# Quench Dynamics in 1-d Integrable Systems

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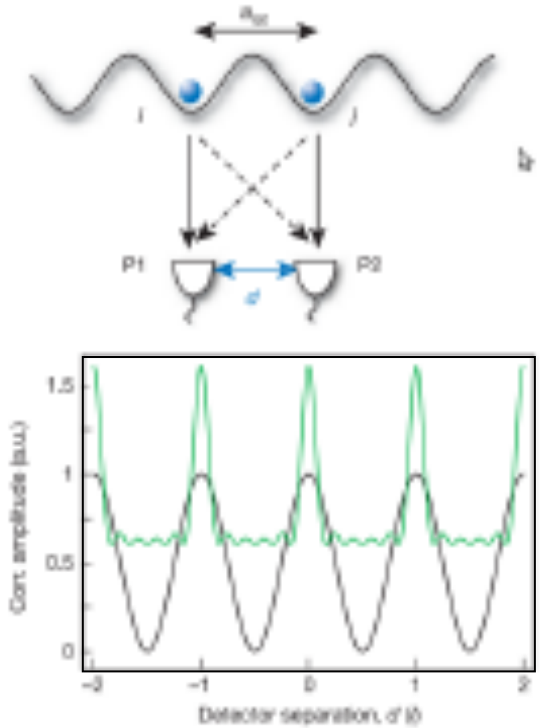
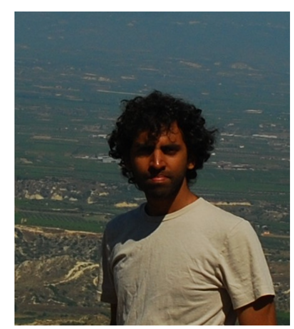
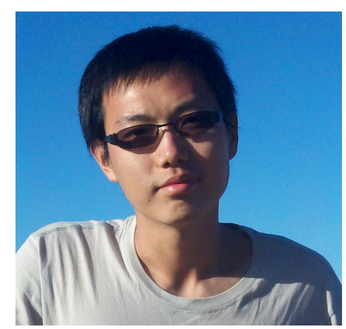



Hanbury-Brown Twiss effect

*I. Bloch et al.*





**Garry Goldstein** **Deepak Iyer** **Wenshuo Liu**

Nordita, Stockholm - Aug, 2014

# Quenching and Time Evolution

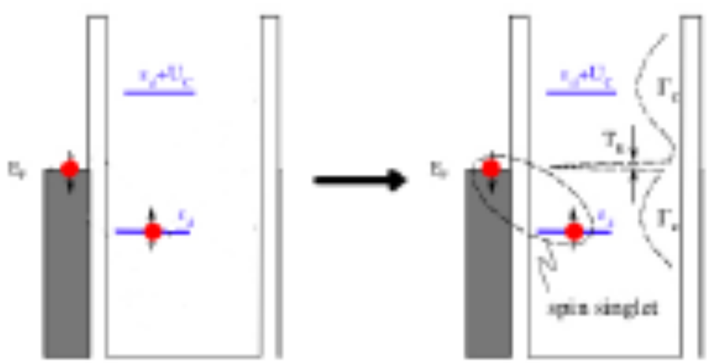
- **Prepare an isolated quantum many-body system in state $|\Phi_0\rangle$,** typically eigenstate of $H_0$
- **At $t=0$ turn on interaction $H_1$, and evolve system with $H = H_0 + H_1$ :**

$$|\Phi_0, t\rangle = e^{-iHt}|\Phi_0\rangle$$

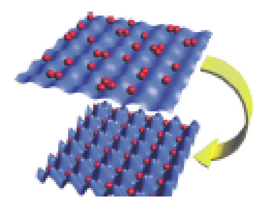
- **Many experiments: cold atom systems, nano-devices, molecular electronics, photonics :** *new systems, old questions*

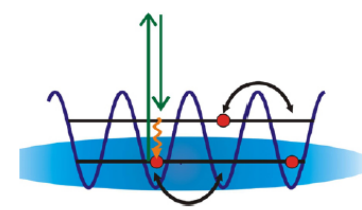
**Time evolution of observables:** $\langle O(t)\rangle = \langle \Phi_0, t|O|\Phi_0, t\rangle$

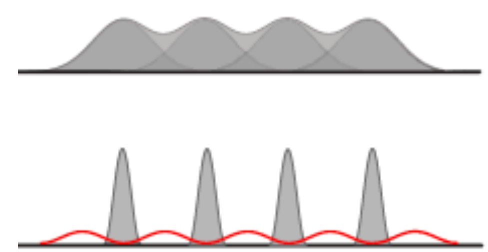
**- Manifestation of interactions**




Measure time evolution of the Kondo peak.
- Time resolved photo emission spectroscopy








**Mott insulator ⟷ superfluid**

# Closed systems: quenching – long time limit, thermalization

**Time evolution and statistical mechanics:**

$$\langle A(t)\rangle = \langle\Phi_0|e^{iHt}Ae^{-iHt}|\Phi_0\rangle = \sum_{\alpha,\beta}\langle\Phi_0|\alpha\rangle A_{\alpha\beta}\langle\beta|\Phi_0\rangle e^{i(E_\alpha - E_\beta)t}$$

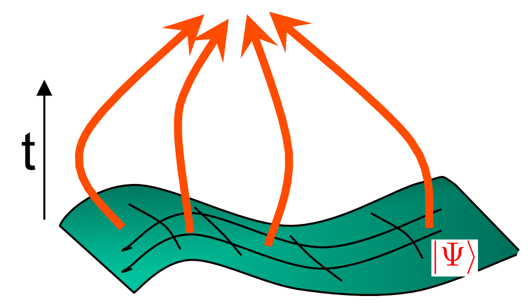
- **Long time limit and thermalization: (Gibbs ensemble - GE )**

- is there a limit for local op. $\bar{A} = \lim_{t\to\infty}\langle A(t)\rangle$? (equilibration)

- is there a density operator $\rho$ such that $\bar{A} = Tr(\rho A)$?

- does it depend only on $E_0 = \langle\Phi_0|H|\Phi_0\rangle$, not on $|\Phi_0\rangle$? (ETH)

$$\bar{A} = \sum_\alpha |C_\alpha|^2 A_{\alpha\alpha} \stackrel{?}{=} \langle A\rangle_{\mathrm{microcan}}(E_0) \equiv \frac{1}{N_{E_0,\Delta E}}\sum_{\alpha\in\Delta} A_{\alpha\alpha}$$




$|\Psi\rangle \in \Delta \leftrightarrow E_0 - \Delta E \le E_\alpha \le E_0 + \Delta E$

- **Scenarios of thermalization (ETH** and others)

  - Diagonal matrix elements of physical operators $A_{\alpha\alpha} = \langle\psi_\alpha|A|\psi_\alpha\rangle$ do not fluctuate much around constant energy surface (**ETH-*eigenstate thermalization hypothesis***, Deutsch 92, Srednicki 94)

  - Overlaps $|C_\alpha|^2 = |\langle\psi_\alpha|\Phi_0\rangle|^2$ do not fluctuate on the energy surface for reasonable IC

  - Both fluctuate but are uncorrelated

- **Thermalization, Integrability, Non-Boltzmannian ensembles (GETH)** Rigol, Cazalilla..

*If conservation laws are present – how do they affect dynamics of thermalization?* *GE → GGE?*

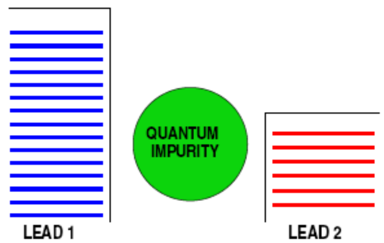
- **If initial state sufficiently non-translational invariant?** *currents, entropy prod, NESS*

# Breaking Translation Invariance: quenching and non-thermalization

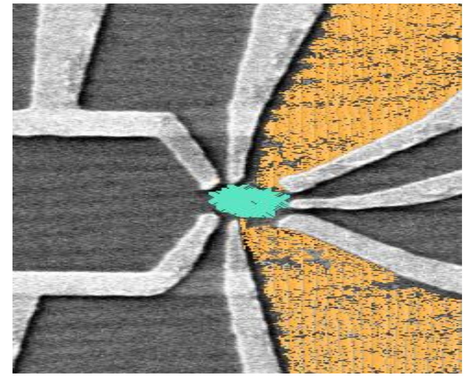
## Nonequilibrium currents

Goldhaber-Gordon *et al*, Cronenwett *et al*, Schmid *et al*

- **Two baths or more:**
  - time evolution in a nonequilibrium set up: $t \ll L/v$



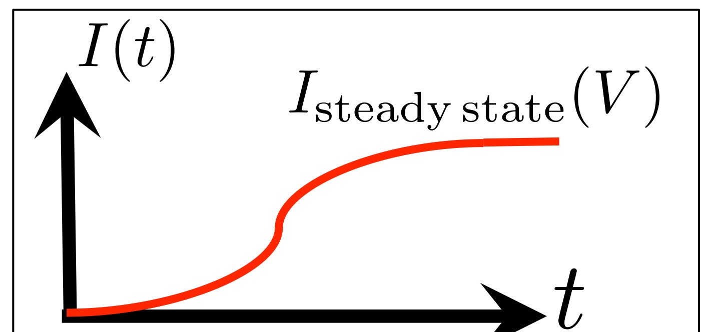

$H = H_{leads} + H_{imp} + H_{leads-imp}$



**Quench**

- $t \leq 0$, leads decoupled, system described by: $\rho_o$
- $t = 0$, couple leads to impurity
- $t \geq 0$, evolve with $H(t) = H_0 + H_1$

Interplay - strong correlations and nonequilibrium



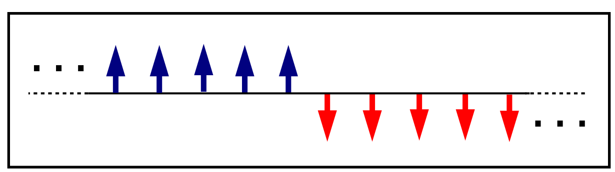

- **What is the time evolution of the current** $\langle I(t) \rangle$ **?**

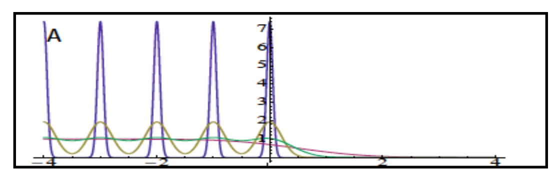
- **Long time limit: Under what conditions is there a steady state? Dissipation mechanism?**

- **Steady state – is there a non thermal** $\rho_s$ **? Voltage dependence?**
- **New effects out of equilibrium? New scales? Phase transitions, universality?**

- **Domain wall:** spin currents, NESS

  $t \to \infty, L \to \infty$



# Time Evolution and the Bethe Ansatz

- A given state $|\Phi_0\rangle$ can be formally time evolved in terms of a complete set of energy eigenstates $|F^\lambda\rangle$

$$|\Phi_0\rangle = \sum_\lambda |F^\lambda\rangle\langle F^\lambda|\Phi_0\rangle \longrightarrow |\Phi_0, t\rangle = e^{-iHt}|\Phi_0\rangle = \sum_\lambda e^{-i\epsilon_\lambda t}|F^\lambda\rangle\langle F^\lambda|\Phi_0\rangle$$

If $H$ integrable $\longrightarrow$ eigenstates $|F^\lambda\rangle$ are known via the Bethe-Ansatz

*BA wave function:* $F^\lambda(\vec{x}) = \sum_{Q\in S_N} A^Q \prod_j f_{\lambda_{Qj}}(x_j)$ *S-matrix* $S_{ij}(\lambda_i - \lambda_j) = \dfrac{\lambda_i - \lambda_j + ic}{\lambda_i - \lambda_j - ic}$

- Use Bethe Ansatz to study quench evolution and nonequilibrium
- New technology is necessary:

- **Standard approach**: **PBC** $\longrightarrow$ **Bethe Ansatz eqns** $\longrightarrow$ **spectrum** $\longrightarrow$ **thermodynamics**

- **Non equilibrium entails *additional* difficulties:**

**i. Compute overlaps (form factors)** **ii. Sum over complete basis** **iii. Take limits**

*J. S. Caux et al, P. Calabrese at al*

# The contour representation

Instead of $|\Phi_0\rangle = \sum_\lambda |F^\lambda\rangle\langle F^\lambda|\Phi_0\rangle$ introduce (directly in infinite volume):

Contour representation of $|\Phi_0\rangle$

$$\boxed{|\Phi_0\rangle = \int_\gamma d^N\lambda\, |F^\lambda\rangle(F^\lambda|\Phi_0\rangle}$$

**V. Yudson**, sov. phys. *JETP* (1985)

Computed S-matrix of Dicke model

with: $|F^\lambda\rangle$ Bethe eigenstate

$|F^\lambda)$ obtained from Bethe eigenstate by setting $S = I$ - One quadrant suffices

$\gamma$ contour in momentum space $\{\lambda\}$ determined by **pole structure** of $S(\lambda_i - \lambda_j)$

Note: in the infinite volume limit momenta $\{\lambda\}$ are not quantized
- no Bethe Ansatz equations, $\{\lambda\}$ free parameters

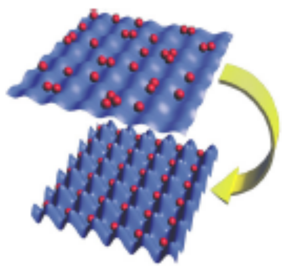
then:

$$\boxed{|\Phi_0, t\rangle = \int_\gamma d^N\lambda\, e^{-iE(\lambda)t}|F^\lambda\rangle(F^\lambda|\Phi_0\rangle}$$

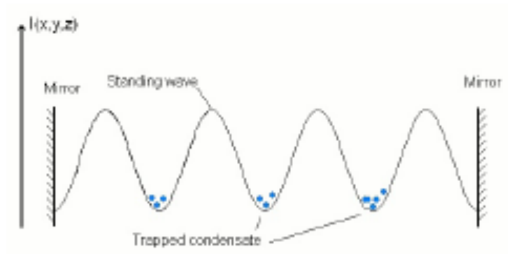
*Apply to:* 1. Bose gas - (a) finite number of particles (b) thermodynamic limit
2. XXZ spin chain 3. Super-radiance (Dicke model)

# 1. Boson Systems - experiments

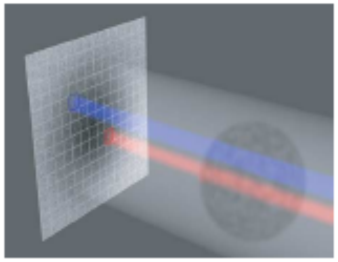
## Bosons in optical traps



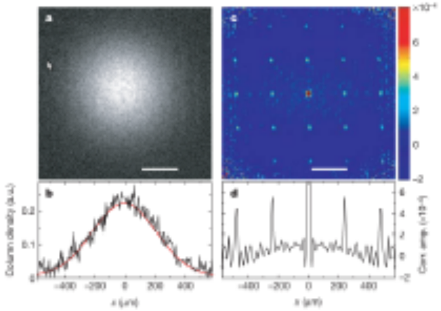
Superfluid Mott insulator transition




Mott insulator – initial condition



Imaging of density cloud using a CCD



Density and noise correlation functions

*Bloch et al* (Nature 2005, Rev Mod Phys 2008)

# Interacting bosonic system

## Bosons in a 1-d  with short range interactions

$$H = -\int dx b^\dagger(x)\partial^2 b(x) + c\int dx\, b^\dagger(x)b(x)b^\dagger(x)b(x)$$

$c$ **- coupling constant**

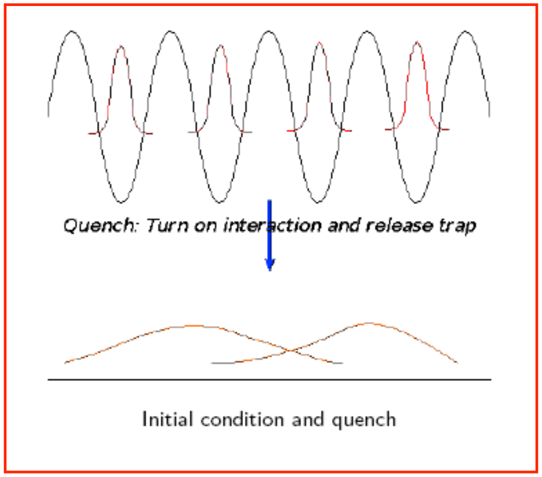
$c > 0$ *repulsive*
$c < 0$ *attractive*

*Can be tune by Feshbach resonance*

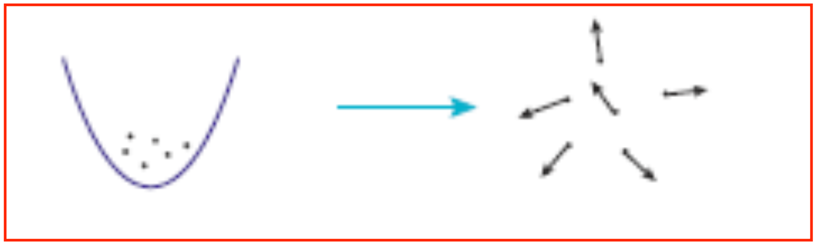
**Equivalently:**

$$H = -\sum_{j=1}^{N}\partial_{x_j}^2 + c\sum_{i<j}\delta(x_i - x_j)$$

- **Initial condition I :  bosons in a periodic optical lattice**




- **Initial condition II : bosons in a trap - condensate**



# Bosonic system – BA solution

**The N-boson eigenstatestate (Lieb-Liniger '67)**

$$|\lambda_1, \cdots, \lambda_N\rangle = \int_y \prod_{i<j} Z^y_{ij}(\lambda_i - \lambda_j) \prod_j e^{i\lambda_j y_j}\; b^\dagger(y_j)|0\rangle$$

