Quantum Quenches in 1D Bose Gases: Glimmers of Quantum KAM

arXiv:1407.7167
PRL 109, 175301 (2012)

Robert Konik

August 22, 2014

Conference on Quantum Engineering of States and Devices, Nordita

Giuseppe Brandino, Universiteit van Amsterdam
J.-S. Caux, Universiteit van Amsterdam

BROOKHAVEN NATIONAL LABORATORY

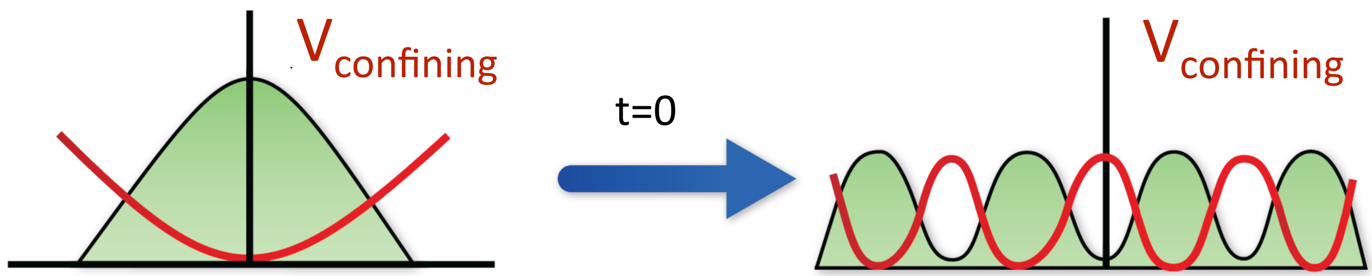
a passion for discovery

Office of Science
U.S. DEPARTMENT OF ENERGY

Quantum quenches

system prepared in an eigenstate of some initial Hamiltonian → ($t=0$) → system forced to time evolve (non-trivially) with some final Hamiltonian

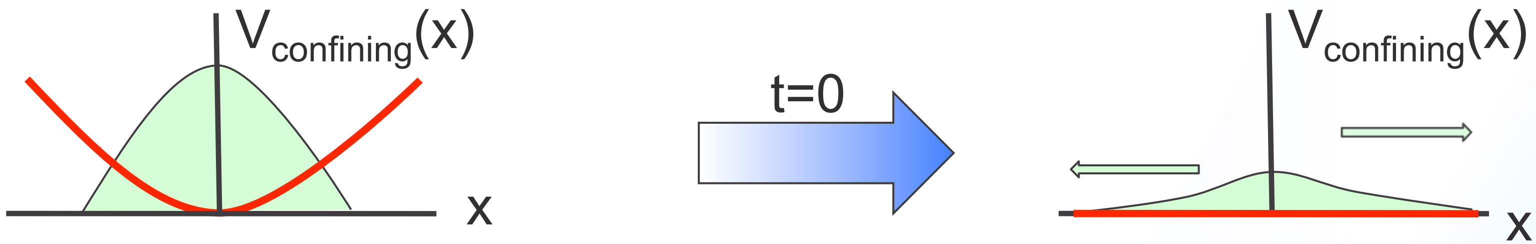
We will be interested in quenches in 1D Bose gases involving changes in the one-body potential of the gas:



gas in a parabolic potential, t<0 gas in a cosine potential, t>0

Integrable quantum quenches in 1D Bose gases

First consider a quantum quench where we prepare the gas in the ground state of a trap and at t=0, we release the trap:



gas in a parabolic potential, t<0

gas unconfined, t>0

For t>0 the gas is governed by the Lieb-Liniger model, an integrable model.

$$H = -\frac{1}{2m}\sum_{j=1}^{N}\frac{\partial^2}{\partial z_j^2} + 2c\sum_{1\le j<k}\delta(z_j - z_k)$$

In the absence of a confining potential, the dynamics of the gas are governed by an infinite set of conserved charges, Q_i, i = 1, 2, 3, …..

Conserved Quantities, Q_i, in the Lieb-Liniger Model

The N-particle eigenfunctions of Lieb-Liniger model are characterized by N distinct rapidities, λ_i, which are solutions of the Bethe equations:

$$|s\rangle = |\lambda_1, \cdots, \lambda_N\rangle; \qquad e^{i\lambda_i L} = \prod_{i \neq j} \frac{\lambda_i - \lambda_j + ic}{\lambda_i + \lambda_j - ic}$$

Once we know the λ_i' s, it is straightforward to write down the action of the charges on the state: Korepin and Davies, arXiv:1109..6604

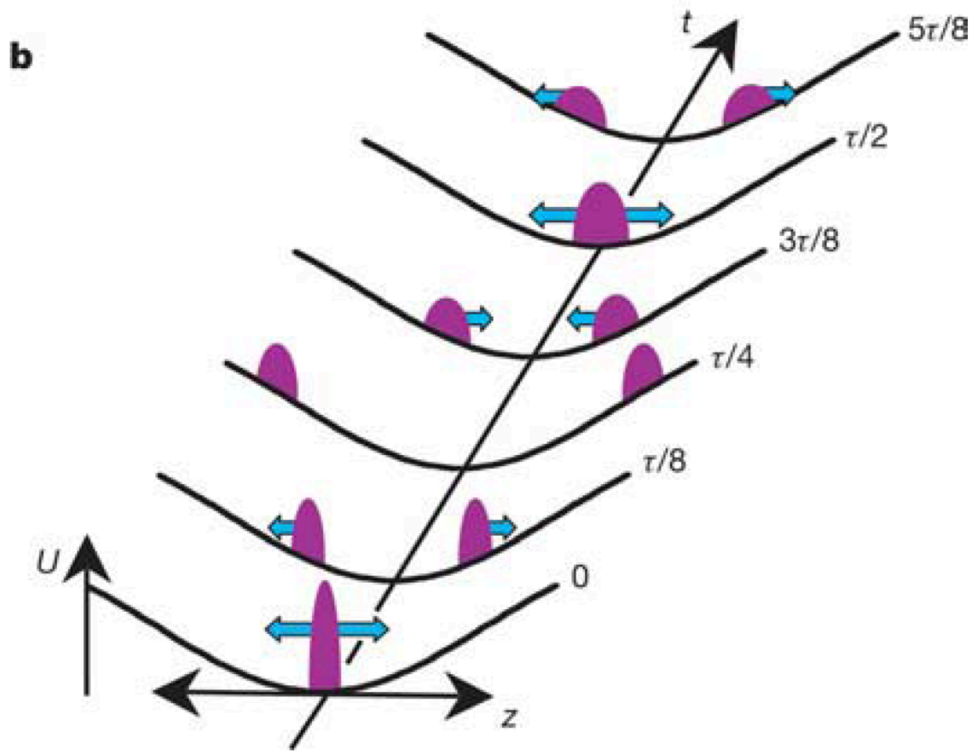
$$P|\lambda_1, \cdots, \lambda_N\rangle = (\sum_{i=1}^{N} \lambda_i)|\lambda_1, \cdots, \lambda_N\rangle$$

$$H|\lambda_1, \cdots, \lambda_N\rangle = (\sum_{i=1}^{N} \lambda_i^2)|\lambda_1, \cdots, \lambda_N\rangle$$

$$Q_n|\lambda_1, \cdots, \lambda_N\rangle = (\sum_{i=1}^{N} \lambda_i^n)|\lambda_1, \cdots, \lambda_N\rangle$$

Quantum Newton's Cradle

T. Kinoshita, T. Wenger, and D. Weiss, Nature 440, 900 (2006)

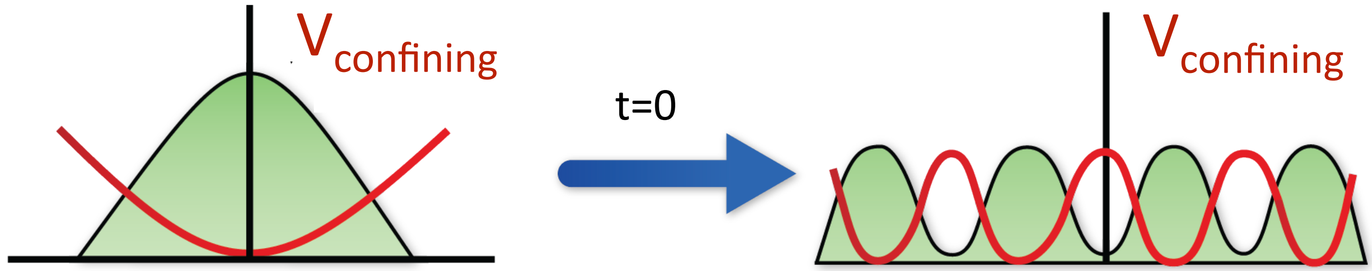


Counter-propagating clouds of 1D Bose condensates are seen not to thermalize.

