# Stability of 2D Topological Insulators

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## Outline

- Chiral and non-chiral Topological States of Matter
- Stability of edge states of 2D Topological Insulators: $\mathbb{Z}_2$ anomaly
- Partition function: electromagnetic and gravitational (discrete) responses
- Topological Insulators with interacting & non-Abelian edges

# Topological States of Matter

- System with bulk gap but non-trivial at energies below the gap
- global effects and global degrees of freedom (edge states, g.s. degeneracy)
- described by topological field theory: Chern-Simons theory etc.
- quantum Hall effect is chiral (B field, chiral edge states)
- quantum spin Hall effect is non-chiral (edge states of both chiralities)
- other systems: Chern Insulators, Topological Insulators,
  Topological Superconductors, etc.
- Topological Band Insulators (free fermions) have been observed in 2 & 3 D

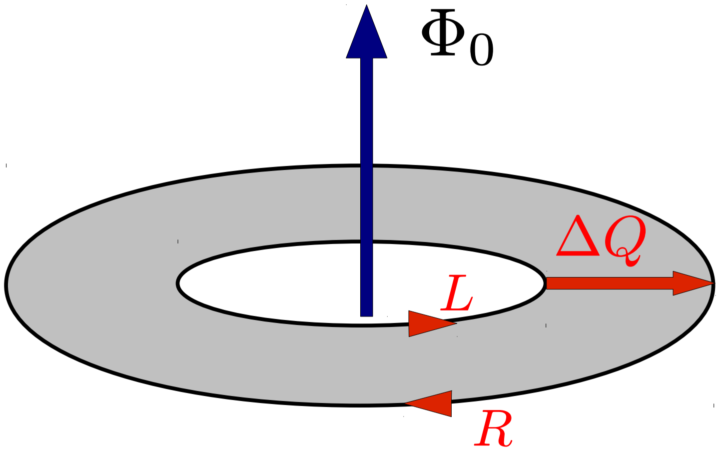
→ « excitement »

# Questions/Answers

- Q: systems of interacting fermions? (e.g. fractional topological insulators)
- A: use quantum Hall modeling and CFTs
- Q: but non-chiral edge states are stable?
- A: generically NO
- Q: stability with Time Reversal symmetry?
- A: YES/NO; there is a $\mathbb{Z}_2$ symmetry; if this is anomalous, they are stable

# Chiral Topological States

## Quantum Hall effect




- chiral edge states
- no Time-Reversal symmetry (TR)
- Laughlin's argument: $\nu = \frac{1}{3}$

$$\Phi \to \Phi + \Phi_0, \quad H\left[\Phi + \Phi_0\right] = H\left[\Phi\right]$$

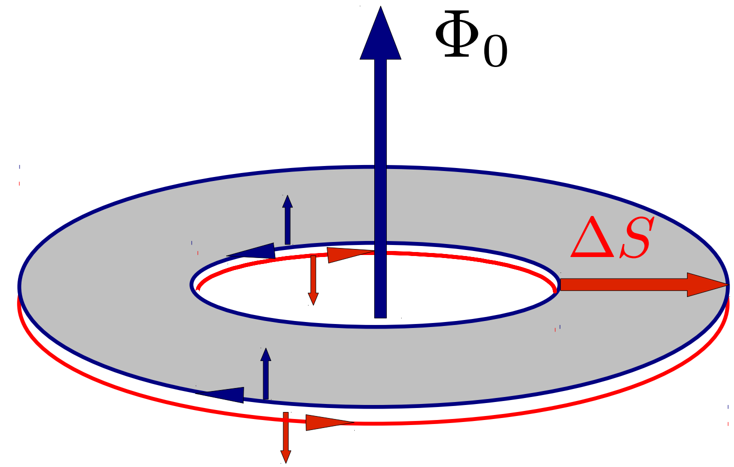
$$Q_R \to Q_R + \Delta Q = \nu, \qquad \Delta Q = \int dtdx\ \partial_t J_R^0 = \nu \int F = \nu\, n$$ chiral anomaly

- $\Phi \to \Phi + n\Phi_0$ spectral flow between charge sectors $\{0\} \to \{\frac{1}{3}\} \to \{\frac{2}{3}\} \to \{0\}$
- edge chiral anomaly = response of topological bulk to e.m. background
- chiral edge states cannot be gapped ⟷ topological phase is stable
- anomalous response extended to other systems and anomalies in any D=1,2,3,......