- **Eigenstates labeled by Momenta** $\lambda_1, \cdots, \lambda_N$

- ***Thermodynamics***: *impose PBC* $\longrightarrow$ *BA eqns* $\longrightarrow$ *momenta* $\begin{cases} \textit{real} & c > 0 \\ \textit{n-strings} & c < 0 \end{cases}$ $\boxed{\vdots}$

***- Dynamics** (infinite volume): momenta unconstrained*

- **Dynamic factor:** $Z^y_{ij}(\lambda_i - \lambda_j) = \dfrac{\lambda_i - \lambda_j - ic\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic} \quad = \begin{cases} 1 & y_i > y_j \\ S^{ij} & y_i < y_j \end{cases}$

**-** *The 2-particle S-matrix:* $S_{ij}(\lambda_i - \lambda_j) = \dfrac{\lambda_i - \lambda_j + ic}{\lambda_i - \lambda_j - ic}$ *enters when the particles cross*

- poles of the S-matrix at: $\lambda_i = \lambda_j + ic$

- **The energy eigenvalues**

$$H|\lambda_1, \cdots, \lambda_N\rangle = \sum_j \lambda_j^2\, |\lambda_1, \cdots, \lambda_N\,\rangle$$

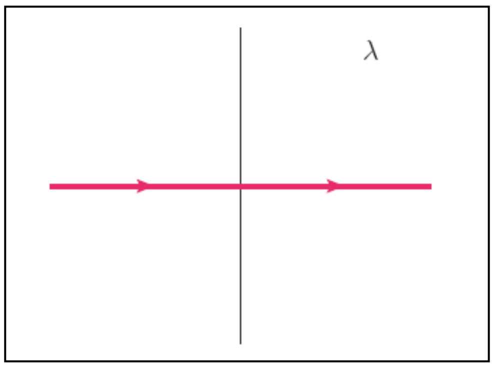
# bosonic system: contour representation

**"Central theorem"**

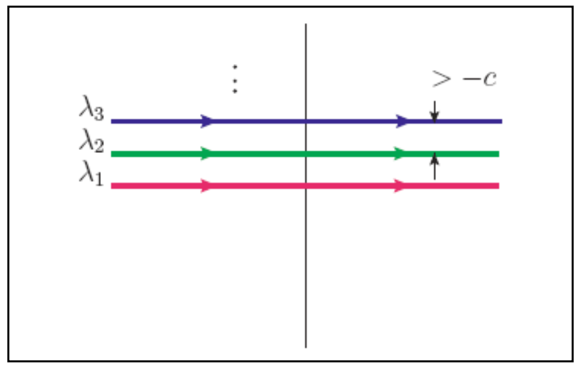
denote: $\theta(\vec{x}) = \theta(x_1 > x_2 > \cdots .x_N)$

$$|\Phi_0\rangle = \int_x \Phi_0(\vec{x}) b^\dagger(x_N)\cdots b^\dagger(x_1)|0\rangle =$$

$$= \int_{x,y}\int_\lambda \theta(\vec{x})\Phi_0(\vec{x}) \prod_{i<j} \frac{\lambda_i - \lambda_j - ic\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic} \prod_j e^{i\lambda_j(y_j - x_j)} b^\dagger(y_j)|0\rangle$$




**Repulsive c > 0**




**Attractive c < 0,**

*contour accounts for strings, bound states*

**It time evolves to:**

$$\boxed{|\Phi_0, t\rangle = \int_x \int_y \int_\lambda \theta(\vec{x})\Phi_0(\vec{x}) \prod_{i<j} \frac{\lambda_i - \lambda_j - ic\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic} \prod_j e^{-i\lambda_j^2 t} e^{i\lambda_j(y_j - x_j)} b^\dagger(y_j)|0\rangle}$$

- **Expression contains full information about the dynamics of the system**

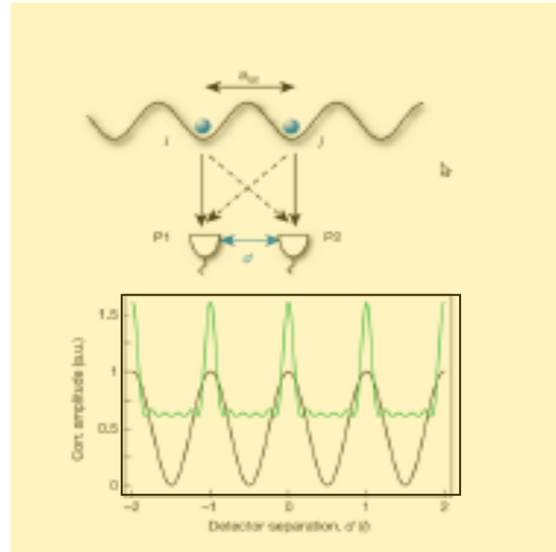
# What to calculate?

- **We shall study:**

**1. Evolution of the density**

$$C_1(x,t) = \langle \rho(x,t) \rangle$$ Time of Flight experiment

**competition** between quantum broadening and attraction



Hanbury-Brown Twiss effect

*Measure:* $C_2(x_1,x_2,t)$

**- two sources:** originally stars

*Free bosons* $C_2(x,-x) \sim \cos x$

*Free Fermions* $C_2(x,-x) \sim -\cos x$

**2. Evolution of noise correlation**

$$C_2(x_1,x_2;t) = \frac{\langle \rho(x_1,t)\rho(x_2,t)\rangle}{\langle \rho(x_1,t)\rangle\langle \rho(x_2,t)\rangle} - 1$$

***time dependent Hanbury Brown - Twiss effect***

- repulsive bosons evolve into fermions
- attractive bosons evolve to a condensate

**- Two bosons:**

*Similar, but time dependent*

**- Many bosons:**

*More structure: main peaks, sub peaks*

**Effects of interactions?**

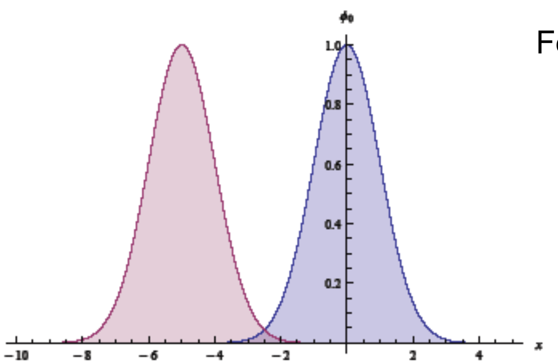
# Evolution of repulsive bosons into fermions: HBT

## Long time asymptotics - repulsive:

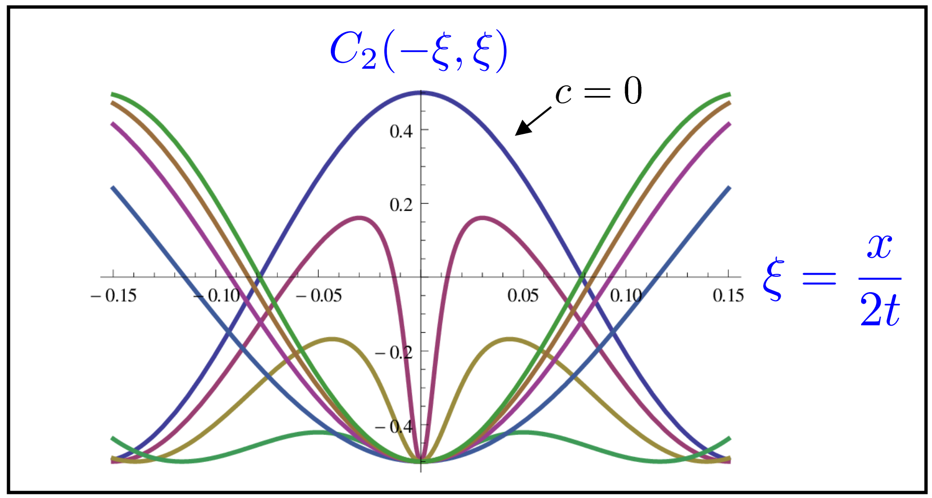
- Bosons turn into fermions as time evolves (for any $c > 0$)
- Can be observed in the noise correlations: (dependence on $t$ only via $\xi = x/2t$ )

$$C_2(x_1, x_2, t) \to C_2(\xi_1, \xi_2) = \frac{\langle \rho(\xi_1)\rho(\xi_2)\rangle}{\langle \rho(\xi_1)\rangle\langle \rho(\xi_2)\rangle} - 1,$$

$c/a = 0, .3, \cdots, 4$




Fermionic correlations evolve




- Fermionic dip develops for any repulsive interaction on time scale set by $1/c^2$

# Emergence of an asymptotic Hamiltonian

## Long time asymptotics - repulsive:

- **Bosons turn into fermions as time evolves (for any $c > 0$ )** (cf. Buljan et al. '08)

$$
\begin{aligned}
|\Phi_0, t\rangle &= \int_x \int_y \int_\lambda \theta(\vec{x})\Phi_0(\vec{x}) \prod_{i<j} \frac{\lambda_i - \lambda_j - ic\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic} \prod_j e^{-i\Sigma_j \lambda_j^2 t - \lambda_j(y_j - x_j)} \prod_j b^\dagger(y_j)|0\rangle \\
&= \int_x \int_y \int_\lambda \theta(\vec{x})\Phi_0(\vec{x}) \prod_{i<j} \frac{\lambda_i - \lambda_j - ic\sqrt{t}\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic\sqrt{t}} e^{-i\Sigma_j \lambda_j^2 - \lambda_j(y_j - x_j)/\sqrt{t}} \prod_j b^\dagger(y_j)|0\rangle \\
&\to \int_x \int_y \int_\lambda \theta(\vec{x})\Phi_0(\vec{x}) e^{-i\Sigma_j \lambda_j^2 - \lambda_j(y_j - x_j)/\sqrt{t}} \prod_{i<j} sgn(y_i - y_j) \prod_j b^\dagger(y_j)|0\rangle \\
&= e^{-iH_0^f t} \int_{x,k} \mathcal{A}_x\, \theta(\vec{x})\Phi_0(\vec{x}) \prod_j c^\dagger(x_j)|0\rangle.
\end{aligned}
$$

$\mathcal{A}_x$ *antisymmetrizer*

**where**

$H_0^f = -\int_x c^\dagger(x)\partial^2 c(c)$

***- In the long time limit repulsive bosons for any $c > 0$ propagate under the influence of* Tonks – Girardeau *Hamiltonian (hard core bosons=free fermions)***

***- The state equilibrates, does not thermalize***

***- Argument valid for any initial state $\Phi_0$***

- **Scaling argument fails for attractive bosons** (instead, they form bound states)

# Evolution of a bosonic system: saddle point app

## Corrections to long time asymptotics -

**Stationary phase approx at large times** (carry out $\lambda$ - integration)

- **Repulsive** – only stationary phase contributions (on real line); (cf. Lamacraft 2011)

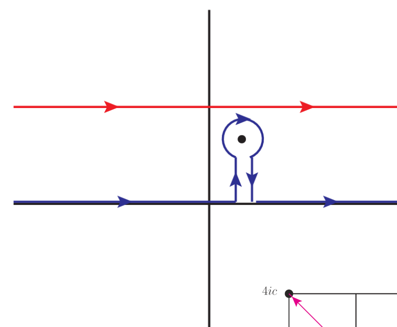
$$\phi\left(\xi \equiv \frac{y}{2t}, x, t\right) = S_\xi \frac{1}{(4\pi i t)^{\frac{N}{2}}} \prod_{i<j} \frac{\xi_i - \xi_j - ic\, sgn(\xi_i - \xi_j)}{\xi_i - \xi_j - ic} e^{\sum_j i\xi_j^2 t - i\xi_j x_j}$$

- **Attractive** – contributions from stationary phases and poles.

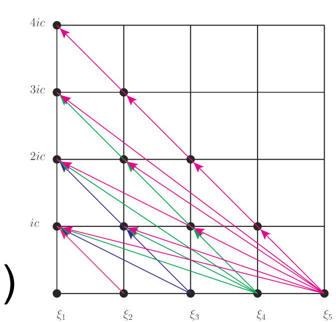
e. g for two particles:

Pole contributions from deformation of contours – formation of bound states

$$\phi(\xi, x, t) = S_\xi\Big[\frac{1}{4\pi i t} \frac{\xi_1 - \xi_2 - ic\, sgn(\xi_1 - \xi_2)}{\xi_1 - \xi_2 - ic} e^{\sum_j i\xi_j^2 t - i\xi_j x_j} + \\ + \frac{2c\theta(\xi_2 - \xi_1)}{\sqrt{4\pi i t}} e^{i\xi_1^2 t - i\xi_1 x_1 - i(\xi_1 - ic)^2 t + i(\xi_1 - ic)(2t\xi_2 - x_2)}\Big]$$



Bound states (string solutions) appear naturally



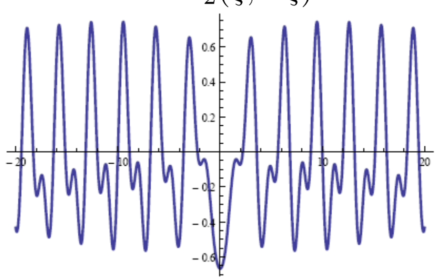

- repulsive correlations depend on $\xi = \frac{y}{2t}$ only (light cone propagation)
- attractive correlations maintain also $t$ dependence (bd. states provide scales)

# Evolution of a bosonic system: noise correlations

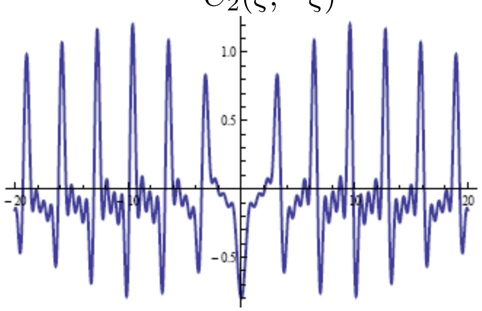
## Noise correlations – starting from a lattice