It is believed that the long time dynamics are controlled by a non-Gibbsian thermodynamic ensemble (Rigol, Olshanii)

Gas is only quasi-integrable here. What happens if we break integrability more strongly?

Non-integrable quenches: release of gas into weak cosine potential



gas in a parabolic potential, t<0   gas in a cosine potential, t>0

For t>0, because the gas is in a cosine potential, the dynamics are no longer integrable.

Is the behavior of the gas now completely ergodic? Or is there a smooth crossover from quantum integrable to quantum chaotic?

Another way of asking this question is whether there is some sort of quantum KAM theorem operating here.

Classical KAM theorem

What does classical KAM say? Take a Hamiltonian weakly deformed from its integrable point:

$$H_{\text{full}}(p_i, q_j) = H_{\text{integrable}}(p_i) + \epsilon H_{\text{pert}}(p_i, q_j)$$

p_i: action variables
q_j: angle variables

When ε=0, all solutions are quasi-periodic, i.e. lie on invariant tori

$$\dot{q}_j = \frac{\partial H_{\text{integrable}}}{\partial p_j} \equiv \omega_j$$

KAM say that when the perturbation is turned on, certain quasi-periodic trajectories for ε=0 where the frequencies, ω_j, satisfy a non-resonancy condition continue to exist as solutions for finite strength of the pertubation.

Classically this seems to promise a smooth integrable to ergodic crossover.

Nekhoroshev estimates

Nekhoroshev says that for *any* trajectory of the full Hamiltonian, the time dependence of the action variables is restricted to

$$|p_j(t) - p_j(0)| < \epsilon^{\frac{1}{2n}}$$

n is the number of degrees of freedom

for times, t, less than

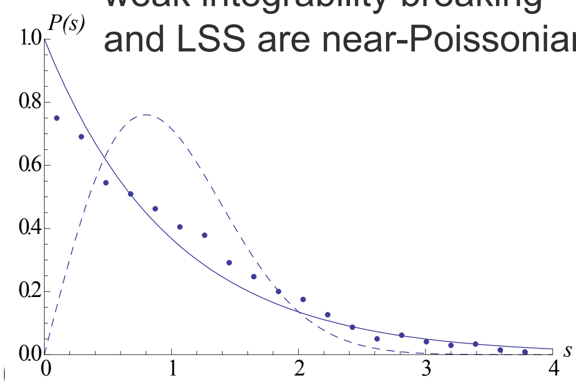
$$t < \exp(c(\frac{1}{\epsilon})^{\frac{1}{2n}})$$

We have found a construction for the quantum case of Lieb-Liniger that exists in this spirit.

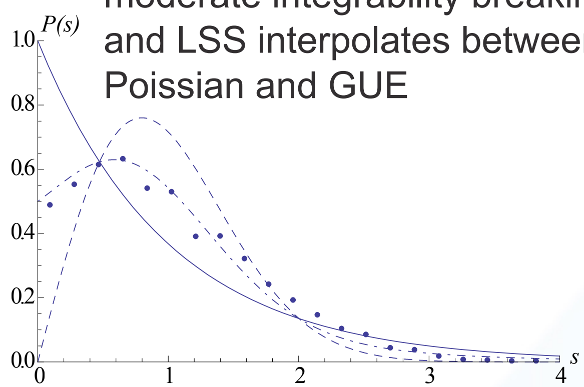
Crossing over from integrability to chaos in a quantum system

Berry-Tabor conjecture: Energy level spacing statistics (LSS) indicate whether model is quantum integrable or quantum chaotic

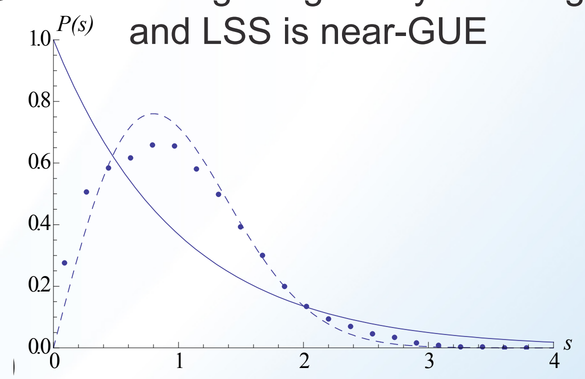
small magnetic field –
weak integrability breaking
and LSS are near-Poissonian



moderate magnetic field –
moderate integrability breaking
and LSS interpolates between
Poissian and GUE



strong magnetic field –
strong integrability breaking
and LSS is near-GUE



tricritical Ising + magnetic field

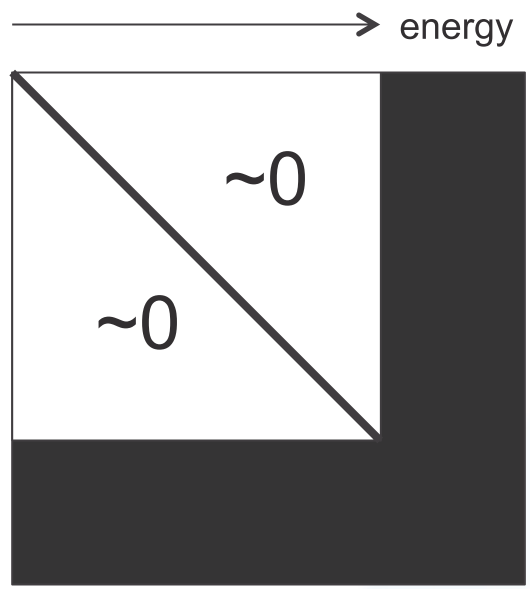
Integrable models have LSS that are Poissonian:

Non-integrable models have LSS that are from a Gaussian ensemble:

Quasi-conserved operators in a non-integrable setting

We will demonstrate that it is possible to construct a sequence of operators $Q_{eff,i}$ *i=1,2,3,….* that are quasi-conserved on the low energy Hilbert space, i.e. are diagonal on the low energy Hilbert space:

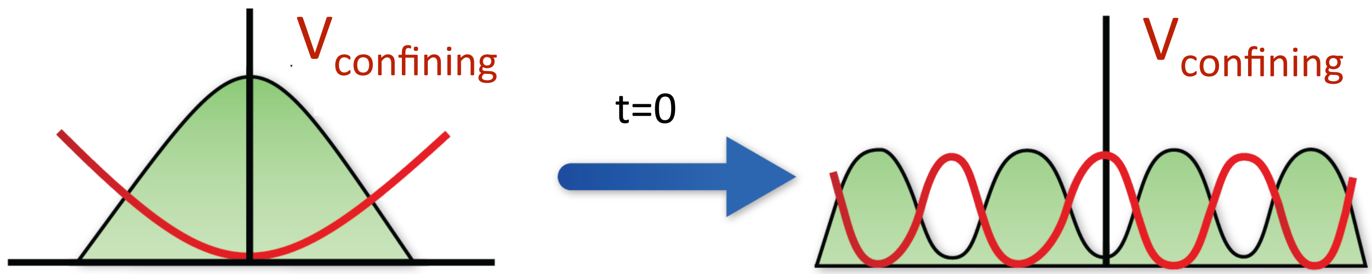
$$Q_{eff,i} =$$



The quality of this conservation is controlled by the strength of the integrability-breaking perturbation.

Time evolution of quantities post-quench

We construct such nearly conserved charges by exploiting our ability to compute the time dependence of observables.



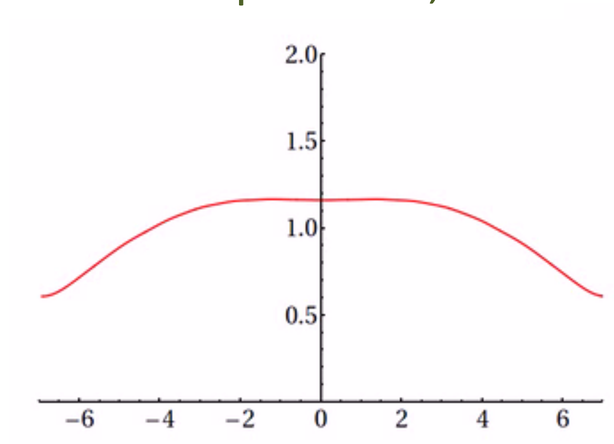
gas in a parabolic potential, t<0

gas in a cosine potential, t>0

So for example, we can compute the time evolution of the density profile of the gas post-quench.