(S. Ryu, J. Moore, A. Ludwig '10)

# Non-chiral topological states

## Quantum Spin Hall Effect

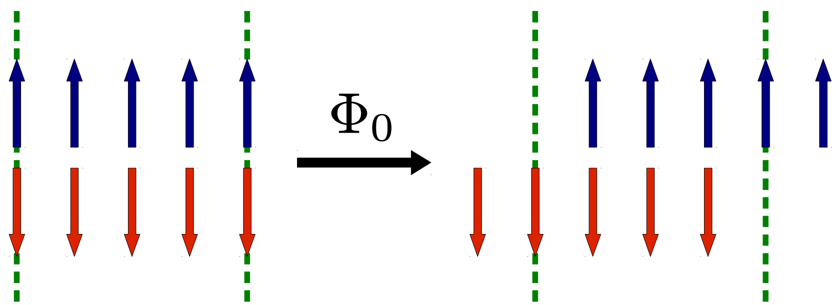



- take two $\nu = 1$ Hall states of spins ↑↓
- system is Time Reversal invariant:

$$\mathcal{T}: \ \psi_{k\uparrow} \to \psi_{-k\downarrow} \ , \qquad \psi_{k\downarrow} \to -\psi_{-k\uparrow}$$

- non-chiral CFT with $U(1)_Q \times U(1)_S$ symmetry
- adding flux pumps spin → $U(1)_S$ anomaly

(X-L Qi, S-C Zhang '08)




$$\Delta Q = \Delta Q^{\uparrow} + \Delta Q^{\downarrow} = 1 - 1 = 0$$

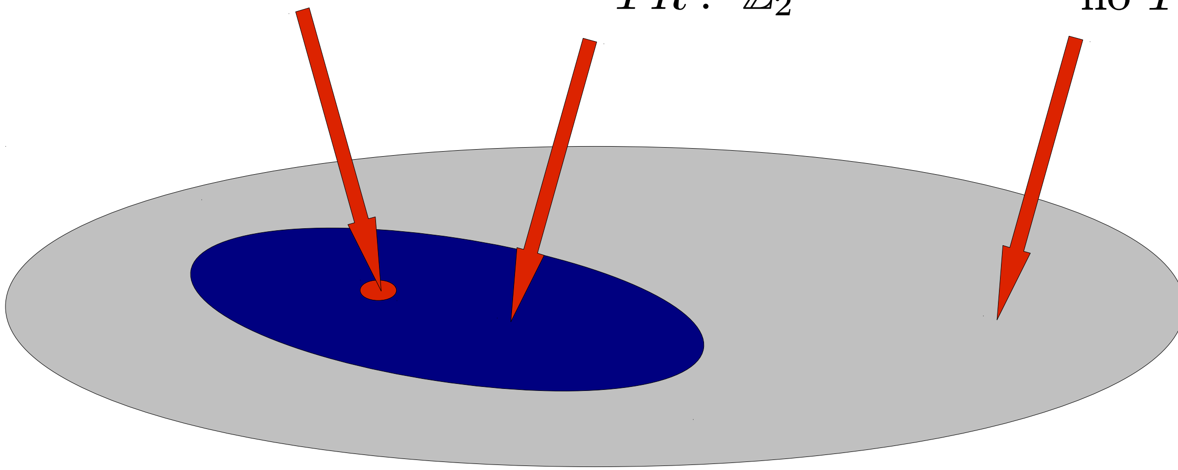
$$\Delta S = \tfrac{1}{2} - \left(-\tfrac{1}{2}\right) = 1$$

$$\Delta S = \Delta Q^{\uparrow} = \nu^{\uparrow} = 1$$

- in Topological Insulators $U(1)_S$ is explicity broken by Spin-Orbit Coupling etc.
- no currents $\sigma_H = \sigma_{sH} = \kappa_H = 0$
- but TR symmetry keeps $\mathbb{Z}_2$ symmetry $(-1)^{2S}$