**Repulsive bosons**

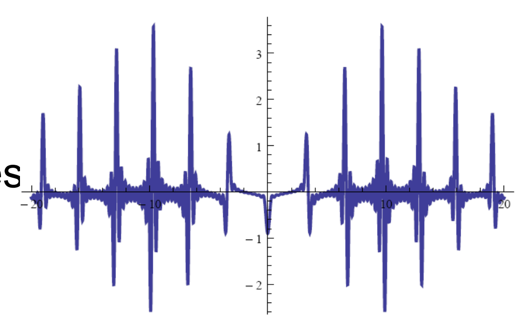
3 particles




5 particles




10 particles

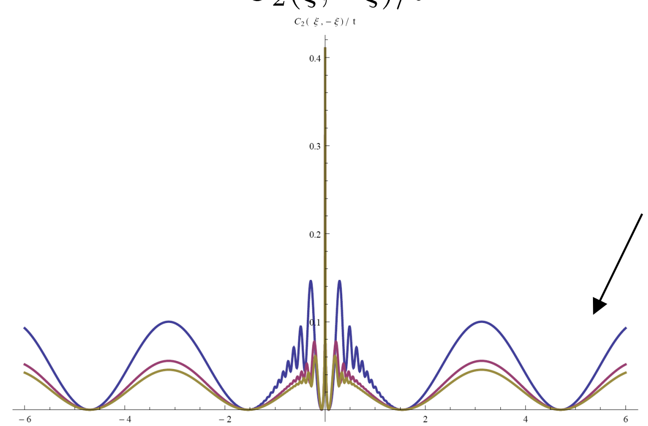


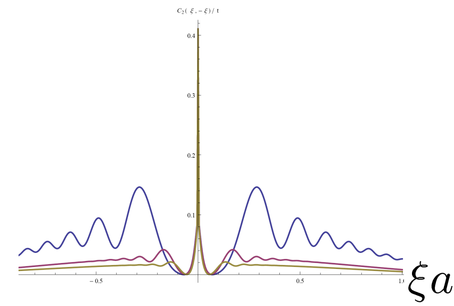

**Fermionic dip as** $\xi \to 0$

**Structure emerges at** $\xi a = \sigma$

**Attractive bosons**

2 particles

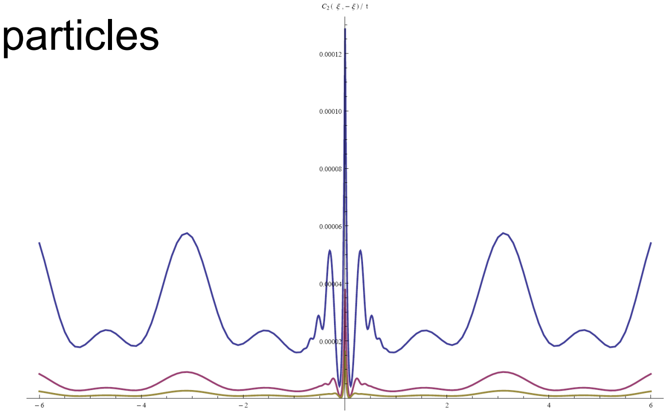



Blue - short times
Violet - longer
Magenta - longest

**central peaks increase with time - weight in the bound states increases**

3 particles



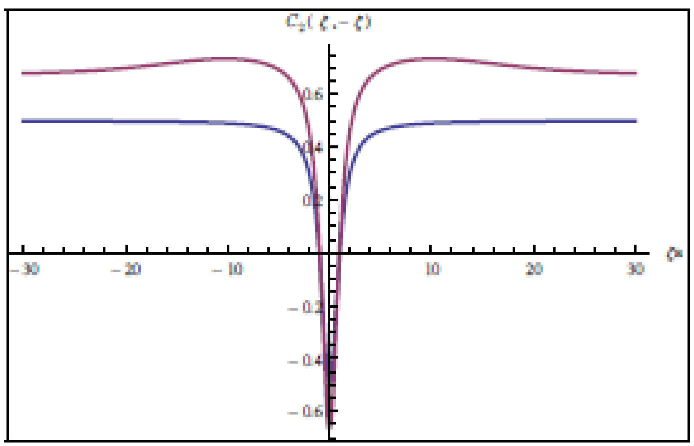

Blue - short times
Violet - longer
Magenta - longest

**peaks diffuse – momenta redistribute**

# Evolution of a bosonic system: noise correlation

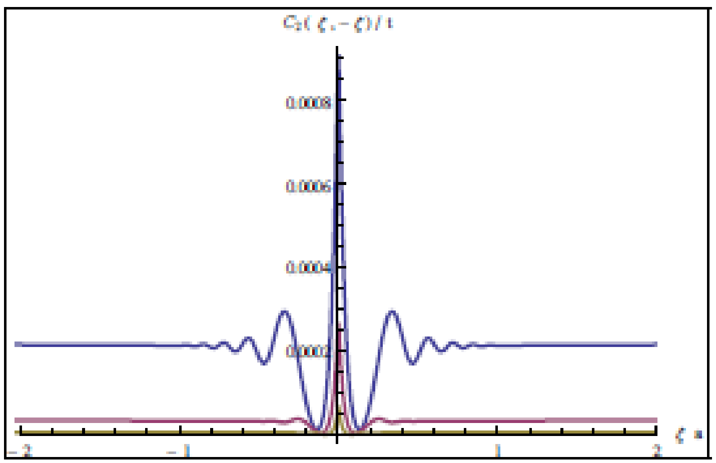
**Noise correlations – starting from a condensate**

**Repulsive bosons**




*Two (blue) and three bosons,*

**Attractive bosons**



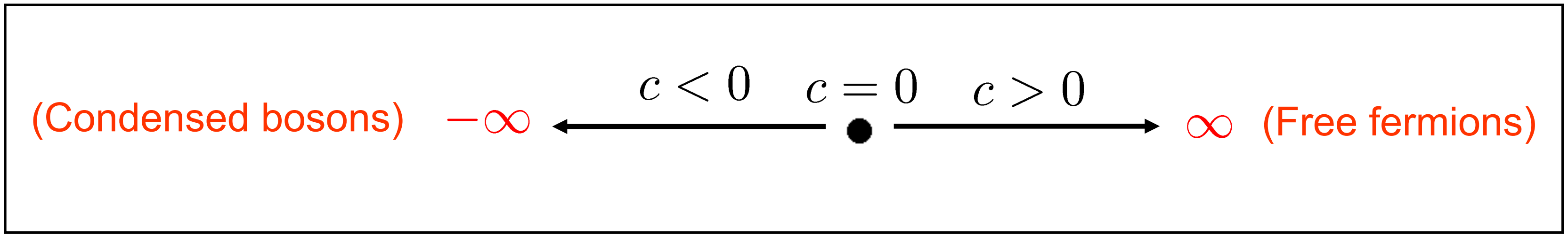

*Three bosons, at times:* $tc^2 = 20; 40; 60$

# Time evolution “Renormalization Group”

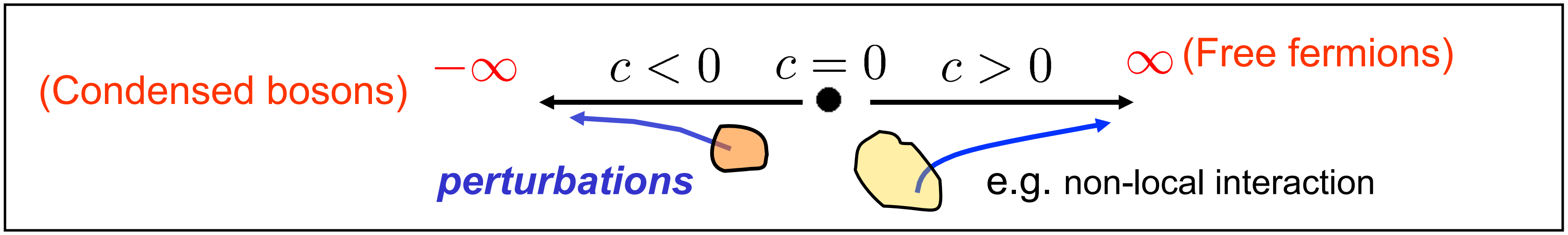
## “Dynamic” RG interpretation

- **Universality out of equilibrium**
- **Can view time evolution as RG flow** $t \sim \ln(D_0/D)$

**- As time evolves the weight of eigenstate contributions varies, time successively “integrates out” high energy states**

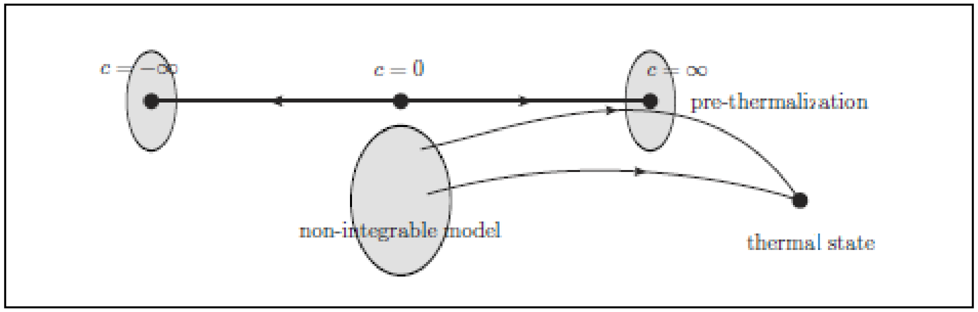



- **Are there “basins of attraction” for perturbations flowing to dynamic fixed points**




**What is beyond** $c = \infty$ **?**

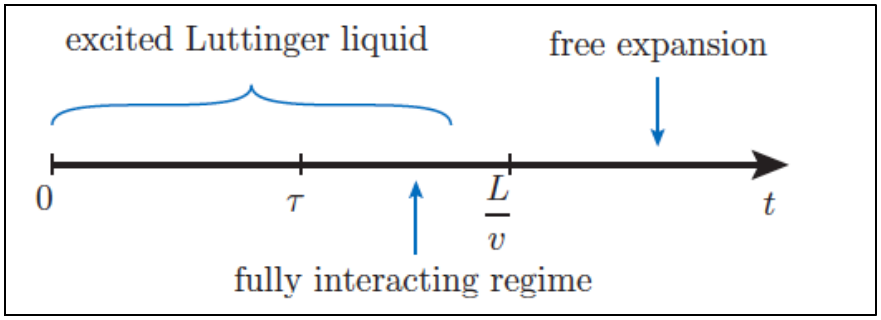
**Thermalization?**



# Evolution of the LL bosonic system – thermodynamic limit

**Wish to study the system in the limit:**

$$N, L \to \infty, \quad n = N/L \; fixed, \quad t \ll L/v_{typ}$$




**Finite size Yudson representation** :

- Claim: express any finite-size initial state (defined in a quadrant, e.g. bosons)

$$|\Phi_0\rangle = \int_{-L/2}^{L/2} dx_N \int_{-L/2}^{x_2} dx_1 \Phi\left(x_1..x_N\right) b^\dagger\left(x_N\right)..b^\dagger\left(x_1\right)|0\rangle\,,$$

in finite-size Yudson form (valid for repulsive interactions $c > o$)

$$|\Phi_0\rangle = \sum_{n_1=-\infty}^{\infty} \ldots \sum_{n_N=-\infty}^{\infty} \mathcal{N}\left(\{k_n\}\right)^{-1} |k_{n_1} \ldots k_{n_N}\rangle \left(k_{n_1} \ldots k_{n_N}\right| |\Phi_0\rangle\,.$$

*- Less powerful than before: need solutions for the Bethe momenta, but do not need overlaps*

# Time evolution of an observable

**The time evolution of an observable -** $\langle\Phi_0|\Theta(t)|\Phi_0\rangle$

- The matrix elements of $\Theta(t)$ defined in a basis: $b^\dagger(x_1)..b^\dagger(x_N)|0\rangle$

$$G(\Theta,t;x_1..x_N;y_1..y_N) = \langle 0|b(y_1)..b(y_N)\, e^{iHt}\Theta e^{-iHt}\, b^\dagger(x_1)..b^\dagger(x_N)|0\rangle$$

*then:*

$$\langle\Phi_0|\Theta(t)|\Phi_0\rangle = \int d\vec{x}\, d\vec{y}\, \Phi_0^*(\vec{x})\, G(\Theta,t;\vec{x},\vec{y})\, \Phi(\vec{y})$$

- In the Yudson representation:

$$\begin{aligned}
&G\left(\Theta,t;x_1,x_2,....x_N;y_1,...y_N\right) = \\
&= \sum\nolimits_{n_1,...n_N} \frac{1}{\mathcal{N}\left(k_{n_1}....k_{n_N}\right)} \langle 0|\, b\left(y_1\right)...b\left(y_N\right)|k_{n_1}..k_{n_N}\right) \times \\
&\quad \times \left\langle k_{n_1}...k_{n_N}\right|\Theta\left|q_{n_1},..q_{n_N}\right\rangle \times \sum\nolimits_{n_1,..n_N} \frac{1}{\mathcal{N}\left(q_{n_1}...q_{n_N}\right)} \times \\
&\quad \times \left(q_{n_1},....q_{n_N}\right| b^\dagger\left(x_1\right)...b^\dagger\left(x_N\right)|0\rangle \prod\nolimits_i e^{i\left(k_{n_i}^2-q_{n_i}^2\right)t}
\end{aligned}$$

- Inserted Yudson representation twice - overlaps simple plane waves

- Need matrix elements: $\left\langle k_{n_1}....k_{n_N}\right|\Theta\left|q_{n_1}...q_{n_N}\right\rangle$

*Korepin, Bogoliubov, Izergin, Slavnov,..*

# Time evolution of an observable

**Consider:** $\boxed{\Theta = e^{\alpha Q_{xy}}}$ $Q_{xy} \equiv \int_x^y b^\dagger(z)\, b(z)\, dz$ *Charge between x and y.*

- *Thermodynamic limit* $L, N \to \infty, \quad t \to \infty \ll L/v_{typ}$

- After a long long calculation …..

$$
\begin{aligned}
&\langle \exp(\alpha Q_{xy}(t)) \rangle = 1 + \int dX dY F_{\alpha,xy}(X,Y,t) \times \\
&\qquad \times \left\langle b^\dagger(Y) \exp\left[ i \int_X^Y dz \pi b^\dagger(z) b(z) \right] b(X) \right\rangle + \\
&+ \int dX_1 dX_2 dY_1 dY_2 \times F_{\alpha,xy}(X_1,Y_1,t) F_{\alpha,xy}(X_2,Y_2,t) \times \\
&\times \left\langle sgn(Y_2 - Y_1) b^\dagger(Y_1) b^\dagger(Y_2) e^{i \int_{X_1}^{Y_1} dz \pi b^\dagger(z) b(z)} \times \right. \\
&\left. \times\ sgn(X_2 - X_1) e^{i \int_{X_2}^{Y_2} dz \pi b^\dagger(z) b(z)} b(X_1) b(X_2) \right\rangle - \\
&- \frac{i\alpha}{\pi^2 c} \int dX_1 dY_1 dX_2 dY_2 \{ F_{\alpha,x}(X_1,Y_1,t) G_{\alpha,x}(X_2,Y_2,t) - \\
&- F_{\alpha,y}(X_1,Y_1,t) G_{\alpha,y}(X_2,Y_2,t) \} \langle sgn(y_2 - y_1) \\
&\cdot sgn(x_2 - x_1) \cdot b^\dagger(y_1) b^\dagger(y_2) e^{i \int_{X_1}^{Y_1} dz \pi b^\dagger(z) b(z)} \cdot \\
&\left. \cdot e^{i \int_{X_2}^{Y_2} dz \pi b^\dagger(z) b(z)} b(X_1) b(X_2) \right\rangle - \\
&- \frac{i\alpha}{2\pi^2 c} \int dX dY G_{\alpha,X}(X,Y,t) \left\langle b^\dagger(Y) e^{i \int_X^Y dz \pi b^\dagger(z) b(z)} \cdot \right. \\
&\left. \cdot \int_{-\infty}^{\infty} dv\, sgn(x - v) \rho(v)\, sgn(v - Y)\, sgn(v - X) \right\rangle + \\
&+ \frac{i\alpha}{2\pi^2 c} \int dX dY G_{\alpha,y}(X,Y,t) \left\langle b^\dagger(Y) e^{i \int_X^Y dz \pi b^\dagger(z) b(z)} \cdot \right. \\
&\left. \cdot \int_{-\infty}^{\infty} dv\, sgn(y - v) \rho(v)\, sgn(v - Y)\, sgn(v - X) \right\rangle + \ldots.
\end{aligned}
$$

***All expectation values taken w.r.t initial state***

$\boxed{\text{up to: } \alpha^3, 1/c^2}$

# Time evolution of an observable

**Where:**

$$F_{\alpha,xy}(X,Y,t) \equiv i\frac{e^{\alpha}-1}{2\pi}\frac{\exp\left(-\frac{i}{4t}\left(Y^2-X^2\right)\right)}{Y-X}\times\left[\exp\left(\frac{i\,(Y-X)\cdot x}{2t}\right)-\exp\left(\frac{i\,(Y-X)\cdot y}{2t}\right)\right]$$

$$G_{\alpha,x}(X,Y,t) \equiv \exp\left(i\frac{(Y-X)\,x}{2t}\right)\frac{\exp\left(-i\left(Y^2-X^2\right)/2t\right)}{t}$$

$$G_{\alpha,y}(X,Y,t) \equiv \exp\left(i\frac{(Y-X)\,y}{2t}\right)\frac{\exp\left(-i\left(Y^2-X^2\right)/2t\right)}{t}$$

$$F_{\alpha,x}(X,Y,t) \equiv \frac{\exp\left(-\frac{i}{4t}\left(Y^2-X^2\right)\right)}{Y-X}\exp\left(\frac{i\,(Y-X)\cdot x}{2t}\right)$$

$$F_{\alpha,y}(X,Y,t) \equiv \frac{\exp\left(-\frac{i}{4t}\left(Y^2-X^2\right)\right)}{Y-X}\exp\left(\frac{i\,(Y-X)\cdot y}{2t}\right)$$

- ***The time evolution is expressed:***

***i. in terms of these time dependent functions***
***ii. correlation functions in initial states***

- ***Valid for any initial state***

# Time evolution- flow chart

## Expression valid for any time and any initial state

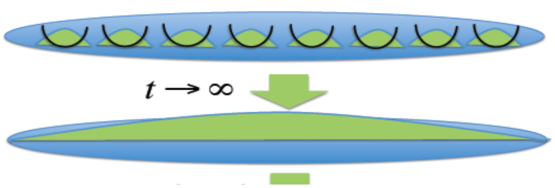
**- Conclusions:**

Initial state translational invariant

*example*: **Mott insulator**

Equilibrates to GGE (c >0)

*Universal correlations (if low YY entropy)*

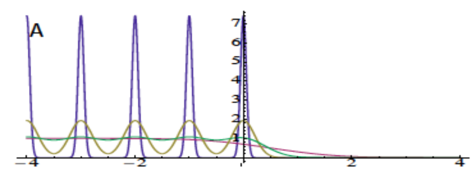



Equilibrates but not to GGE (c < 0)

*- GGE fails when bound states (strings) are present: LL, XXZ, Hubbard, Anderson…*

Initial state not translational invariant

*example*: **Domain wall**




*System does not equilibrate:* currents, *local* entropy production

# Time evolution – Diaginal ensemble, GGGE and GGE

- ***For translationally invariant initial states,*** *c>0*

*i. The system equlibrates, the limit* $t \to \infty$ *is well defined*

*ii. The system equilibrates to a* ***diagonal*** *ensemble*

$$\rho_D = \sum_\lambda |\langle \Phi(t=0)|\lambda\rangle|^2 \, |\{\lambda_i\}\rangle \langle \{\lambda_i\}|$$