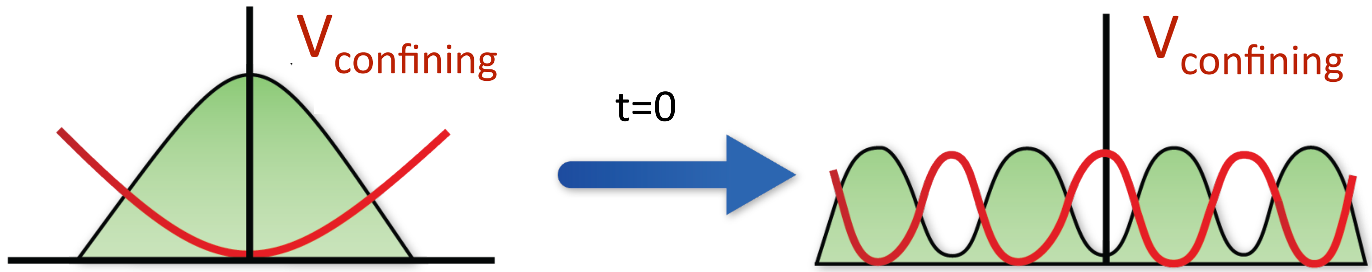
We do so using a numerical renormalization group (NRG) that exploits the integrability of Lieb-Liniger.

J.-S. Caux and RMK: PRL 109, 175301 (2012)



Density profile of gas with 14 particles as a function of time with c =10

Time evolution post-quench



gas in a parabolic potential, t<0 gas in a cosine potential, t>0

If we can write initial condition as a linear combination of eigenstates of the gas in the cosine potential,

$$\left|\psi(0)\right\rangle = \sum_{\substack{\text{cosine}\\\text{eigenstates}}} c_{\text{cosine}} \left|E_{\text{cosine}}\right\rangle$$

we can determine the time dependence of the wavefunction at generic times via

$$\left|\psi(t)\right\rangle = \sum_{\substack{\text{cosine}\\\text{eigenstates}}} c_{\text{cosine}} e^{iE_{\text{cosine}}t} \left|E_{\text{cosine}}\right\rangle$$

Numerical renormalization group for determining low-lying spectrum of perturbed integrable models

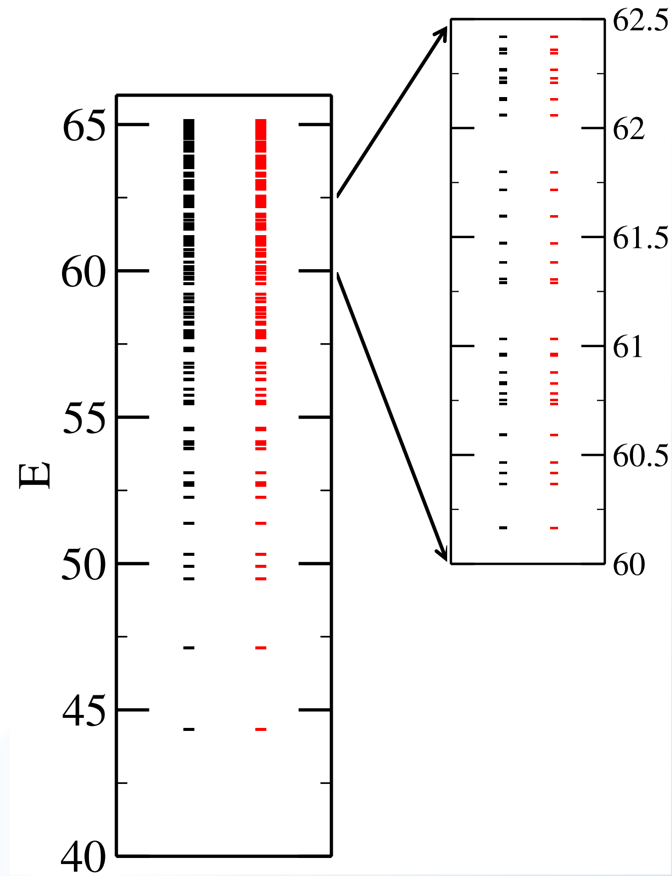
We use a numerical renormalization group designed to describe perturbations of integrable models:

$$H = \underbrace{H_{\text{Integrable/CFT}}}_{\text{Lieb-Liniger model}} + \underbrace{\Phi_{\text{Perturbation}}}_{\text{trapping potential}}$$

This method works on the same principle as Wilson's NRG for quantum impurity problems.

It works well on a wide variety of perturbed integrable and conformal models. It works best for perturbations that are relevant in the RG sense.

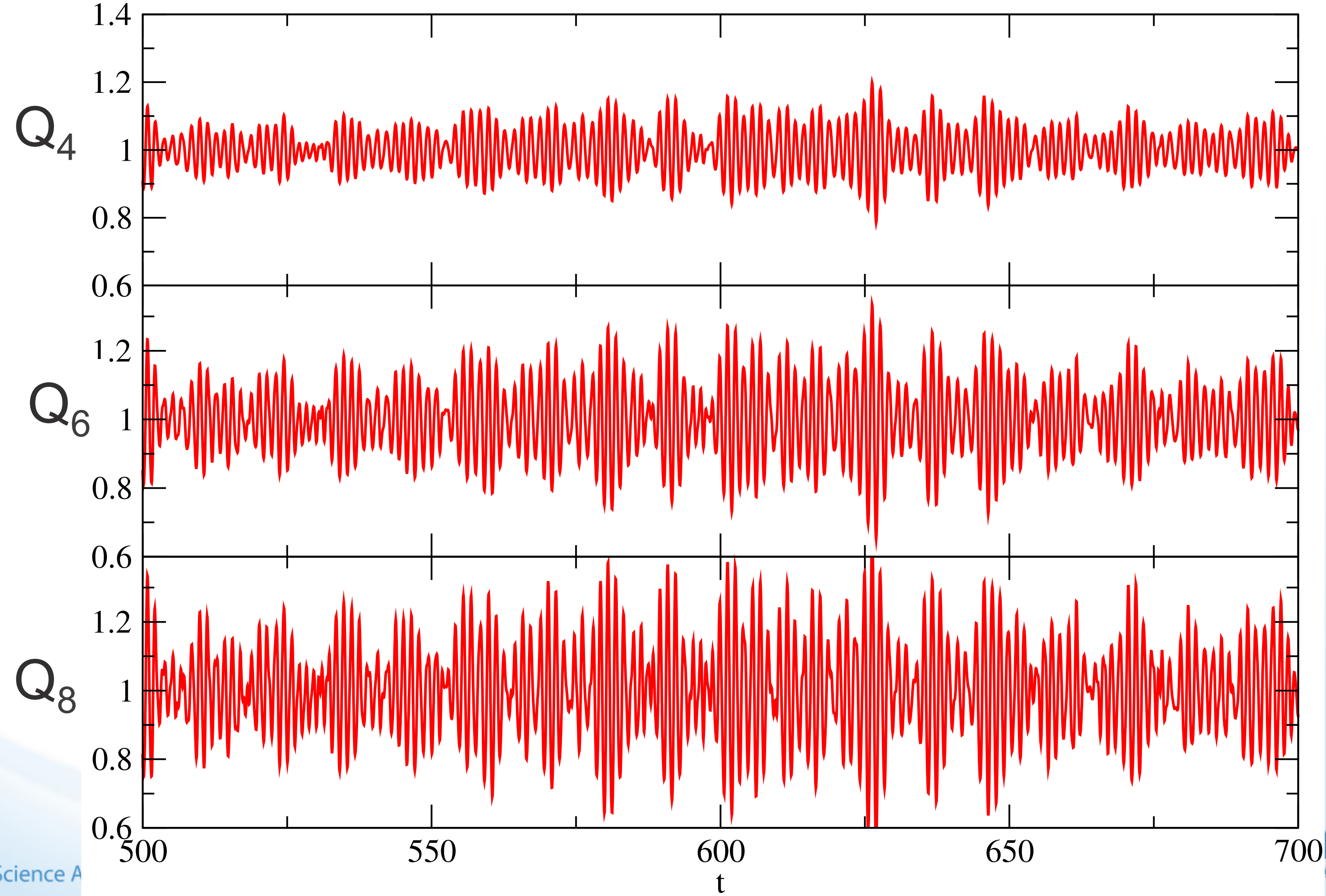
The key idea is that the unperturbed Lieb-Liniger eigenbasis is used as a computational basis.



