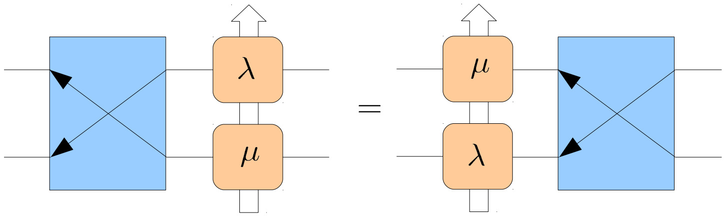
$$e^{-i\delta t H} \approx$$

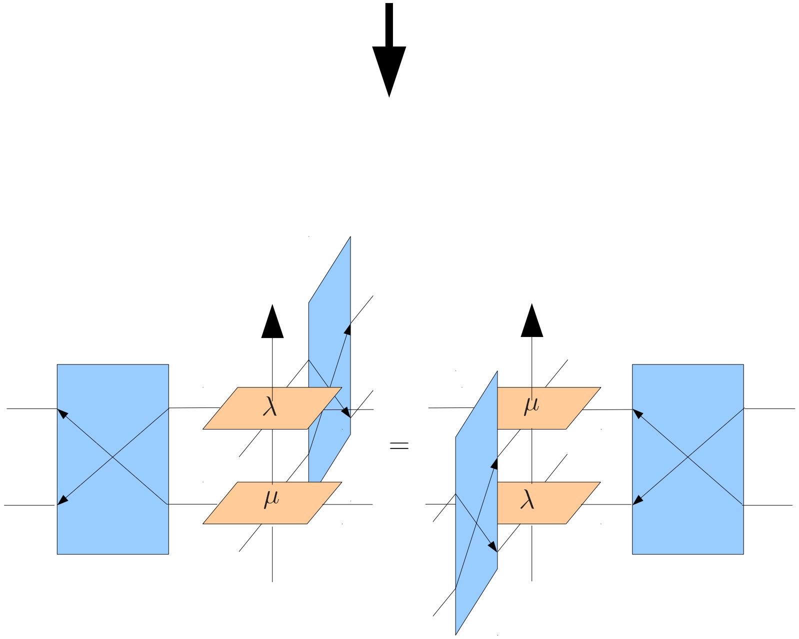


Time evolution in ***one*** dimension***:*** ***Matrix Product Operator (MPO)***



- Two-dimensional models:





Summary

- Bethe wavefunction can be
  - represented as tensor network
  - approximated by a MPS with low bond dimension

- Makes possible the calculation of arbitrary expectation values like 2-site and higher order correlations.

Outlook

- Nested Bethe Ansatz: supersymmetric tJ-Model
  Fermi-Hubbard Model

- XYZ Model