*iia. The system obeys GGE (if no long range correlations present in initial state)*

$$\hat{\rho}_{GGE} = Z^{-1} exp\left[ -\sum_m \alpha_m I_m \right]$$

*The conserved charges:* $I_m|\lambda\rangle = \sum_i \lambda_i^m |\lambda\rangle$

*with* $Tr\left[I_m \widehat{\rho}\right] = \langle I_m \rangle (t=0)$

*No long range correlations:* $\langle I_{m_1} I_{m_2} \ldots \rangle = \langle I_{m_1} \rangle \langle I_{m_2} \rangle \ldots$

*iib. The system obeys "generalized" GGE (if long range correlations present)*

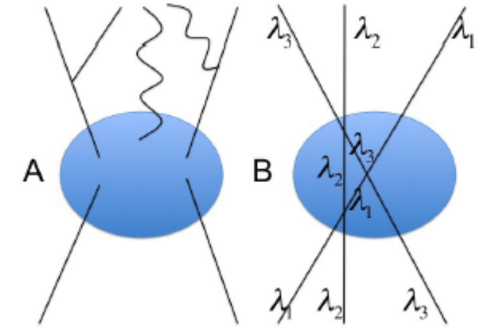
$$\widehat{\rho}_{GGGE} = \widetilde{Z}^{-1} \exp\left[ -\sum_{m_1 m_2 \ldots} \alpha_{m_1 m_2 \ldots} I_{m_1} I_{m_2} \ldots \right]$$

- ***For sufficiently non-translationally invariant initial states (e.g. domain wall)***

*iii. System does not equilibrate, does not reach diagonal ensemble*

# Timee volution – Diagonal ensemble, GGGE and GGE

**More on GGE:**




*i. The diagonal element can be Taylor expanded*

$$\langle\{k_i\}|\,\Theta\,|\{k_i\}\rangle = c_0 + c_1 \sum k_i + c_{1,1} \sum k_i k_j + c_2 \sum k_i^2 + ..$$

*ii. So for trans. invariant initial states :* $\langle\Theta\rangle\,(t\to\infty) = Tr\rho_D\Theta$

$$\rho_D = \sum p_{\{k\}}\,|\{k_i\}\rangle\,\langle\{k_i\}| \quad \textit{with} \qquad p_k = |\langle\{k\}|\Phi_0\rangle|^2$$

*iii. Thus:* $\langle\Theta\rangle \to c_0 + c_1\langle I_1\rangle + c_{1,1}\langle I_1^2\rangle + c_2\langle I_2\rangle + ..$

*with:* $\langle I_1\rangle = \sum p_{\{\lambda\}} \sum k_i, \;\; \langle I_1^2\rangle = \sum p_{\{k\}} \sum k_i k_j, \;\; \langle I_2\rangle = \sum p_{\{k\}} \sum k_i^2 \;\; \ldots$

*iv. Equivalently:* $\rho_D = \widehat{\rho}_{GGGE} = \widetilde{Z}^{-1}\exp\left[-\sum_{m_1 m_2\ldots} \alpha_{m_1 m_2\ldots} I_{m_1} I_{m_2}\ldots.\right]$

*with* $\{\alpha_{m_1 m_2\ldots}\}$ *determined from* $Tr\left[I_{m_1} I_{m_2}\ldots\rho_{GGGE}\right] = \langle I_{m_1} I_{m_2}\ldots\rangle\,(t=0)$

*GGE* $\neq$ *GGGE when long range correlations present in initial state*

*v. GGGE* $\longrightarrow$ *GGE for short range correlations in initial state*

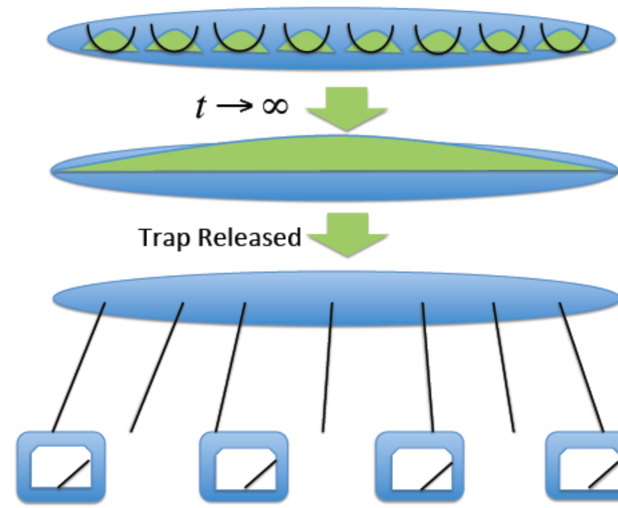
$$\langle I_{m_1} I_{m_2}\ldots.\rangle = \langle I_{m_1}\rangle\,\langle I_{m_2}\rangle\ldots$$

# Time evolution- interaction quench from a Mott state

*Example:* **Quenching from a Mott insulator to a Lieb-Liniger Liquid:** $t \to \infty$ **GGE**

$$\left|\Phi_0\right\rangle = \prod_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi\left(x + jl\right) b^{\dagger}\left(x\right)\left|0\right\rangle$$

*with* $\varphi\left(x\right) = \frac{e^{-x^2/\sigma}}{\left(\pi\sigma/2\right)^{1/4}}$




**- For GGE** *(Caux):* $tr\left[\Theta\rho_{GGE}\right] = \left\langle \vec{k}_0 \middle| \Theta \middle| \vec{k}_0 \right\rangle$

*with the eigenstate* $\left|\vec{k}_0\right\rangle$ *saisfying:*

$$\rho_t\left(k\right) = \frac{1}{2\pi} + \frac{1}{2\pi}\int dqK\left(k,q\right)\rho_p\left(q\right), \qquad K\left(k,q\right) = \frac{2c}{c^2 + \left(k-q\right)^2}$$

*and:*

$$L\int dk\rho_p\left(k\right)k^n = I_n\left(t=0\right) .= \frac{L}{l}\left(\frac{2}{\sigma}\right)^{\frac{n}{2}} \frac{n!}{2^{\frac{n}{2}}\left(\frac{n}{2}!\right)} \longrightarrow \begin{cases} \rho_p\left(k\right) = \dfrac{\sigma^{\frac{1}{2}}}{\pi^{\frac{1}{2}} l}\exp\left(-\dfrac{k^2\sigma}{2}\right) \\ \rho_t\left(k\right) \cong \dfrac{1}{2\pi} \quad \text{for} \quad l \gg \sqrt{\sigma} \end{cases}$$

**- The occupation probability** $f\left(k\right) \equiv \frac{\rho_p\left(k\right)}{\rho_t\left(k\right)} \cong \frac{2\sqrt{\pi\sigma}}{l}\exp\left(-\frac{k^2\sigma}{2}\right)$

# Time evolution – interaction quench from a Mott state

**Can compute various correlation functions:**

1. $$\left\langle b^{\dagger}(0)\, b^{\dagger}(0)\, b(0)\, b(0)\right\rangle \cong 2\int \frac{dk_1}{2\pi}\int \frac{dk_2}{2\pi} f(k_1)\, f(k_2)\, \frac{(k_2-k_1)^2}{(k_2-k_1)^2+c^2} + \ldots..$$

$$= \frac{2}{l^2} - \frac{2\sqrt{\pi c^2\sigma}}{l^2}\left[\exp\left(\frac{\sigma c^2}{4}\right) Erfc\left(\sqrt{\frac{\sigma c^2}{4}}\right)\right] \longrightarrow \begin{cases} \cong \dfrac{2}{l^2} & c^2\sigma \ll 1 \\ \cong \dfrac{1}{l^2 c^4 \sigma^2} & c^2\sigma \gg 1 \end{cases}$$

*Suppression of density correlations, measurable by Time of Flight experiments*

2. $$\left\langle b^{\dagger}(0)\, b^{\dagger}(0)\, b^{\dagger}(0)\, b(0)\, b(0)\, b(0)\right\rangle \cong 6\int dk_1 dk_2 dk_3 f(k_1)\, f(k_2)\, f(k_3)\, \frac{(k_2-k_1)^2}{(k_2-k_1)^2+c^2}\frac{(k_3-k_1)^2}{(k_3-k_1)^2+c^2}\frac{(k_3-k_2)^2}{(k_3-k_2)^2+c^2}$$

*Strong suppression of three body decay rates, measurable through trap loss or third moment of particle number (Bouchoule‘10)*

$$\longrightarrow \begin{cases} \cong \dfrac{6}{l^3} & c^2\sigma \ll 1 \\ \cong \dfrac{9\times 2^{\frac{9}{2}}}{l^3 c^6 \sigma^3} & c^2\sigma \gg 1 \end{cases}$$

3. $$\left\langle \rho(x)\, \rho(0)\right\rangle \cong \rho^2 + \frac{1}{4\pi^2 e^2 l^2}\exp\left(-\frac{x^2}{\sigma}\right)$$ *for* $l \gg \sqrt{\sigma}$

*Gaussian decay of density-density function*

# Time evolution – low entropy intial state,

Generically:

*If a system equilibrates and reaches GGE – what are the experimental signatures?*

***i.*** *GGE can be reduced to a pure state* $\rho_{GGE} \cong \left|\vec{k}_0\right\rangle \left\langle \vec{k}_0\right|$

***ii.*** Define $\Theta_\beta\left(x\right) \equiv \exp\left(\beta \int_0^x b^\dagger\left(z\right) b\left(z\right) dz\right)$

- for hard core bosons (Tonks-Girardeau gas)

$$tr\rho_{GGE}\Theta_\beta(x) = \left\langle \vec{k}_0 \right| \Theta_\beta\left(x\right) \left| \vec{k}_0 \right\rangle = \det\left(I + \frac{e^\beta - 1}{\pi} \frac{1}{\sqrt{1+e^{\varepsilon(k)}}} \frac{\sin\left(k-q\right)\frac{x}{2}}{k-q} \frac{1}{\sqrt{1+e^{\varepsilon(q)}}}\right)$$

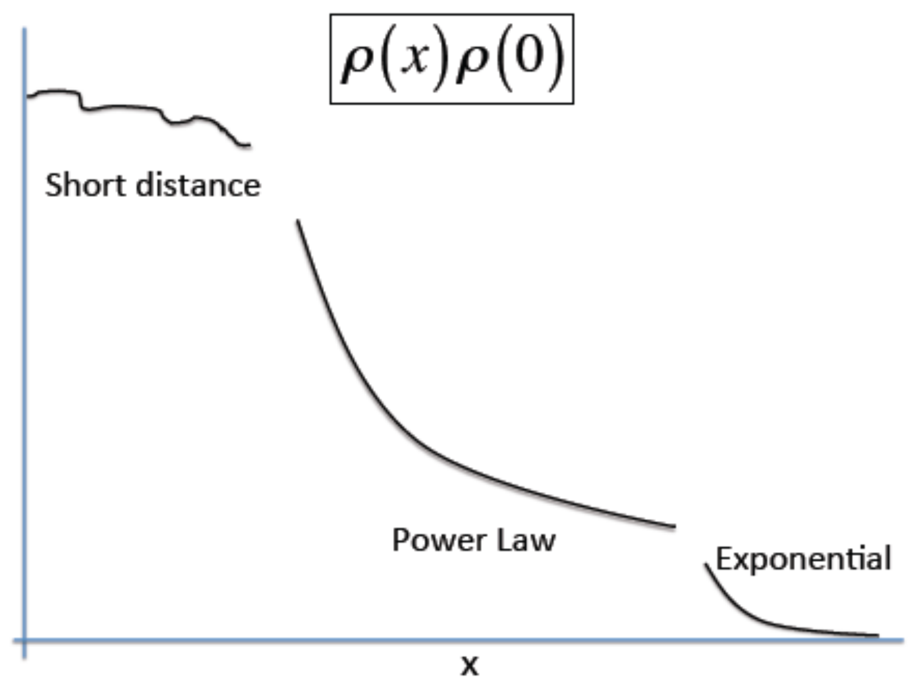
*Slavnov '10*

***iii.*** Expand in $\beta, x$ :

$$\rightarrow \quad tr\hat{\rho}_{GGE}\rho(x)\rho(0)$$

***iv.*** *Generic initial state low YY entropy*

***Note:*** *exponential decay: finite YY entropy of initial state ~ finite T*



# Failure of GGE for models with strings

- *Thus far: Evolution of repulsive (c>0) Lieb-Liniger* $\longrightarrow$ *GGE, GGGE, etc*
- *Not so for c<0, attractive Lieb-Liniger*

**Failure of GGE for systems with bound states**

e.g. *Attractive Lieb- Liniger, XXZ, Gaudin-Yang, Hubbard, sine-Gordon…*

- *Momenta fall into n-string configurations (bound states):* $k_j = k_0 + \frac{ic}{2}(n-2j)\,,\ j = 1 \ldots n$
- *Described by n-string densities,* $\rho_p^n(k)$

- *GGE determine by:* $\langle I_i \rangle_{final} \equiv Tr\left\{\rho_{GGE} I_i\right\} = \langle I_i\,(t=0)\rangle \equiv \langle I_i \rangle_{initial} = I_i^0$

***Need to solve:***

$$\boxed{I_i\left\{\rho_p^1, \rho_p^2, ..\right\} = \sum_{n=0}^{\infty}\sum_{l=0}^{i} J_l^n \left(\frac{ic}{2}\right)^{i-l} \sum_{j=0}^{n} (n-2j)^{i-l} = I_i^0}$$

*Contribution to* $I^i$ *of single n-string centered at* $k_0$ *:* $\sum_{j=0}^{n}\left(k_0 + \frac{ic}{2}(n-2j)\right)^i = \sum_{l=0}^{i} k_0^{i-l}\left(\begin{array}{c} i \\ l \end{array}\right)\left(\frac{ic}{2}\right)^i \sum_{j=1}^{n}(n-2j)^i$

*Integral over positions :* $J_i^n = \int dk\, \rho_p^n(k)\, k^i$

- ***Claim: There are infinitely many solutions, each corresponding to different correlation functions***

$\rightarrow$ $\boxed{\textit{\textbf{GGE fails}}}$ $\rightarrow$ *Need full time evolution, no shortcut available*

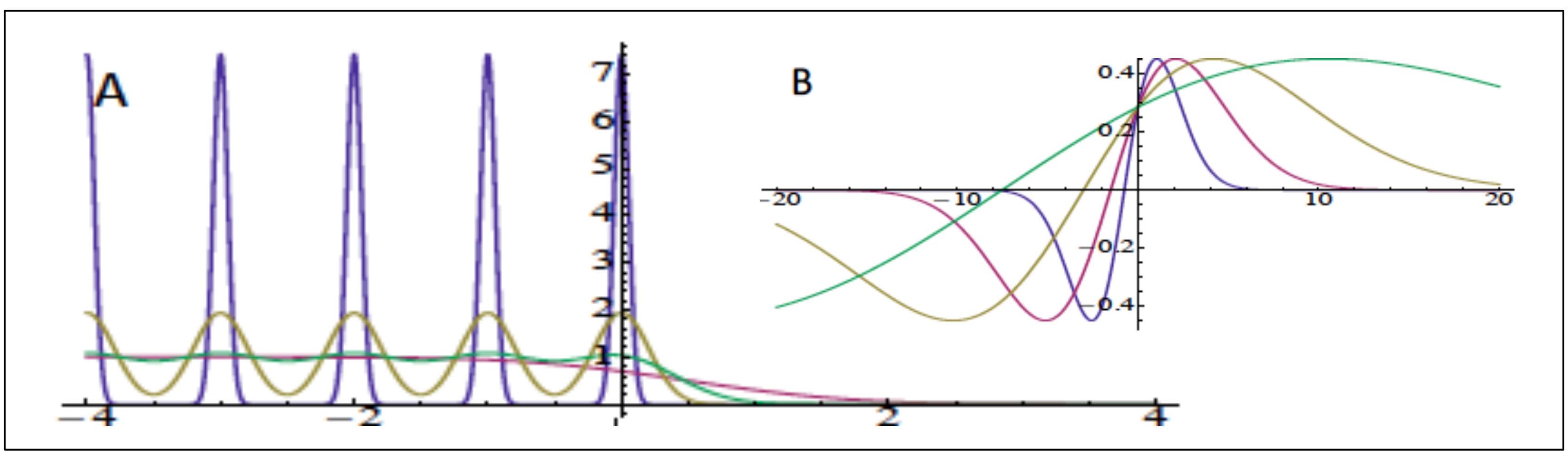
# Time evolution – systems that do not equilibrate

***Example**: time evolution from a non-trans. invariant initial state (no equilibration)*

$$|\Psi(t=0)\rangle = \prod_{j=0}^{\infty}\int_{-\infty}^{\infty}\varphi(x+jl)\,b^{\dagger}(x)\,|0\rangle \qquad \textit{with:} \quad \varphi(x)=\frac{e^{-x^2/\sigma}}{(\pi\sigma/2)^{1/4}}$$