Excitation spectrum of N=14, c=7200 1D Bose gas in a cosine potential of amplitude $A=0.35E_F$. Analytics (red), NRG Numerics (black)

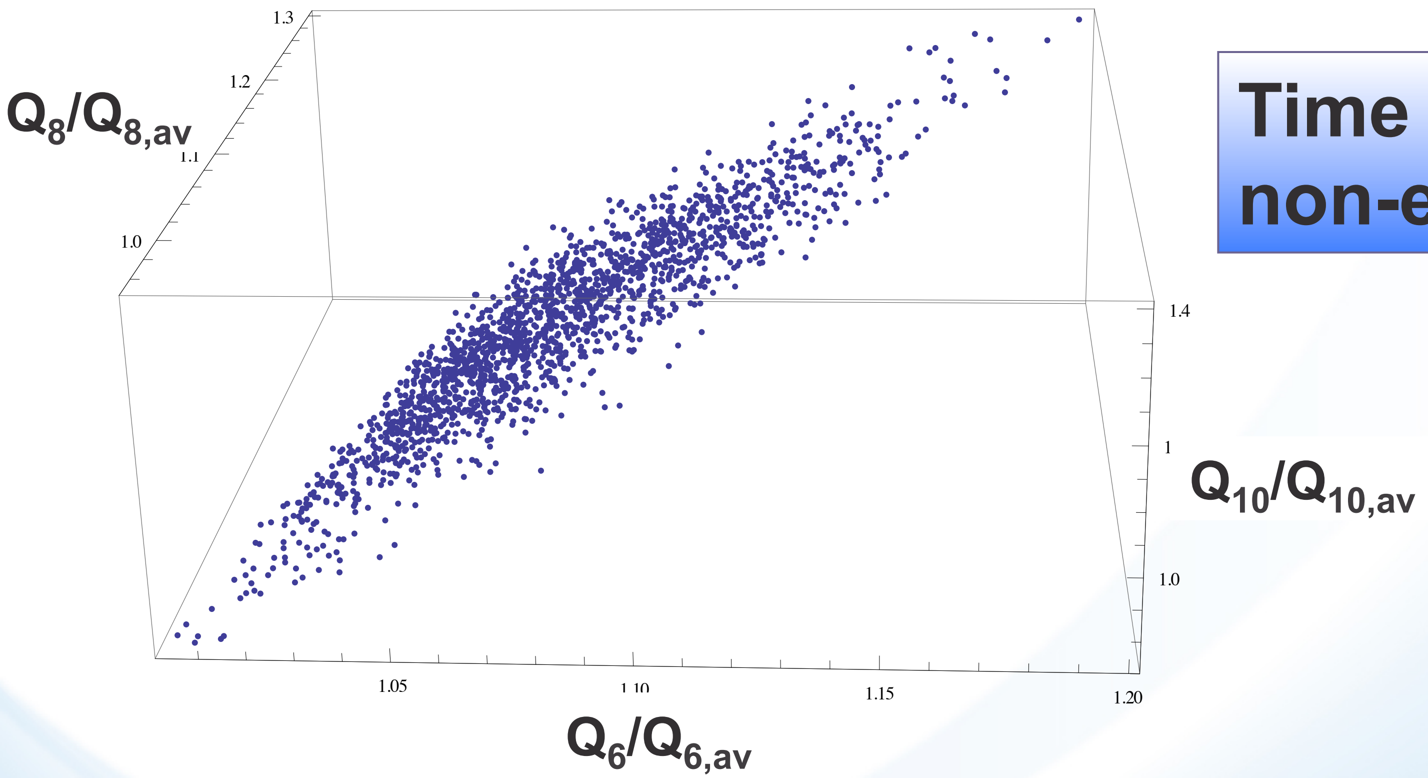
Time evolution of (formerly) conserved charges

So like for the density operator, we can compute the time evolution of the expectation value of the Lieb-Liniger charges:



Poincaré sections of time evolution of Q_i

Section of Q_6, Q_8, and Q_{10} for a fixed value of Q_4

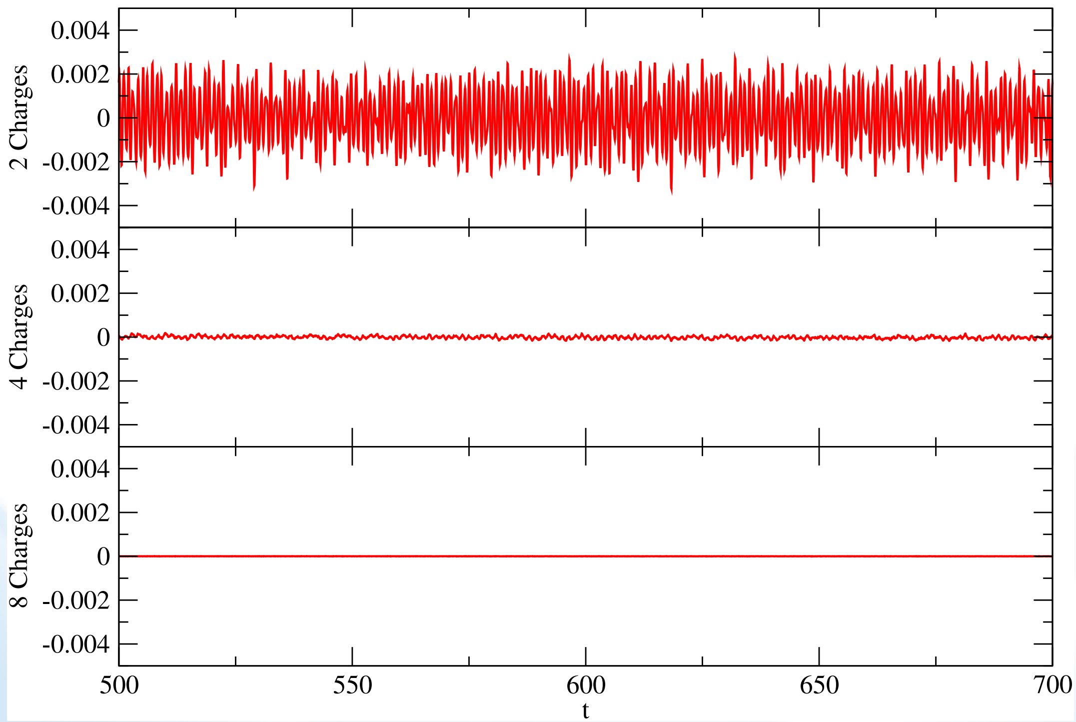


Time evolution is non-ergodic

New quasi-conserved charges

We can construct charges that are linear combinations of the Lieb-Liniger charges that are quasi-conserved.

$$Q_{eff}(t) = a_0 + \sum_{i=1}^{N_Q} a_i Q_{2i}$$



The more charges in the linear combination, N_Q, the better the time invariance.

$$\overline{(\Delta Q_{eff})^2} \sim e^{-[\Lambda(N_Q)/(m\omega_0)]^2} \frac{2^{N-1}\Lambda(N_Q)^{2N-2}}{(m\omega_0)^{2N-2}(N-1)!}$$

$$\Lambda(N_Q) = \frac{2\pi(N_Q-2)}{L};$$

A_{cosine} weak

$$\overline{(\Delta Q_{eff})^2} \sim 1 - \left(\frac{2\pi N_Q}{L\sqrt{2mA_{cosine}}}\right)^N$$

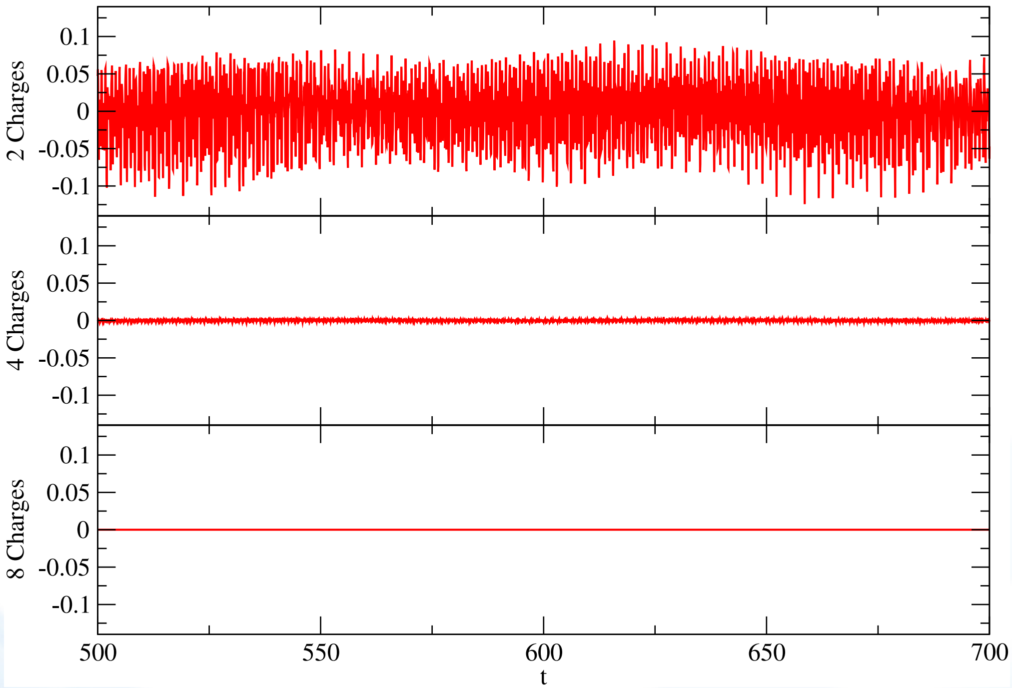
A_{cosine} strong

N_Q = # of charges in lin. comb.
N = number of particles in gas
L = system length
m = mass of particle
ω_0 = frequency of pre-quench trap
A_{cosine} = amplitude of cosine

New conserved charges: more than one

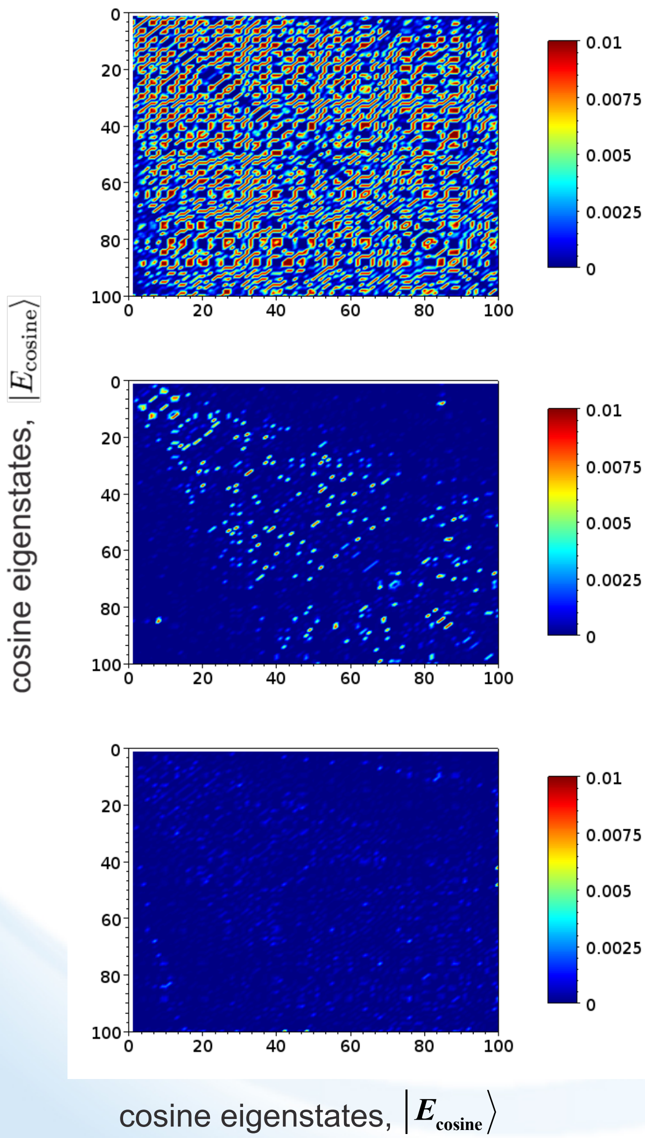
We can construct more than one charge:

$$Q_{eff}(t) = a_0 + \sum_{i=N_Q+1}^{2N_Q} a_i Q_{2i}$$



Again the more charges in the linear combination the better the time invariance.

Charges are conserved as operators



$$\langle E'_{\text{cosine}} | Q_2 | E_{\text{cosine}} \rangle$$

$$\langle E'_{\text{cosine}} | \sum_{i=1}^{4} a_i Q_{2i} | E_{\text{cosine}} \rangle$$

$$\langle E'_{\text{cosine}} | \sum_{i=1}^{8} a_i Q_{2i} | E_{\text{cosine}} \rangle$$

To demonstrate these charges are conserved as operators, we consider their off-diagonal matrix elements.

We see as more charges are used in the linear combination, the off-diagonal elements become successively smaller.

Why can we find quasi-conserved charges?

We have constructed the new effective charges such that a particular expectation value of the charge has (near-)zero time variation.

$$\langle \partial_t Q_{eff}(t) \rangle_{\text{initial condition}} = \sum_i a_i \partial_t \langle Q_i(t) \rangle_{\text{init.cond.}} \approx 0$$

This is similar to what Essler, Kehrein, Manmana, and Robinson (arXiv-1311.4557) and Kollar, Wolf, Eckstein PRB 84, 054304 (2011) did in the case of the spinless fermions.

But we have also shown that these effective charges are (nearly) zero as an operator equality for all times:

$$\partial_t Q_{\text{eff}} = \underbrace{i[V_{\text{cosine}}, Q_{\text{eff}}]}_{=0} - \underbrace{t[V_{\text{cosine}}, [V_{\text{cosine}}, Q_{\text{eff}}]]}_{=0} - \underbrace{i\frac{t^2}{2}[V_{\text{cosine}}, V_{\text{cosine}}, [V_{\text{cosine}}, Q_{\text{eff}}]]]}_{=0} + \cdots = 0$$

if restricted to finite energy Hilbert space
if $V_{\text{pertubation}}(=V_{\text{cosine}})$ has a finite number of Fourier modes

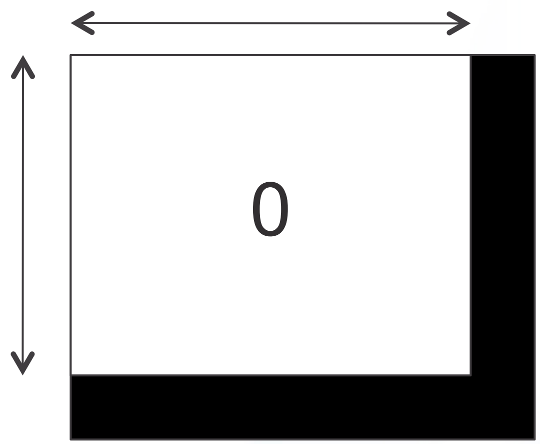
Why can we find quasi-conserved charges?

We can choose the a_i such that for a given quantum number, n_{max}, states $|s\rangle, |s'\rangle$ involving quantum numbers $n_i<n_{max}$ are such that

first order

$$\langle s|[V_{\text{cosine}}, Q_{eff}]|s'\rangle = 0$$

states involving quantum numbers less than n_{max}



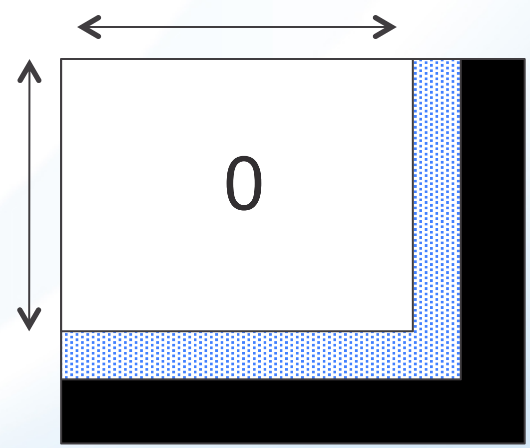
number of charges needed for a given n_{max}:

$$N_Q = n_{max} + 2$$

second order

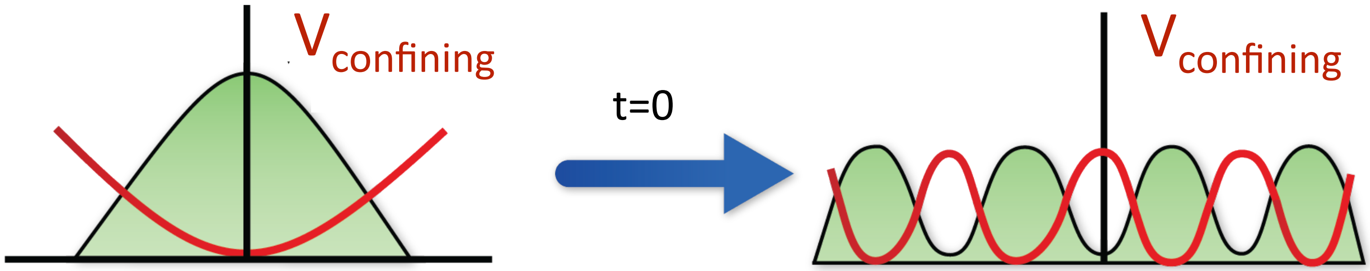
$$\langle s|[V_{\text{cosine}}, [V_{\text{cosine}}, Q_{eff}]|s'\rangle = 0$$

states involving quantum numbers less than $n_{max}-n(k)_{cosine}$



third order is 0 for states with quantum numbers less than $n_{max}-2n(k)_{cosine}$, etc.