*The density :*

$$\rho(x,t)\rightarrow\begin{cases}\frac{1}{l}\left(1+\sum_{s=1}^{\infty}e^{-\pi^2(8t^2/\sigma+\sigma/2)s^2/l^2}\cos(2\pi s x/l)\right) & x\rightarrow-\infty\\[2ex] \frac{1}{l}\left(\frac{1}{2}Erfc\left(\frac{x}{\sqrt{A}}\right)+\right. \\ +\frac{16}{\pi c l}e^{-x^2/A}\left(\frac{1}{2}\sqrt{\pi}\frac{x}{\sqrt{A}}Erfc\left(\frac{x}{\sqrt{A}}\right)-\frac{1}{2}e^{-x^2/A}\right) \\ \left.+\frac{16}{\pi c l}\frac{\pi}{2}\left(1-\frac{1}{2}Erfc\left(\frac{x}{\sqrt{A}}\right)\right)Erfc\left(\frac{x}{\sqrt{A}}\right)\right) & x\sim t,\quad A\equiv 8t^2/\sigma+\sigma/2\\[2ex] \frac{1}{l}\left(\frac{1}{2}-\frac{4\pi}{cl}\right) & \sigma\ll|x|\ll t\\[2ex] 0 & x\gg\sqrt{8t^2/\sigma}\end{cases}$$




*Domain wall: strongly breaks trans. Invariance*

# 2. The Heisenberg Chain: Theory and Experiment

**The XXZ Hamiltonian**

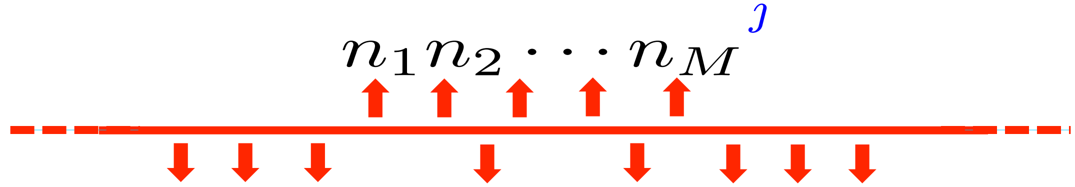
$$H = J\sum_j \sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta(\sigma_j^z\sigma_{j+1}^z - 1)$$

**The phase diagram**


**Initial states:**

$$|\Psi_0\rangle = \theta(n_1 > n_2 > \ldots n_N)\prod_j \sigma_{n_j}^+|0\rangle$$

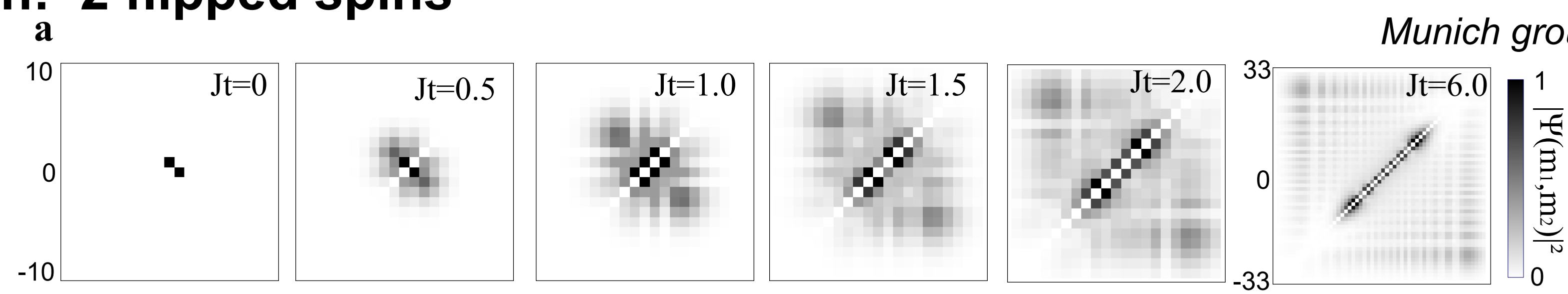
*e. g.*




**Time evolution: 2 flipped spins**

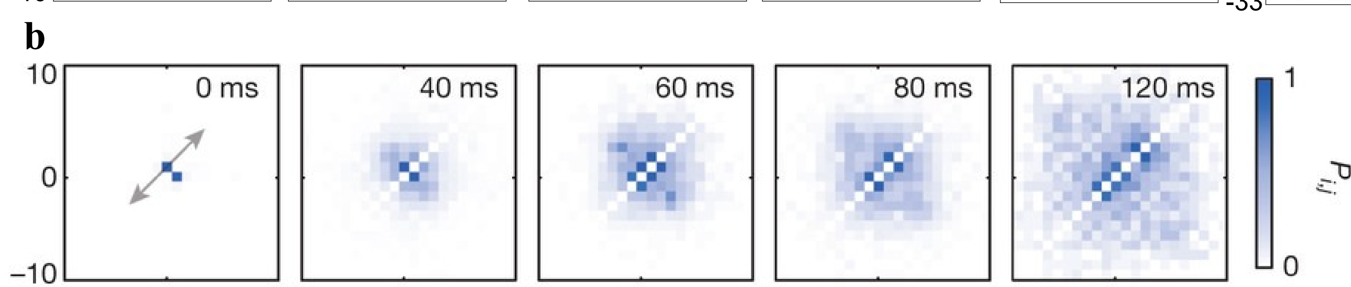
*Munich group '13*

**Theory:**




**Experiment:**

*Munich group '13*



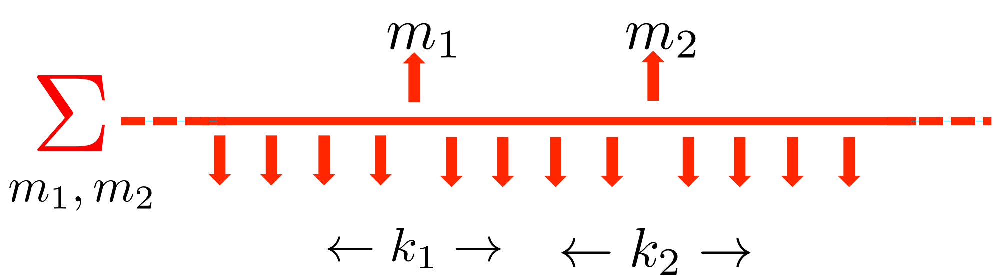
# Eigenstates of the Heisenberg Chain

## Eigenstates of the XXZ (M flipped spins)

$$|k\rangle = \sum_{\{m_j\}} \mathcal{S} \prod_{i<j} [\theta(m_i - m_j) + s(k_i, k_j)\theta(m_j - m_i)] \prod_j e^{ik_j m_j} \sigma^+_{m_j} |0\rangle$$

$$s(k_i, k_j) = e^{i\phi(k_i, k_j)} = -\frac{1 + e^{ik_i + ik_j} - 2\Delta e^{ik_i}}{1 + e^{ik_i + ik_j} - 2\Delta e^{ik_j}}$$

$$E = 4J \sum_{j=1}^{M} (\Delta - \cos k_j)$$



# Time evolution of the XXZ magnet

*i.* Critical region $-1 < \Delta < 0$ $\quad \Delta = -\cos\mu \quad (0 < \mu < \frac{\pi}{2})$

**Reparametrize:** $\Delta \to \mu, \quad k \to \alpha$

$$e^{ik} \to \frac{\sinh\frac{i\mu-\alpha}{2}}{\sinh\frac{i\mu+\alpha}{2}} \longrightarrow \quad s(k_1,k_2) \to \frac{\sinh(\frac{\alpha_1-\alpha_2}{2}-i\mu)}{\sinh(\frac{\alpha_1-\alpha_2}{2}+i\mu)}$$

$$E(k) \to E(\alpha) = \frac{4J\sin^2\mu}{\cosh\alpha - \cos\mu}$$

**The contour expression of the initial state:**

$$|\Psi_0\rangle = \theta(n_1 > n_2 > ...n_N)\prod_j \sigma^+_{n_j}|0\rangle$$

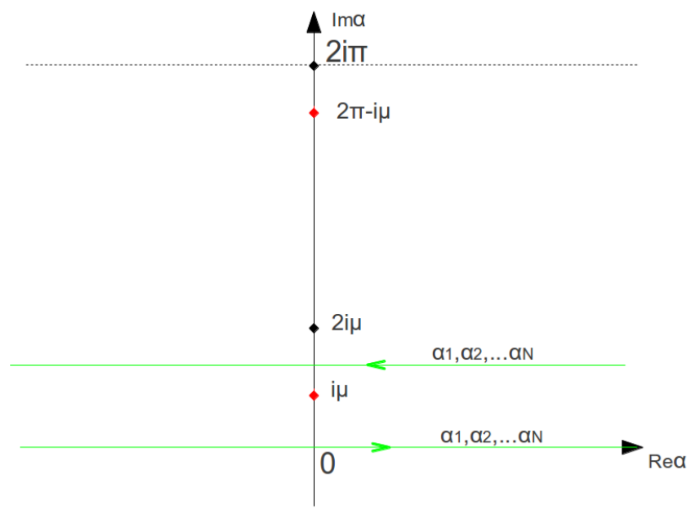
*Expanded in terms of eigenstates*

$$|\Psi_0\rangle = \sum_{\{m_j\}}\int_{\gamma_j}\prod_j[\frac{d\alpha_j}{2\pi}\frac{\sin\mu}{2\sinh\frac{\alpha_j+i\mu}{2}\sinh\frac{\alpha_j-i\mu}{2}}]\prod_j[\frac{\sinh(\frac{i\mu-\alpha_j}{2})}{\sinh(\frac{i\mu+\alpha_j}{2})}]^{m_j-n_j}$$

$$\times\prod_{i<j}[\theta(m_i-m_j)+\frac{\sinh(\frac{\alpha_i-\alpha_j}{2}-i\mu)}{\sinh(\frac{\alpha_i-\alpha_j}{2}+i\mu)}\theta(m_j-m_i)]\prod_j\sigma^+_{m_j}|0\rangle$$

# Evolution of the XXZ magnet

**The contour:**




**The time evolved state:**

$$|\Psi(t)\rangle = \sum_{\{m_j\}} \int_{\gamma_j} \prod_j [\frac{d\alpha_j}{2\pi} \frac{\sinh\lambda}{2\sin\frac{\alpha_j+i\lambda}{2}\sin\frac{\alpha_j-i\lambda}{2}}] \prod_{i<j} [\theta(m_i-m_j) + \frac{\sin(\frac{\alpha_i-\alpha_j}{2}-i\lambda)}{\sin(\frac{\alpha_i-\alpha_j}{2}+i\lambda)}\theta(m_j-m_i)]$$

$$\times \prod_j [\frac{\sin(\frac{i\lambda-\alpha_j}{2})}{\sin(\frac{i\lambda+\alpha_j}{2})}]^{m_j-n_j} e^{-iE(\alpha_j)t} \sigma^+_{m_j} |0\rangle$$

# Evolution of the XXZ magnet

*ii.* $\Delta < -1$ *Ferromagnetic regime*

$$\Delta = -\cosh\lambda \rightarrow \lambda > 0$$

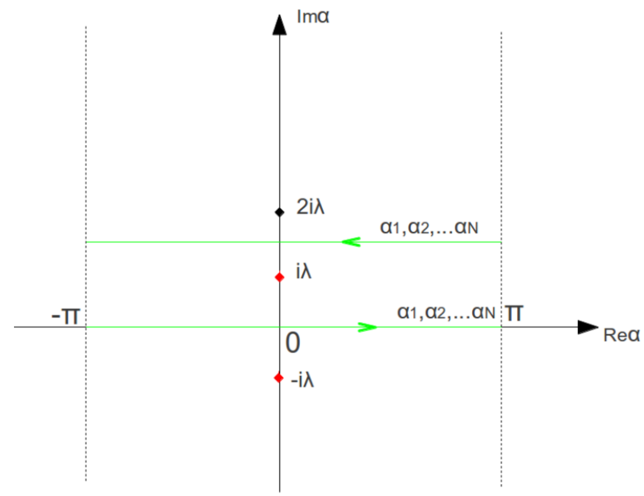
**Reparametrize:**

$$e^{ik} \rightarrow \frac{\sin\frac{i\lambda-\alpha}{2}}{\sin\frac{i\lambda+\alpha}{2}}$$

$$s(k_1,k_2) \rightarrow \frac{\sin(\frac{\alpha_1-\alpha_2}{2}-i\lambda)}{\sin(\frac{\alpha_1-\alpha_2}{2}+i\lambda)}$$

$$E(k) \rightarrow E(\alpha) = -\frac{4J\sinh^2\lambda}{\cos\alpha-\cosh\lambda}$$

**The contour:**




**The time evolved state**

$$|\Psi(t)\rangle = \sum_{\{m_j\}} \int_{\gamma_j} \prod_j \Big[\frac{d\alpha_j}{2\pi}\frac{\sinh\lambda}{2\sin\frac{\alpha_j+i\lambda}{2}\sin\frac{\alpha_j-i\lambda}{2}}\Big] \prod_{i<j}\Big[\theta(m_i-m_j)+\frac{\sin(\frac{\alpha_i-\alpha_j}{2}-i\lambda)}{\sin(\frac{\alpha_i-\alpha_j}{2}+i\lambda)}\theta(m_j-m_i)\Big]$$

$$\times \prod_j \Big[\frac{\sin(\frac{i\lambda-\alpha_j}{2})}{\sin(\frac{i\lambda+\alpha_j}{2})}\Big]^{m_j-n_j} e^{-iE(\alpha_j)t}\sigma^+_{m_j}|0\rangle$$

# Evolution of the XXZ magnet

**Some results** *- local magnetization and bound states*

*- Spin currents*

**Start from**

$$|\Psi_0\rangle = \sigma^-_{-1}\sigma^-_0\sigma^-_{+1}|\Uparrow\rangle$$

**Calculate:**

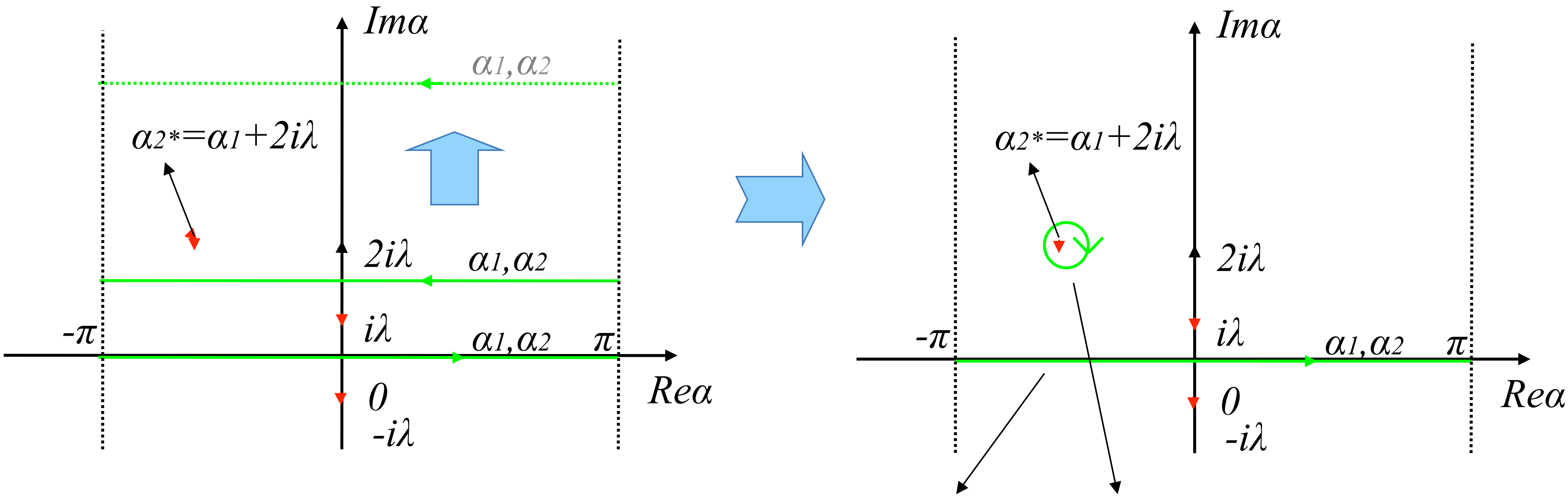
$$M(n,t) = \langle\Psi(t)|\sigma^z_n|\Psi(t)\rangle$$

$$I(n,t) = i\langle\Psi(t)|(\sigma^+_n\sigma^-_{n+1} - \sigma^-_n\sigma^+_{n+1})|\Psi(t)\rangle$$