Conclusions



gas in a parabolic potential, t<0 gas in a cosine potential, t>0

We are able to describe post-quench dynamics in the perturbed Lieb-Liniger model out to long finite times.

Using this, we have been able to construct quasi-conserved quantities taken as linear combinations of the Lieb-Liniger charges.

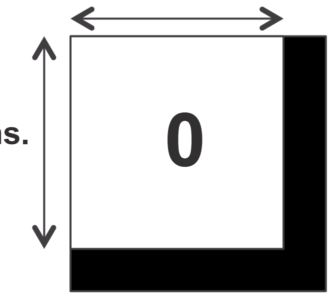
These charges are conserved ***as operators*** when acting on the low-energy Hilbert space.

These operators will govern the long time behavior of correlation functions via Mazur's inequality at low temperatures.

1st order

$$C_1 = \langle s|[V_{\text{cosine}}, Q_{eff}]|s'\rangle = 0$$

states involving quant. nums. less than N_{max}

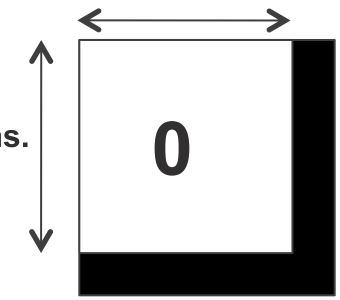


2nd-3rd orders

$$C_2 = \langle s|[V_{\text{cosine}}, C_1]|s'\rangle = 0$$

$$C_3 = \langle s|[V_{\text{cosine}}, C_2]|s'\rangle = 0$$

states involving quant. nums. less than N_{max}

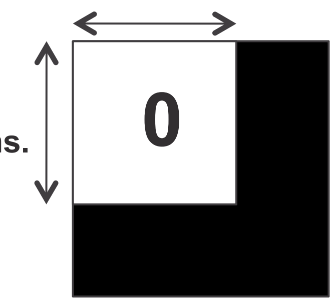


4th-5th orders

$$C_4 = \langle s|[V_{\text{cosine}}, C_3]|s'\rangle = 0$$

$$C_5 = \langle s|[V_{\text{cosine}}, C_4]|s'\rangle = 0$$

states involving quant. nums. less than N_{max}-n_{cos}

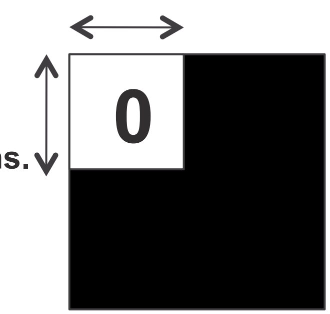


6th-7th orders

$$C_6 = \langle s|[V_{\text{cosine}}, C_5]|s'\rangle = 0$$

$$C_7 = \langle s|[V_{\text{cosine}}, C_6]|s'\rangle = 0$$

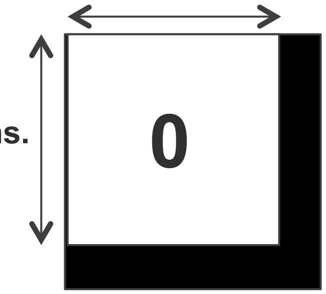
states involving quant. nums. less than N_{max}-2n_{cos}



1^{st} order

$$C_1 = \langle s | [V_{\text{cosine}}, Q] | s' \rangle = 0$$

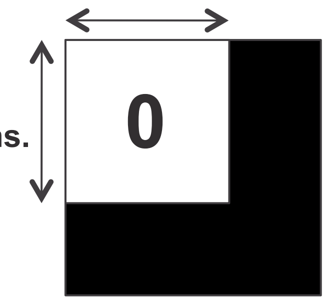
states involving quant. nums. less than N_{max}



2^{nd} order

$$C_2^{1,p-h} = \langle s | [V_{\text{cosine}}^{1,p-h}, C_1] | s' \rangle = 0$$

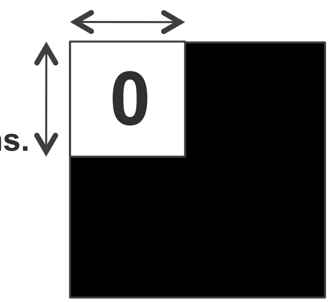
states involving quant. nums. less than N_{max}-n_{cos}



3^{rd} order

$$C_3^{1,p-h} = \langle s | [V_{\text{cosine}}^{1,p-h}, C_2] | s' \rangle = 0$$

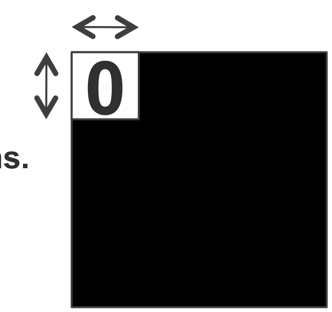
states involving quant. nums. less than N_{max}-2n_{cos}



m^{th} order

$$C_m^{1,p-h} = \langle s | [V_{\text{cosine}}^{1,p-h}, C_{m-1}] | s' \rangle = 0$$

states involving quant. nums. less than N_{max}-mn_{cos}



1st order

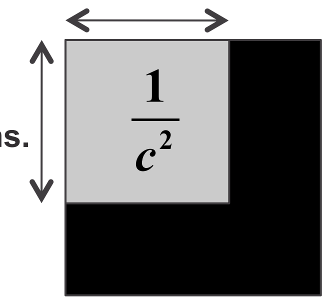
$$C_1 = \langle s | [V_{\text{cosine}}, Q] | s' \rangle = 0$$

states involving quant. nums. less than N_{max}

2nd order

$$C_2 = \langle s | [V_{\text{cosine}}, C_1] | s' \rangle = 0$$

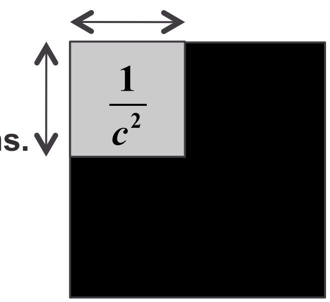
states involving quant. nums. less than N_{max}-n_{cos}



3rd order

$$C_3 = \langle s | [V_{\text{cosine}}, C_2] | s' \rangle = 0$$

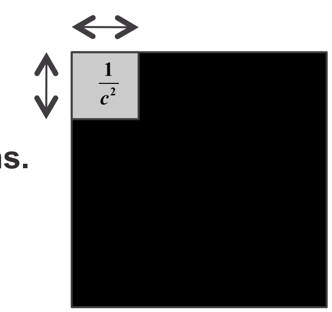
states involving quant. nums. less than N_{max}-2n_{cos}

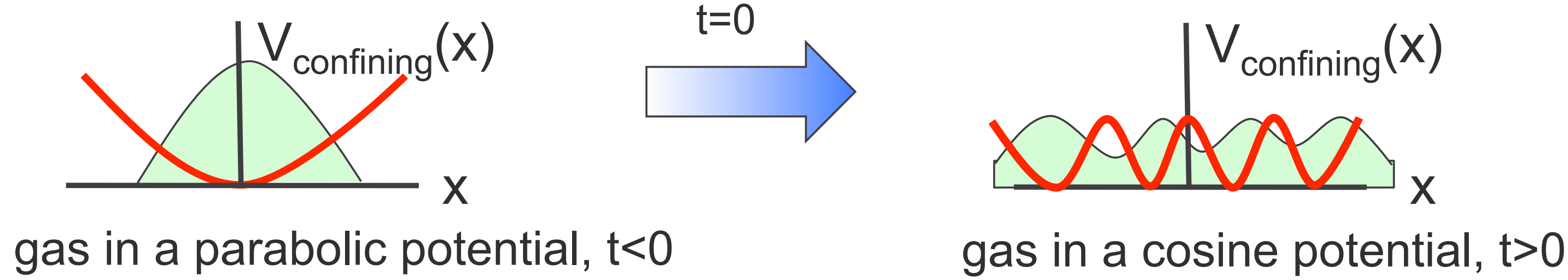


mth order

$$C_m = \langle s | [V_{\text{cosine}}, C_{m-1}] | s' \rangle = 0$$

states involving quant. nums. less than N_{max}-mn_{cos}





$V_{confining}(x)$
x
gas in a parabolic potential, t<0
t=0
$V_{confining}(x)$
x
gas in a cosine potential, t>0

Generalized Gibbs Ensemble

In an attempt to understand this experiment, it was conjectured that the thermalization of this system (and integrable models in general) is not controlled by the Gibbs ensemble

$$\hat{\rho}_{Gibbs} = \frac{e^{-\beta H}}{\mathrm{Tr} e^{-\beta H}}$$

but by a thermodynamic ensemble that knows of all the conserved quantites, Q_i , i=1,…, of the system.

$$\hat{\rho}_{Generalized\ Gibbs} = \frac{e^{-\sum \beta_i Q_i}}{\mathrm{Tr} e^{-\sum \beta_i Q_i}}$$

Rigol, Dunjko, Yurovsky, Olshanii, PRL 98, 050405 (2007)

M. Cazalilla, PRL 97 156403 (2006)
P. Calabrese, F.H.L. Essler, and M. Fagotti, PRL 106, 227203 (2011)
J. Stat. Mech. (2012) P04017 and P07022
D. Rossini, A. Silva, G. Mussardo, and G. Santoro, PRL 102, 127204 (2009)
T. Barthel and U. Schollwöck, PRL 100, 100601 (2008)

D. Fioretti and G. Mussardo, New J. Phys. 12, 055015 (2010).

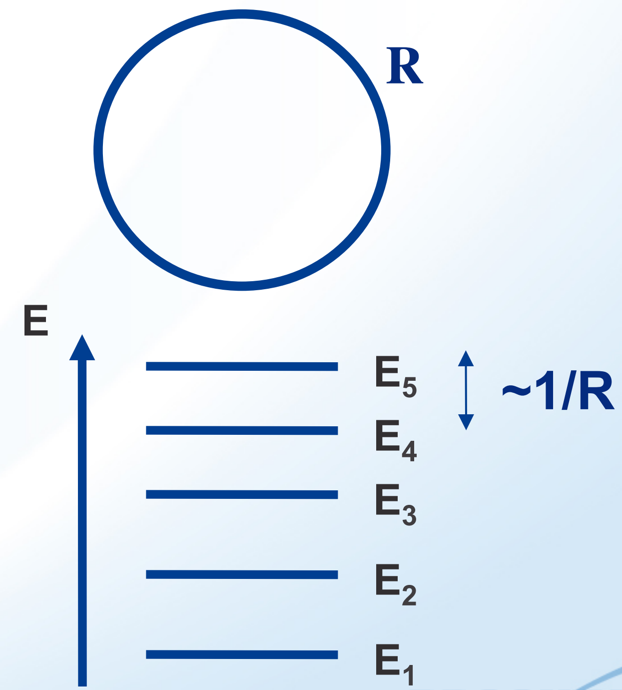
Numerical Renormalization Group for Perturbed Integrable Theories

The method can in principle study any Hamiltonian that takes the form:

$$H = \underbrace{H_{known}}_{\substack{\text{conformal/integrable theory} \\ \text{i.e. Lieb-Liniger model}}} + \underbrace{\Phi_{perturbation}}_{\text{trapping potential}}$$

Consider the model on a finite sized ring of circumference, R

Spectrum of H_{known} then becomes discrete and we can order states in terms of ascending energy.

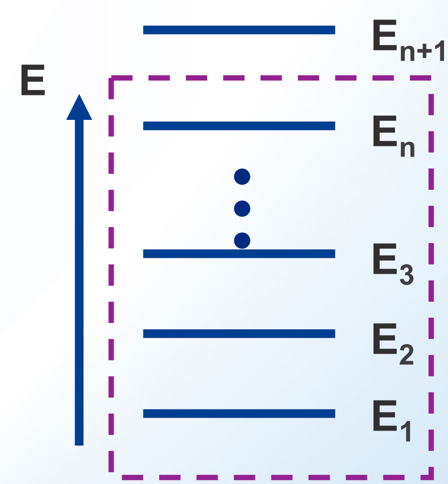


We are able to compute matrix elements with ABACUS (J.-S. Caux).

$$\Phi_{ij} = \left.\langle i | \Phi_{perturbation} | j \rangle\right|_{\mathbf{H}_{\mathbf{Known}}}$$

Truncate Hilbert space, making it finite dimensional. This allows one to write full Hamiltonian as a finite dimensional matrix.

$$\mathbf{H} = \begin{bmatrix} E_1 & \Phi_{12} & \cdots & \Phi_{1n} \\ \Phi_{21} & E_2 & & \vdots \\ \vdots & & \ddots E_{n-1} & \Phi_{n-1n} \\ \Phi_{n1} & \cdots & \Phi_{nn-1} & E_n \end{bmatrix}$$

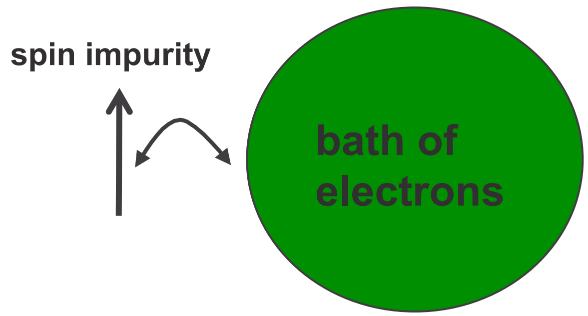


Diagonalize H numerically and extract spectrum

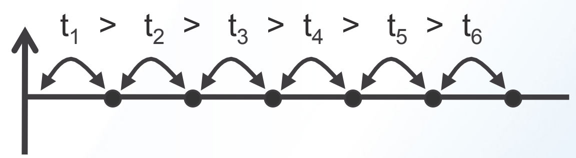
Key idea: Using the “known” basis as a computation basis

Second Step of Numerical Renormalization Group

The next step is to find a way to include states previously tossed away using same idea as the one Wilson applied to the Kondo model:



impurity interacting with bath of electrons

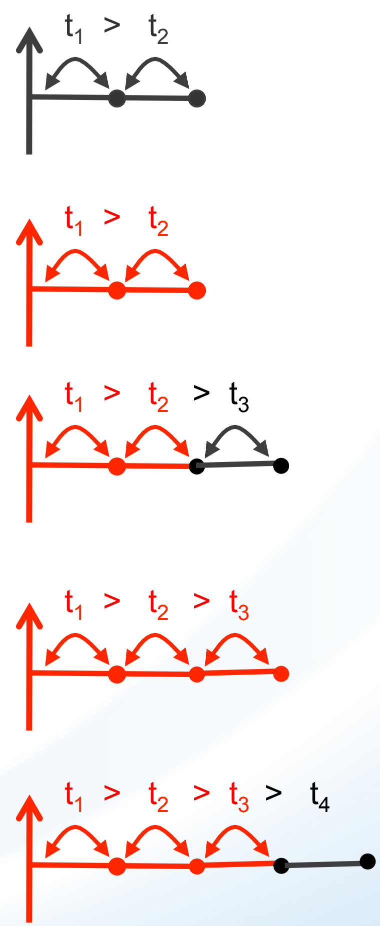


spin impurity living on a semi-infinite lattice where the electrons on the lattice have decaying hopping amplitudes the further they are from the impurity

Second Step of Numerical Renormalization Group

Wilson treated the solution of this lattice problem iteratively:

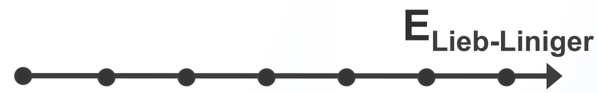
1. First diagonalize small system
2. Throw away high energy eigenstates
3. Add a site to truncated system
4. Diagonalize new system and retruncate
5. Repeat



Numerical Renormalization Group for Continuum Theories

For perturbed integrable models, the same principle applies:

$$H = \underbrace{H_{\text{Known/Exactly Solvable}}}_{\text{Lieb-Liniger}} + \underbrace{\Phi_{\text{Perturbation}}}_{\text{trap potential}} \longrightarrow$$



We "map" our perturbed Lieb-Liniger to a lattice on which are arranged the states of $H_{\text{Lieb-Liniger}}$ in order of increasing energy.

The metric in the field theoretic Hilbert space is different but the idea is the same.

The reason why it works here is that $\Phi_{\text{perturbation}}$ is relevant.