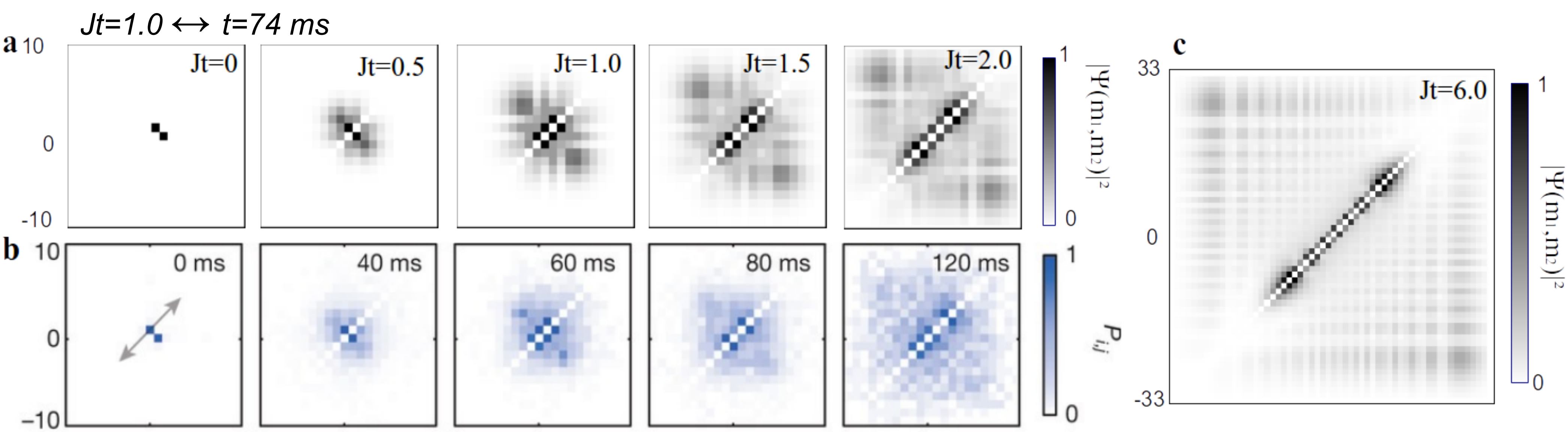
**For different values of anisotropy** $\Delta$

*- as the anisotropy increases the weight of the bound states increases*

# Contour Shift and Bound States




$$\Psi^{1,0}(m_1, m_2; t) = \Psi_{magn}(m_1, m_2; t) + \Psi_{bound}(m_1, m_2; t)$$

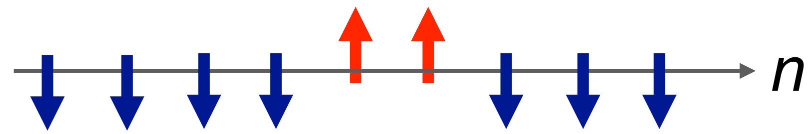



*b.* T. Fukuhara et al, Nature **502**, 76 (2013)

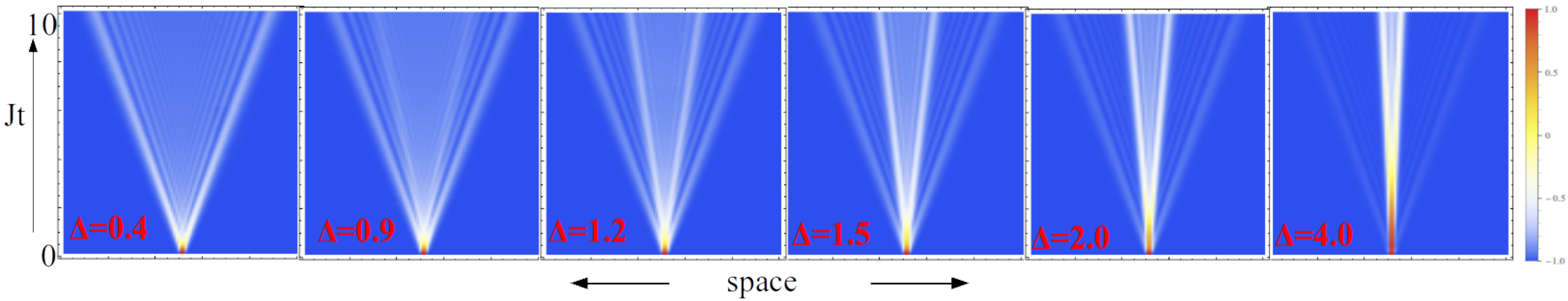
# Observables

- **Local Magnetization** $M(n,t) \equiv \langle \Psi(t) | \sigma_n^z | \Psi(t) \rangle$

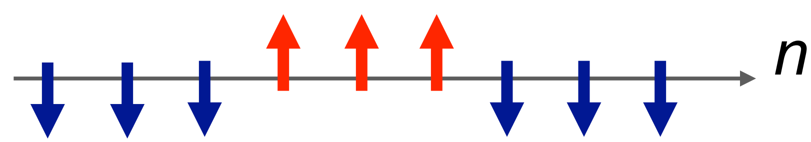
$$|\Psi_0\rangle = \sigma_1^+ \sigma_0^+ | \Downarrow \rangle =$$

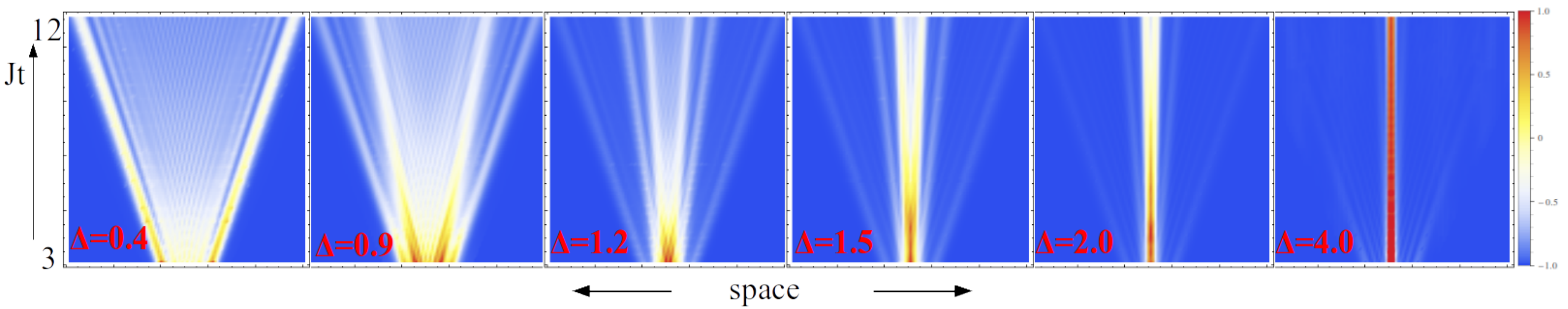



(cf. Ganahl et al. '12)




$$|\Psi_0\rangle = \sigma_1^+ \sigma_0^+ \sigma_{-1}^+ | \Downarrow \rangle =$$

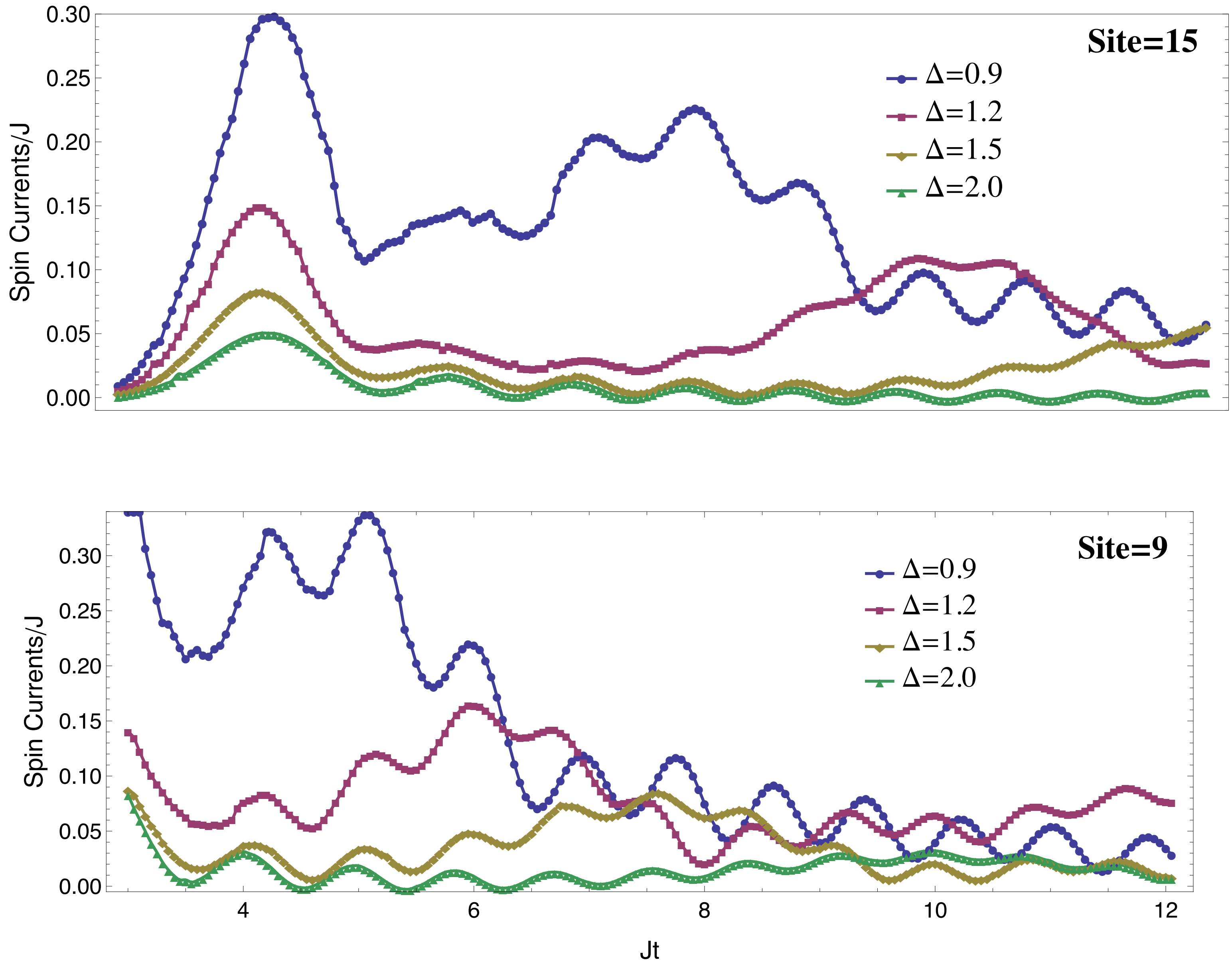



$$\frac{v_{bound}}{v_{magn}} = \frac{\sin \mu}{\sin(n\mu)} (|\Delta| = \cos \mu) \qquad \frac{v_{bound}}{v_{magn}} = \frac{\sinh \lambda}{\sinh(n\lambda)} (|\Delta| = \cosh \lambda)$$

# Evolution of the XXZ magnet

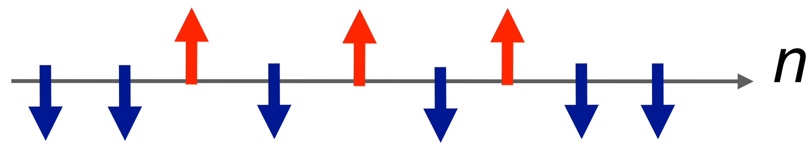
## Spin currents - evolution

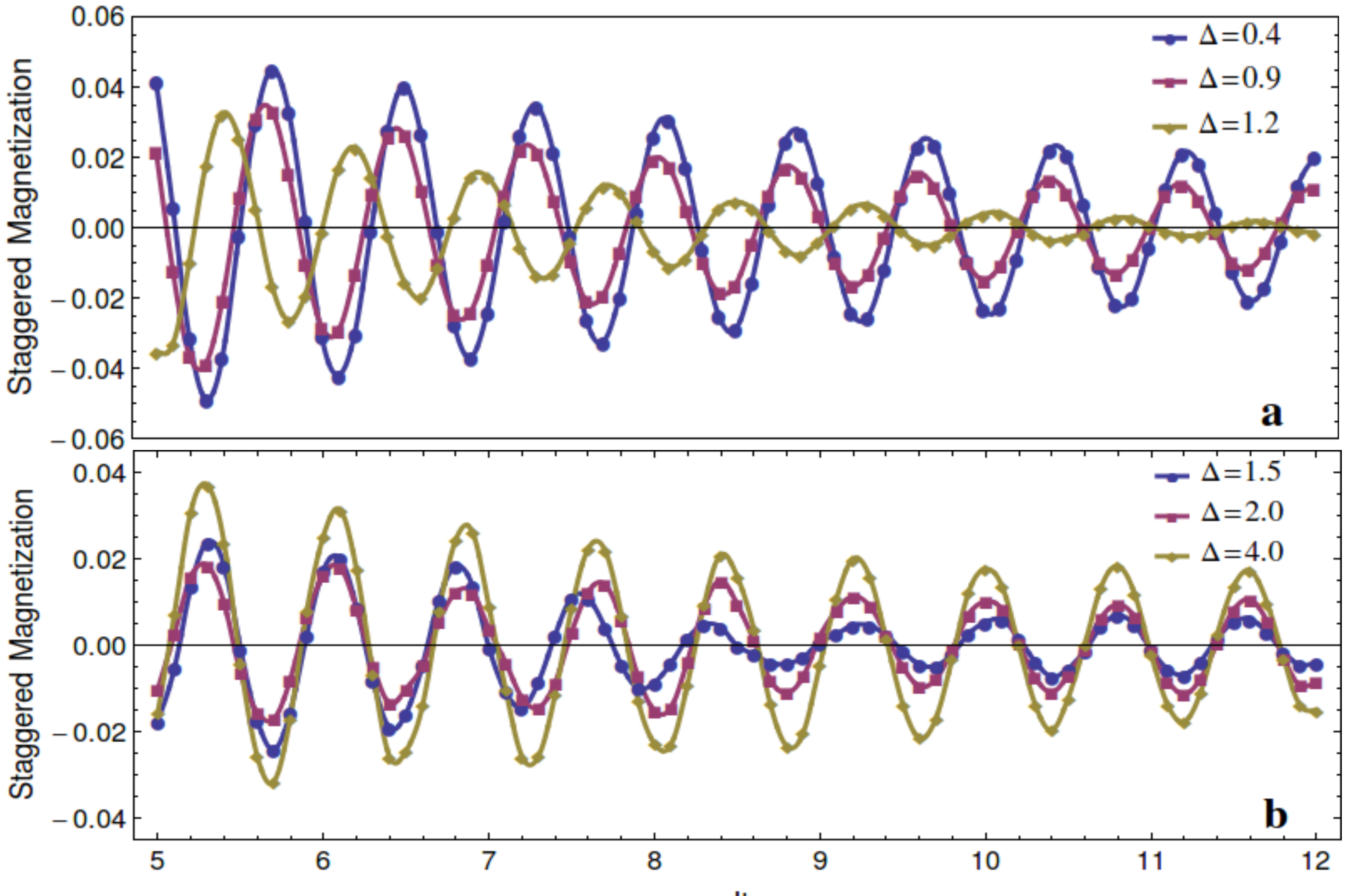


- *Staggerd Magnetization (Order Parameter)* $M_s(t) = \frac{1}{N}\sum_n (-1)^n \langle \sigma_n^z \rangle (t)$

*Quench across a QCP* $\Delta = \infty \rightarrow |\Delta| < 1$

$|\Psi_0\rangle = \sigma_1^+ \sigma_0^+ \sigma_{-2}^+ |\Downarrow\rangle =$

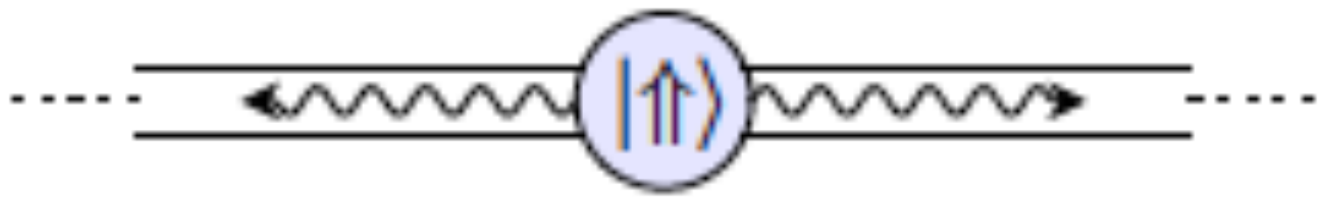


# Evolution of Super-radiance (Dicke model)

*M* 2-level atoms located at x=0 in a waveguide: $s_i^{\pm},\ i = 1 \ldots M$

$$H_{\rm nc} = \int \left( i b_L^\dagger(x) \partial_x b_L(x) - i b_R^\dagger(x) \partial_x b_R(x) \right) - \sqrt{c/2} \left( S^+ (b_L(0) + b_R(0)) + S^- (b_L^\dagger(0) + b_R^\dagger(0) \right)$$



Unfold*:*

*Jaynes-Cummings, Tavis-Cummings model*

$$H = -i \int dx\, b^\dagger(x) \partial_x b(x) - \sqrt{c} \left( S^+ b(0) + S^- b^\dagger(0) \right).$$

$$S^{\pm} = \sum_{i=1}^{M} s_i^{\pm}$$

Prepare system in an excited state *e.g.* Excite $N \leq M$ atoms, no photons

$$|\Phi_0\rangle = \left( \frac{(M-N)!}{M!N!} \right)^{1/2} (S^+)^N |0\rangle$$

# Evolution of Super-radiance (Dicke model)

Time evolution of the photon current:

$$\langle j(z)\rangle_t = \langle \Phi_0 | e^{-iHt} \rho(z) e^{iHt} | \Phi_0 \rangle \quad \textit{with } \rho(z) = b^\dagger(z) b(z)$$

Time evolution of photon number

$$\left\langle \hat{N}_{\mathrm{p}} \right\rangle_t = \int dz \, \langle \rho(z) \rangle_t$$

Expand in eigenstates (Rupasov and Yudson) and use Yudson representation

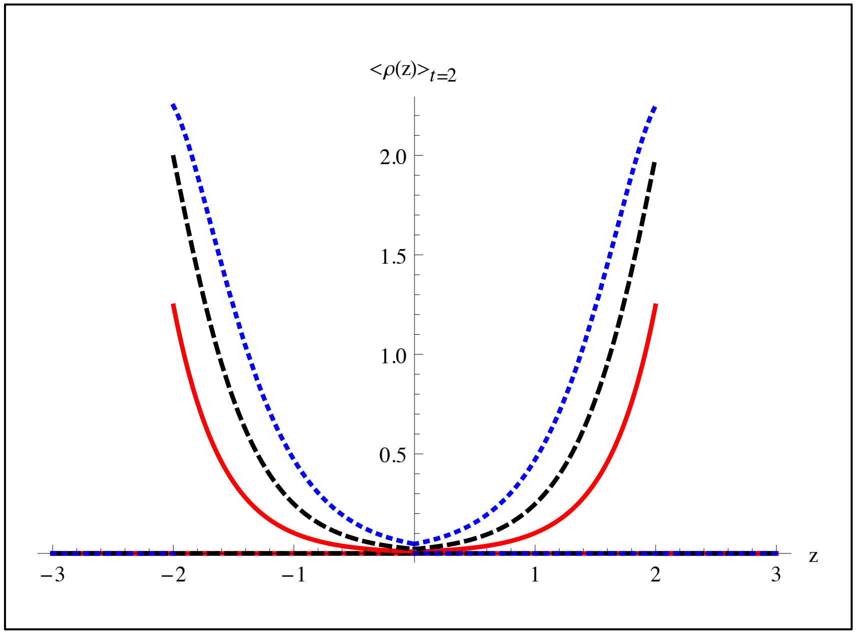
$$|\vec{\lambda}\rangle = \frac{1}{(2\pi)^{\frac{N}{2}} \, N!^{\frac{1}{2}}} \int d^N x \prod_{i<j} \left( 1 - \frac{2ic\theta(x_i - x_j)}{\lambda_i - \lambda_j + ic} \right) \prod_{j=1}^{N} e^{i\lambda_j x_j} f(\lambda_j, x_j) r^\dagger(\lambda_j, x_j) |0\rangle$$

*with*

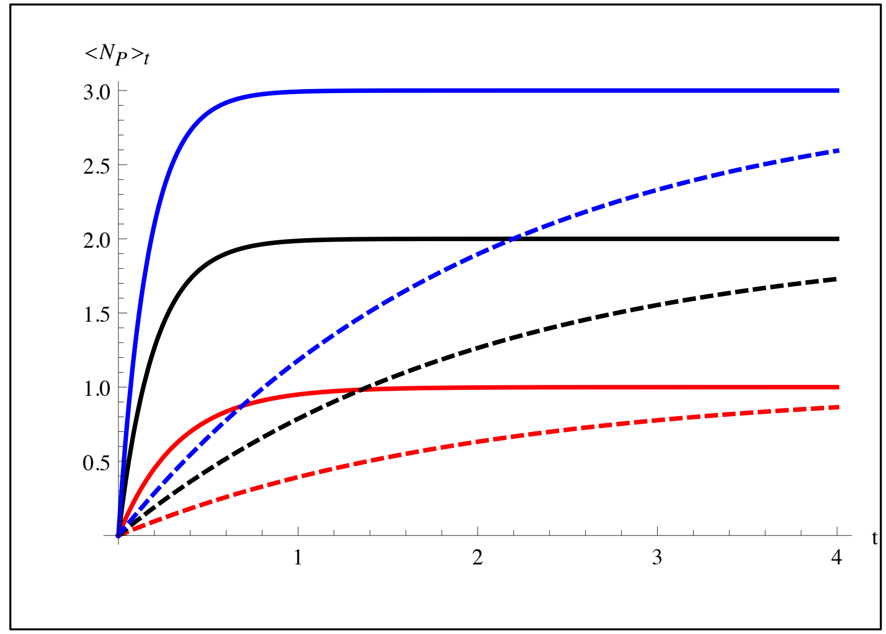
$$r^\dagger(\lambda_j, x_j) = b^\dagger(x_j) - \frac{\sqrt{c}}{\lambda_j} S^+$$ *photon-atom creation operator*

$$f(\lambda_j, x_j) = \frac{\lambda_j - icM/2 \, \mathrm{sgn}(x_j)}{\lambda_j + icM/2}$$ *photon-atom scattering*

# Evolution of Super-radiance (Dicke model)




$\langle j(z, t=2) \rangle$ at t = 2 for
M = 6, N = 3 (blue, dotted),
N = 2 (dashed) and N = 1 (red)
for the non-chiral model




The total photon number $\langle N_p \rangle_t$
for c = 1, M = 6, N = 3 (blue), N = 2
(black) and N = 1
also corresponding expressions
ignoring cooperative eects,

Note:

$$\partial_t \left\langle \hat{N}_{\mathrm{p}} \right\rangle_t = cN^2(1+M-N)e^{-cN(1+M-N)t}$$

Dicke cooperative effect: decay rate $\sim N^2$ (rather than $\sim N$ for incoherent decay)

# Evolution of a bosonic system

## Conclusions:

- **Time evolution at *infinite* volume; no need for spectrum of Hamiltonian or overlaps**
  - **Takes into account existence of bound states w/o sums over strings**
- **Time evolution at *finite* volume, finite density (need spectrum, no need for overlaps)**
- **Asymptotics calculable for all coupling regimes, for all initial states** (asymptotic equilibrium or not)

## To do list:

- **Generalize to other integrable models:**

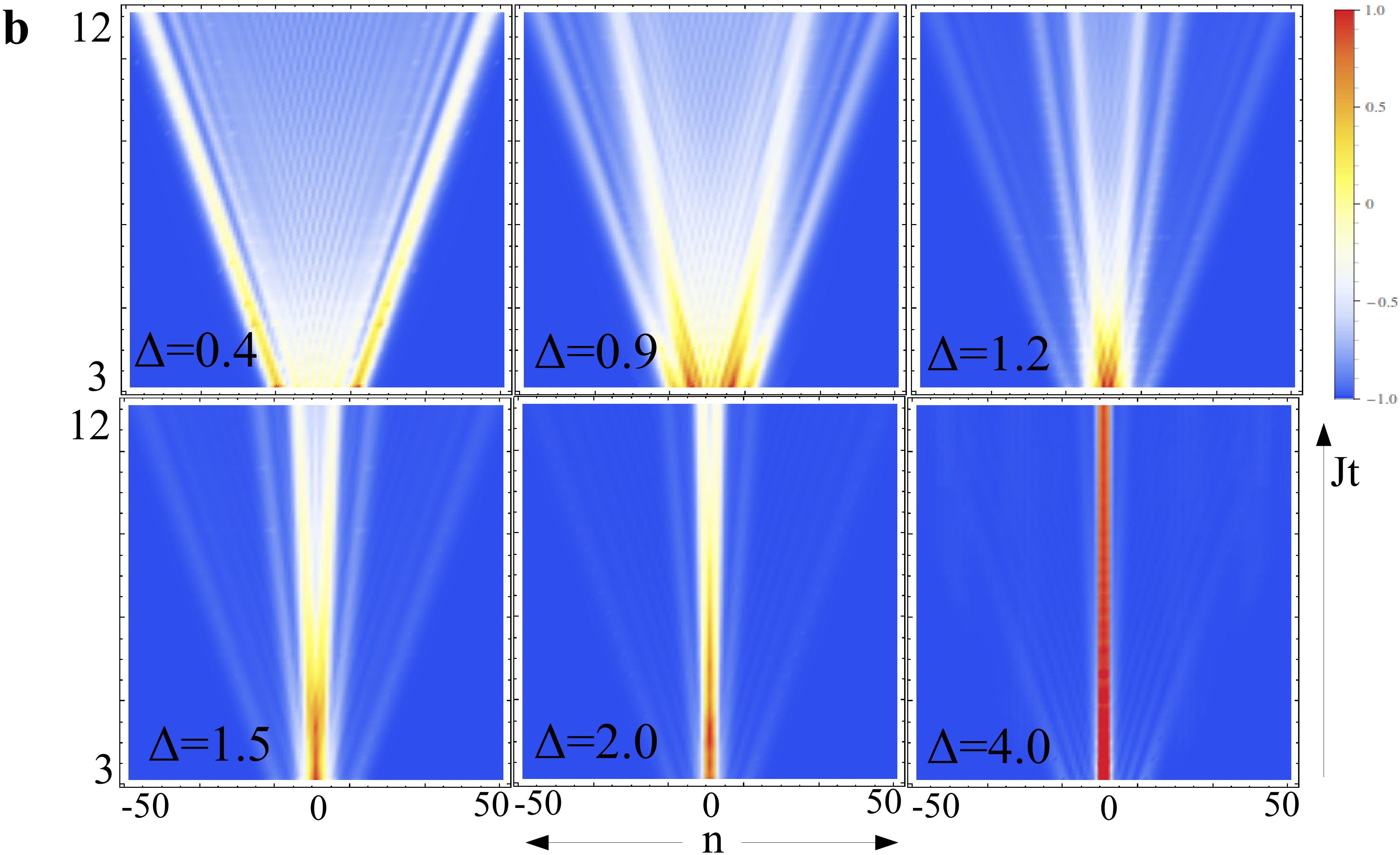
*Anderson model* **(Adrian Culver),** *Lieb-Liniger + impurity, Gaudin-Yang* **(Huijie Guan),** *Sine-Gordon model* **( Roshan Tourani, Garry Goldstein),** *Kondo Model* **(Yuval Vinkler),** *Richardson model* **(Garry Goldstein, Emil Yuzbashyan),** *Hubbard Model* **(Huijie Guan*)***

- **Time evolution at finite temperatures (under discussion)**
- **Approach to nonequilibrium steady state (in progress, with Adrian Culver, Yuval Vinkler)**
- **Numerical tests of *dynamic RG hypothesis* (in progress, t-DMRG)**
- **Correlation functions (Garry Goldstein)**

## Big Questions:

- **What drives thermalization of pure states? Canonical typicality, entanglement entropy (Lebowitz, Tasaki, Short…)**
- **General principles, variational? F-D theorem out-of-equilibrium? Heating? Entanglement?**
- **What is universal? RG Classification?**

# Evolution of the XXZ magnet




*Propagation of free and bound magnons for different Δ, starting from an initial state* $|\Psi_0\rangle = \sigma+\sigma+\sigma+ |\Downarrow\rangle$.

# Evolution of the XXZ magnet

# Evolution of a bosonic system

## Long time asymptotics:

- **General expression – repulsive**

$$|\Phi_0, t\rangle = \int_x \int_y \theta(\vec{x})\Phi_0(\vec{x}) \prod_{i<j} \frac{\xi_i - \xi_j - ic\, sgn(\xi_i - \xi_j)}{\xi_i - \xi_j - ic} \prod_j \frac{1}{\sqrt{4\pi i t}} e^{i\xi_j^2 t - i\xi_j x_j} b^\dagger(y_j)|0\rangle$$

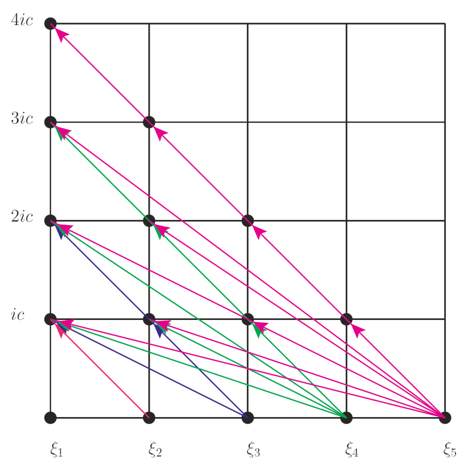
**function of** $\xi = y/2t$ **only, light-like propagation**

*Exp: Bloch et al*
*Nature 2012*

- **General expression – attractive (poles and bound states)**

$$|\Phi_0, t\rangle = \int_x \int_y \theta(\vec{x})\Phi_0(\vec{x}) \sum_{\xi_j^* = \xi_j, \xi_i^* + ic, i<j} \prod_{i<j} \frac{\xi_i^* - \xi_j^* + ic\, sgn(\xi_i - \xi_j)}{\xi_i^* - \xi_j^* + ic} \prod_j \frac{1}{\sqrt{4\pi i t}} e^{-i(\xi_j^*)^2 t + i\xi_j^*(2t\xi_j - x_j)} b^\dagger(y_j)|0\rangle$$

**Pole contributions follow recursive pattern:**




**Pattern corresponds to successive formation and contributions of**
**bound states**

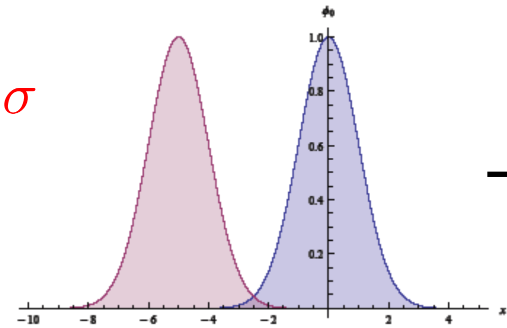
# Evolution of a bosonic system: density

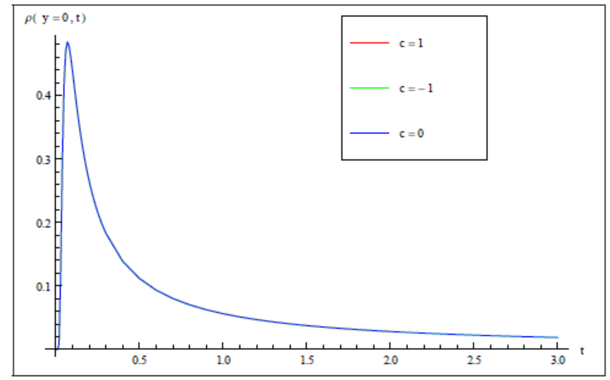
**Density evolution:**
(Time of flight experiment)

$$\langle \rho(x_0,t)\rangle = \langle \Phi_0(t)|b^\dagger(x_0)b(x_0)|\Phi_0(t)\rangle$$

- ***Two bosons***

*i.* **Initial condition:** $a \gg \sigma$



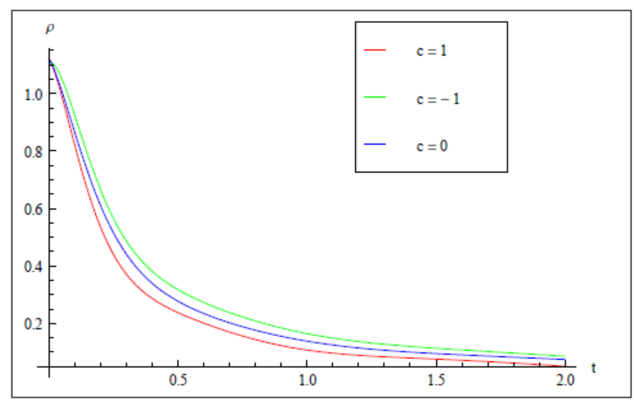




$$|\Phi_0\rangle = \frac{1}{(\pi\sigma^2)^{\frac{1}{2}}}\int_x e^{-\frac{(x_1)^2}{2\sigma^2}} e^{-\frac{(x_2+a)^2}{2\sigma^2}} b^\dagger(x_1)b^\dagger(x_2)|0\rangle$$

$$\rightarrow\ |\Phi_0(t)\rangle_2 = \int_y \int_c \Phi(x_1,x_2)\frac{e^{i\frac{(y_1-x_1)^2}{4t}+i\frac{(y_2-x_2)^2}{4t}}}{4\pi i t}\left(1 - c\sqrt{\pi i t}\,\theta(y_2-y_1)e^{\frac{i}{8t}\alpha^2(t)}\mathrm{erfc}\left(\frac{i-1}{4}\frac{i\alpha(t)}{\sqrt{t}}\right)\right) b^\dagger(y_1)b^\dagger(y_1)|0\rangle$$