J. S. Caux and RK, PRL 109, 175301 (2012): 1D Bose Gases
RK and Y. Adamov, PRL 98, 147205 (2007): perturbed conformal minimal models
RK and Y. Adamov, PRL 102, 097203 (2009); Andrew James and RK, PRB 87, 241103 (2013): 2+1D Systems of Coupled Quantum Ising Chains
RK, PRL 106, 136805 (2011): Semiconducting Carbon Nanotubes
G. Brandino, RK, G. Mussardo, J. Stat. Mech. P07013 (2010) : Level spacing statistics in perturbed CFTs
M. Beria, G. Brandino, A. Luca, RK, G. Sierra, Nucl. Phys. B (2013): Perturbations of $SU(2)_k$ WZW

Excited Energy Spectra of Gas in Cosine Potential

Spectrum

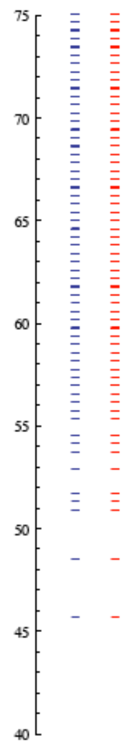
Cosine Potential, N=14 c=7200

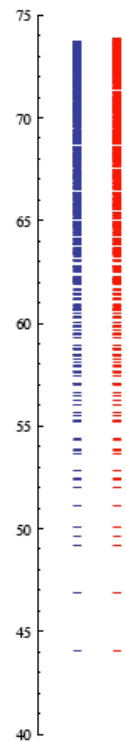
Analytics $\to$ Free fermion representation + $1/c$ corrections

red – analytics   blue - numerics

A_{cos} = 0.1

1200 states
can be described
with 10^(-4)
accuracy





A_{cos} = 3

300 states
can be described
with 10^(-4)
accuracy

Conserved Quantities, Q_i, in the Lieb-Liniger Model

The N-particle eigenfunctions of Lieb-Liniger model are characterized by N distinct rapidities, λ_i, which are solutions of the Bethe equations:

$$|s\rangle = |\lambda_1, \cdots, \lambda_N\rangle; \qquad e^{i\lambda_i L} = \prod_{i \neq j} \frac{\lambda_i - \lambda_j + ic}{\lambda_i + \lambda_j - ic}$$

Once we know the λ_i's, it is straightforward to write down the action of the charges on the state: Korepin and Davies, arXiv:1109..6604

$$P|\lambda_1, \cdots, \lambda_N\rangle = (\sum_{i=1}^{N} \lambda_i)|\lambda_1, \cdots, \lambda_N\rangle$$

$$H|\lambda_1, \cdots, \lambda_N\rangle = (\sum_{i=1}^{N} \lambda_i^2)|\lambda_1, \cdots, \lambda_N\rangle$$

$$Q_n|\lambda_1, \cdots, \lambda_N\rangle = (\sum_{i=1}^{N} \lambda_i^n)|\lambda_1, \cdots, \lambda_N\rangle$$

We use the Lieb-Liniger eigenstates as a computational basis making the computation of the time evolution of the charges straightforward.

Why Can We Find Quasi-Conserved Charges?

Each state is associated with a set of quantum numbers, $\{n_i\}$:

$$|s\rangle = |\lambda_1, \cdots, \lambda_N\rangle; \qquad e^{i\lambda_i L} = \prod_{i \neq j} \frac{\lambda_i - \lambda_j + ic}{\lambda_i + \lambda_j - ic}$$

$$|s\rangle = |\lambda_1, \cdots, \lambda_N\rangle = |n_1, \cdots, n_N\rangle \qquad 2\pi n_i = L\lambda_i + i \sum_{j \neq i} \log\left(\frac{\lambda_i - \lambda_j + ic}{\lambda_i - \lambda_j - ic}\right)$$

We will find a linear combination of charges $Q_{eff} = \sum_i a_i Q_i$ such that we zero out all matrix elements of the form

$$\langle s | [V_{\text{cosine}}, Q_{eff}] | s' \rangle = 0$$

where the states, $|s\rangle, |s'\rangle$ are constructed with quantum numbers $n_i \leq n_{max}$ for some n_{max}.

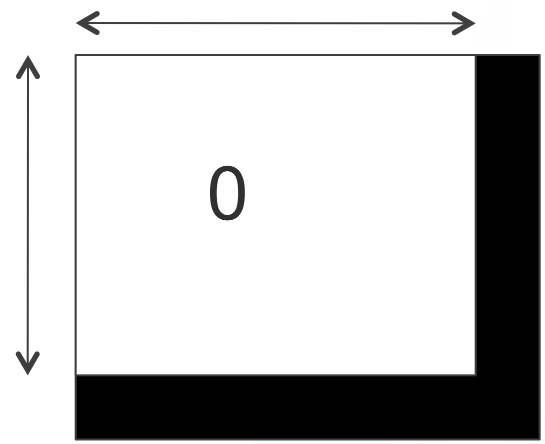
Why Can We Find Quasi-Conserved Charges?

We can choose the a_i such that for a given quantum number, n_{max}, states $|s\rangle, |s'\rangle$ involving quantum numbers $n_i<n_{max}$ are such that

first order

$$\langle s|[V_{\text{cosine}}, Q_{eff}]|s'\rangle = 0$$

states involving quantum numbers less than n_{max}



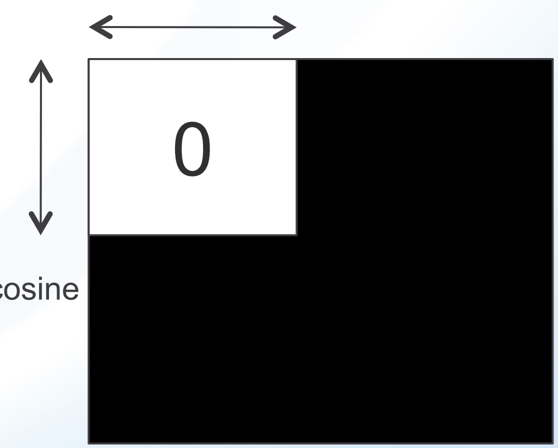
number of charges needed for N particles:

$$N_Q \sim \frac{(2n_{\text{max}})^N}{2(N-1)!}$$

second order

$$\langle s|[V_{\text{cosine}}, [V_{\text{cosine}}, Q_{eff}]|s'\rangle = 0$$

states involving quantum numbers less than n_{max}-$n(k)_{cosine}$



third order is 0 for states with quantum numbers less than n_{max}-$2n(k)_{cosine}$, etc.

Why Can We Find Quasi-Conserved Charges?

Things are considerably better for the c=∞. One needs far fewer charges

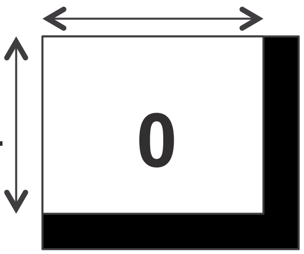
$$N_Q = (n_{max}/2)$$

to zero out a given block and the block shrinks much more slowly as one goes up in order.

1st order

$$C_1 = \langle s | [V_{\text{cosine}}, Q_{eff}] | s' \rangle = 0$$

states involving quant. nums. less than n_{max}

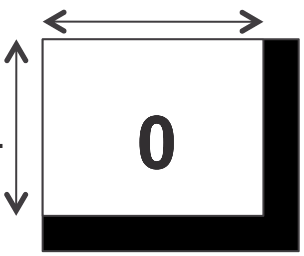


2nd-3rd orders

$$C_2 = \langle s | [V_{\text{cosine}}, C_1] | s' \rangle = 0$$

$$C_3 = \langle s | [V_{\text{cosine}}, C_2] | s' \rangle = 0$$

states involving quant. nums. less than n_{max}

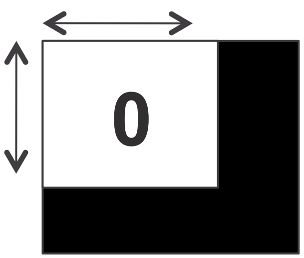


4th-5th orders

$$C_4 = \langle s | [V_{\text{cosine}}, C_3] | s' \rangle = 0$$

$$C_5 = \langle s | [V_{\text{cosine}}, C_4] | s' \rangle = 0$$

states involving quant. nums. less than $n_{max}-n_k$



6th-7th orders

$$C_6 = \langle s | [V_{\text{cosine}}, C_5] | s' \rangle = 0$$

$$C_7 = \langle s | [V_{\text{cosine}}, C_6] | s' \rangle = 0$$

states involving quant. nums. less than $n_{max}-2n_k$