*with* $\alpha(t) = 2ct - i(y_1-x_1) - i(y_2-x_2)$

*ii.* **Initial condition:** $a \ll \sigma$




*Conclusion: Strong dependence on initial state*

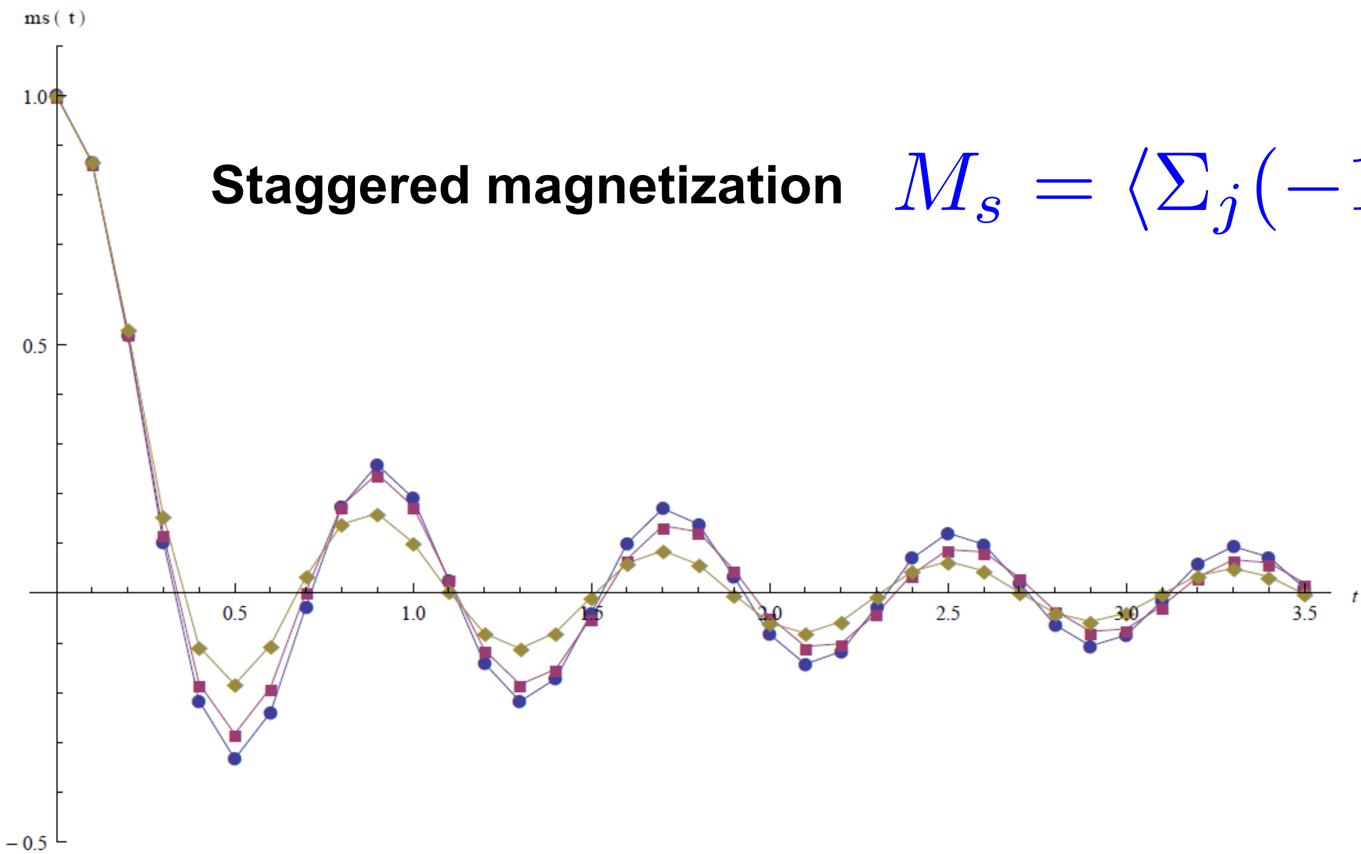
- ***Many bosons*** **:**
  **physical regimes and time scales, asymptotics**

# Evolution of the XXZ magnet

*Quench across a QCP: evolution of an order parameter*

$$\Delta = \infty \rightarrow |\Delta| < 1$$

**Staggered magnetization** $M_s = \langle \Sigma_j (-1)^j \sigma_j^z \rangle$




*cf.* Altman et al. '10

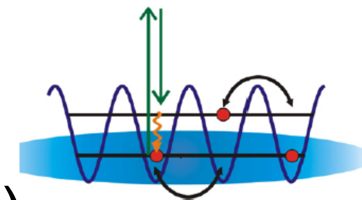
Staggered magnetization as a function of time for $\Delta = 0.01, 0.54, 0.99$

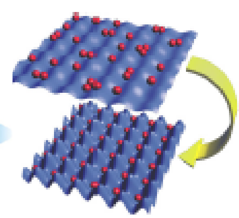
# Quenching and Time Evolution

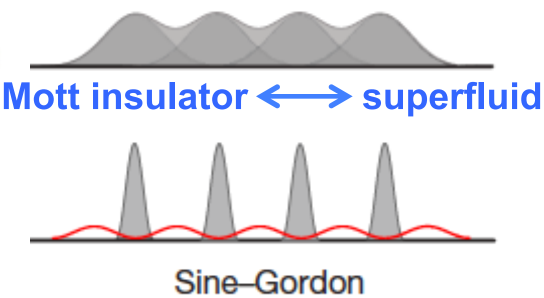
$|\Phi_0\rangle$ $H_0$

**- Many experiments: cold atom systems, nano-devices, molecular electronics, photonics**

**- New technologies, old questions**







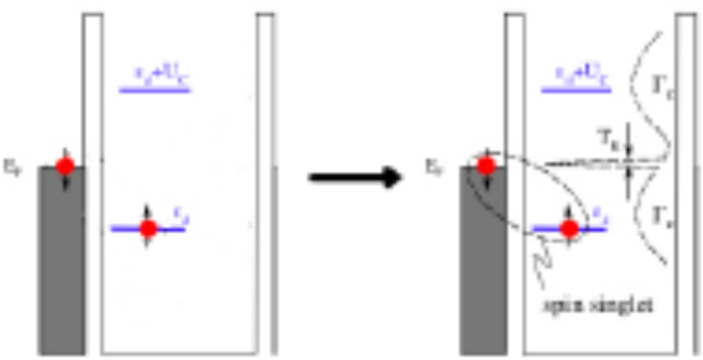

**Questions:** (as an introduction)

- **Time evolution of observables** $\langle O(t)\rangle = \langle \Phi_0, t|O|\Phi_0, t)\rangle$
- **Evolution of correlation functions in quenched systems**

$$\langle \Phi_0|A(t+\tau)B(t)|\Phi_0\rangle = \langle \Phi_0, t)|A(\tau)B|\Phi_0, t\rangle$$

**- Dynamics of evolution of the Kondo resonance in a quantum dot: Anderson model**

**Quench at t = 0 : couple dot to leads**



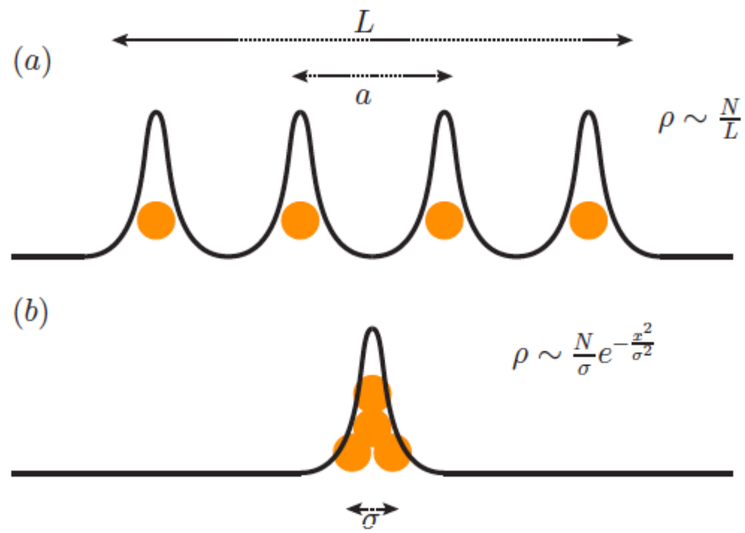

*DOS*

Measure time evolution of the Kondo peak.

- Time resolved photo emission spectroscopy

# Time scales in the Bosonic system

**The system**



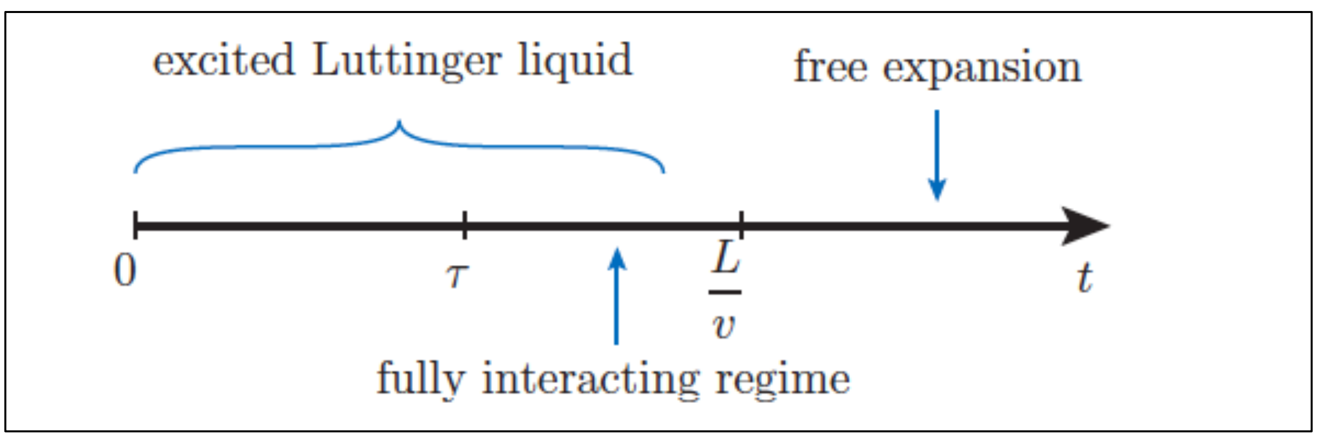

**Time scales:**

*Thermodynamic regime - constant density* $t \ll \frac{L}{v}$

$t \gg \frac{L}{v}$ *Expansion of interacting gas - density decreases*

*Interaction time*

$$\tau \sim \frac{1}{c^2}$$



# The Bethe Ansatz - Review

***Example:*** *the Bethe Ansatz wave functions (for the Lieb-Liniger model)*

$$F^{\lambda_1..\lambda_N}(x_1..x_N) = \prod_{i<J} Z^y_{ij} \prod_j e^{i\lambda_j x_j}$$

*with*

$$Z^y_{ij}(\lambda_i - \lambda_j) = \frac{\lambda_i - \lambda_j - ic\, sgn(y_i - y_j)}{\lambda_i - \lambda_j - ic} = \begin{cases} 1 \\ S^{ij} \end{cases}$$

*S-matrix enters when bosons cross*

*The wave functions satisfy:*

$$H\, F^{\lambda_1..\lambda_N}(x_1..x_N) = \sum_{J=1}^{N} \lambda_j^2 \; F^{\lambda_1..\lambda_N}(x_1..x_N)$$

**vi**. *impose periodic boundary conditions (for thermodynamics)*

$$e^{i\lambda_j L} \prod_{l\neq j} \frac{\lambda_j - \lambda_l - ic}{\lambda_j - \lambda_l + ic} = 1,\; j = 1..N \longrightarrow \{\lambda_j\}$$ *solutions*

**vii.** *Obtain all eigenstates, eigenvalues* $\longrightarrow$ *thermodynamics* $Z = \sum_\alpha e^{-\beta E_\alpha}$

# Keldysh

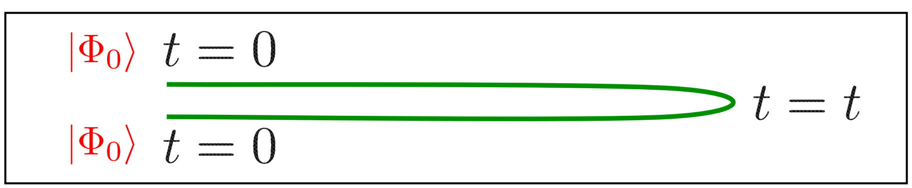
- **Time evolution of expectation values:**

$$O_{\Phi_0}(t) = \langle \Phi_0 | e^{iHt}\, \hat{O}\, e^{-iHt} | \Phi_0 \rangle = \langle \Phi_0, t |\, \hat{O}\, | \Phi_0, t \rangle$$

*Non-perturbative Keldysh:*

$$= \int \mathcal{D}\, b^* \mathcal{D}\, b\, \hat{O}\; e^{-i \int_C [S_0(b,b^*) + S_I(b,b^*)]\, dt}$$

*carried out on the Keldysh contour* $C$ *, with separate fields for the top and bottom lines:*



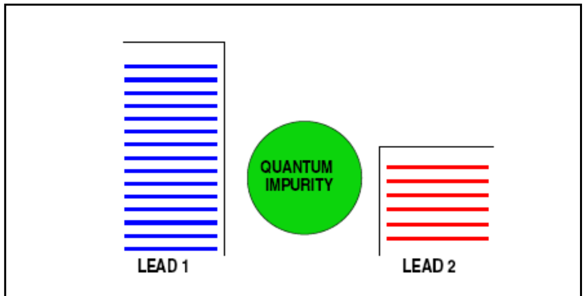
# Breaking Translation Invariance: quenching and non-thermalization

## Nonequilibrium currents

Goldhaber-Gordon *et al*, Conenwett *et al*, Schmid *et al*

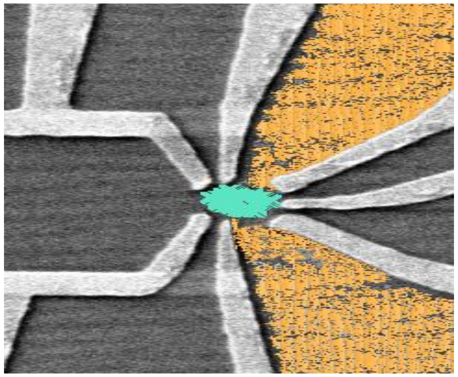
- **Two baths or more**

time evolution in a nonequilibrium set up



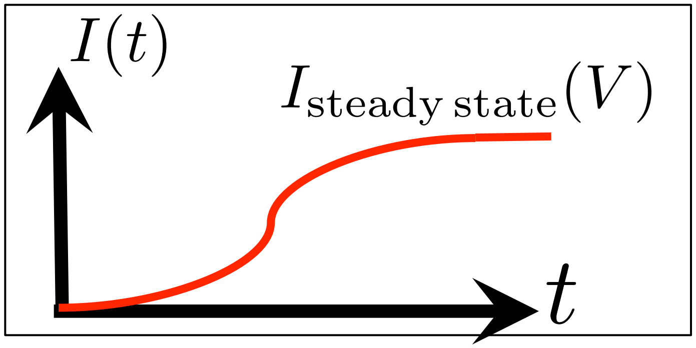

$$H = H_{leads} + H_{imp} + H_{leads-imp}$$

**Quench**



- $t \leq 0$, leads decoupled, system described by: $\rho_o$
- $t = 0$, couple leads to impurity
- $t \geq 0$, evolve with $H(t) = H_0 + H_1$

Interplay - strong correlations and nonequilibrium




- **What is the time evolution of the current $\langle I(t) \rangle$ ?**
- **Long time limit:**
  - **Under what conditions is there a steady state? Dissipation mechanism?**
  - **Steady state – is there a non thermal $\rho_s$ ? Voltage dependence?**
  - **New effects out of equilibrium? New scales? Phase transitions, universality?**

# Quenching in 1-d systems

## Physical Motivation:

- **Natural dimensionality of many systems:**

  **- wires, waveguides, optical traps, edges**

- **Impurities: Dynamics dominated by s-waves, reduces to 1D system**
- **Many experimental realizations: ultracold atom traps, nano-systems..**

## Special features of 1- d : theoretical

- **Strong quantum fluctuations for any coupling strength**
- **Powerful mathematical methods:**

  **- RG methods, Bosonization, CFT methods, Bethe Ansatz approach**

**- Bethe Ansatz approach: allows complete diagonalization of *H***

**- Experimentally realizable:** *Hubbard model, Kondo model, Anderson model, Lieb-Linniger model, Sine-Gordon model, Heisenberg model, Richardson model..*

**- BA $\longrightarrow$ Quench dynamics of many body systems? Exact!**

*Others approaches: Keldysh, t-DMRG, t-NRG, t-RG*

*Much work in context of Luttinger Liquid: Cazalilla et al, Mitra et al*

# The Bethe Ansatz - Review

- **General $N$ - particle state**

$$|F^\lambda\rangle_N = \int d^N x \, F^\lambda(\vec{x}) \prod_{j=1}^{N} \psi^\dagger(x_j)|0\rangle$$

- **Eigenfunctions very complicated in general**
- **The BA - wave function much simpler:**

**Product of single particles wave functions $f_\lambda(x)$ and S-matrices $S_{ij}$ ,**

*Example:*

$$H = -\sum_{j=1}^{N} \partial_{x_j}^2 + c \sum_{i<j} \delta(x_i - x_j)$$

$$f_\lambda(x) \sim e^{i\lambda x}$$

$$S_{ij}(\lambda_i - \lambda_j) = \frac{\lambda_i - \lambda_j + ic}{\lambda_i - \lambda_j - ic}$$

**i.** divide configuration space into N! regions $Q, \{x_{Q1} \leq \cdots, \leq x_{QN}\}$

**ii** particles interact only when crossing: inside a region product of single particle wave funct.

**iii.** assign amplitude $A^Q$ to region $Q$

**iv.** amplitudes related by S-matrices $S_{ij}$ (*e.g.* $A^{132} = S^{23} \, A^{123}$)

$\rightarrow$ $\boxed{F^\lambda(\vec{x}) = \sum_{Q \in S_N} A^Q \prod_j f_{\lambda_{Qj}}(x_j)}$

**v.** do it consistently: ***Yang-Baxter relation***

$$S^{12} \, S^{13} \, S^{23} = S^{23} \, S^{13} \, S^{12}$$