Kramers theorem

# Symmetry Protected Topological Phases




- QSHE is used to describe Topological Insulators with Time-Reversal symmetry but no spin symmetry: $U(1)_S \ \rightarrow \ \mathbb{Z}_2$ of $(-1)^{2S}$

  Main issue: stability of TI ⟷ stability of non-chiral edge states

- e.g. TR symmetry forbids mass term in CFT with odd number of free fermions

$$\mathcal{T} : H_{\text{int.}} = m \int \psi_\uparrow^\dagger \psi_\downarrow + h.c. \ \rightarrow \ -H_{\text{int.}}$$

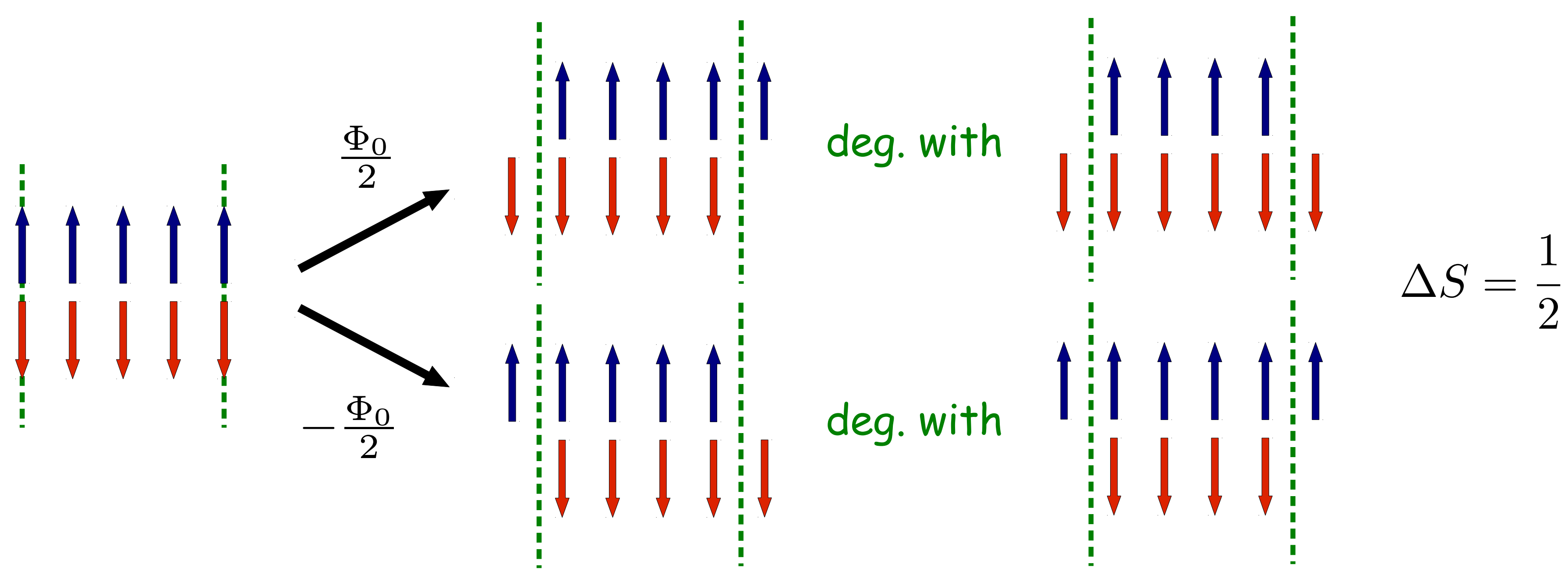
$\mathbb{Z}_2$ classification (free fermions)

# Flux insertion argument

(Fu, Kane, Mele '05-06; Levin, Stern '10-13)

- TR symmetry: $\mathcal{T} H[\Phi] \mathcal{T}^{-1} = H[-\Phi]$ & $H[\Phi + \Phi_0] = H[\Phi]$
- TR invariant points: $\Phi = 0, \frac{\Phi_0}{2}, \Phi_0, \frac{3\Phi_0}{2}, \ldots$
- Kane et al. define a TR-invariant $\mathbb{Z}_2$ polarization (bulk quantity) that:

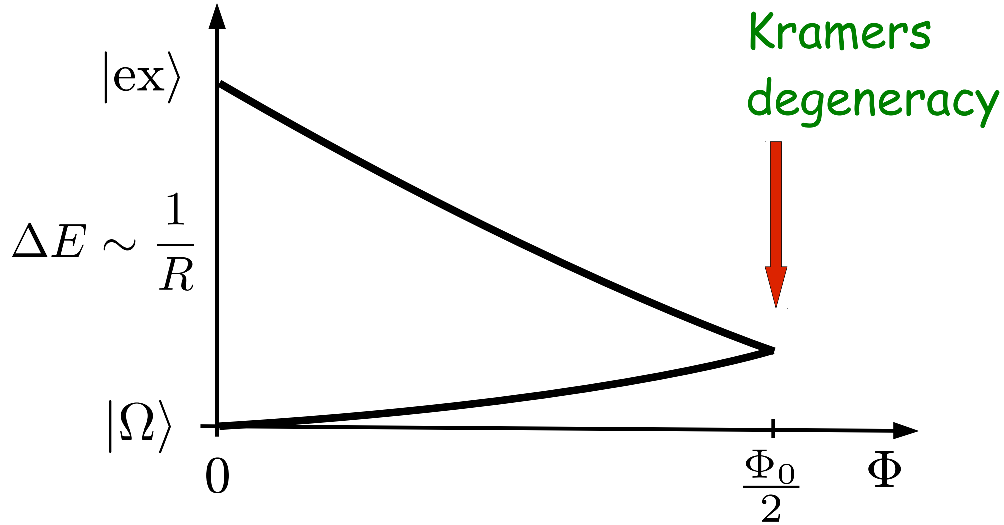
  - is topological and conserved by TR invariance
  - is equal to parity of edge spin $(-1)^{2S} = (-1)^{N_\uparrow + N_\downarrow}$
  - if $(-1)^{2S} = -1$ there exits a pair of edge states degenerate by Kramers theorem



$$\Phi = 0 : \quad (-1)^{2S}|\Omega\rangle = |\Omega\rangle$$

$$\Phi = \frac{\Phi_0}{2} : \; (-1)^{2S}|\Omega\rangle = -|\Omega\rangle$$

$|\mathrm{ex}\rangle$ *gapless edge state*



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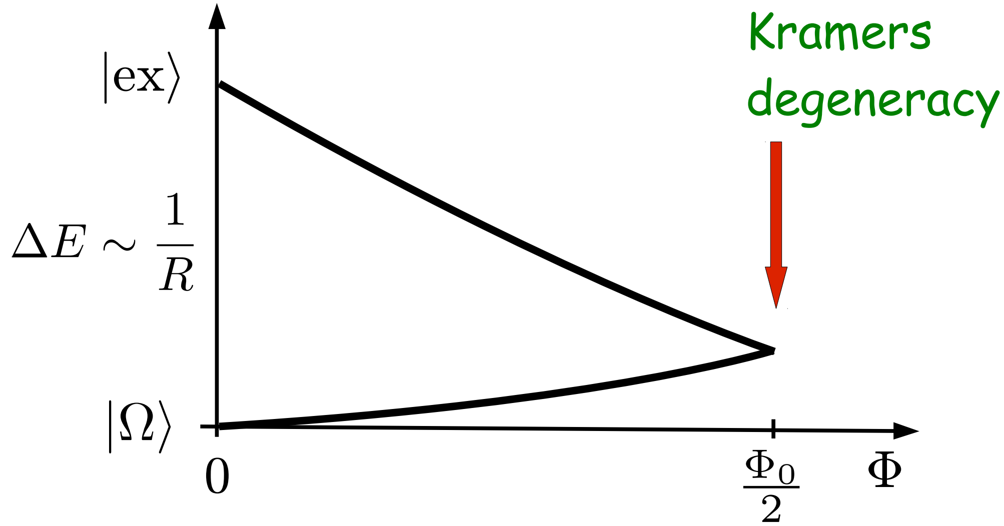


## Conclusions

- topological phase is protected by TR symmetry if $\exists$ edge Kramers pair ( $N_f$ odd)
- spin parity is anomalous, discrete remnant of spin anomaly $U(1)_S \to \mathbb{Z}_2$

Fu-Kane argument is Laughlin's argument for $\mathbb{Z}_2$ anomaly: $(-1)^{2\Delta S} = -1$

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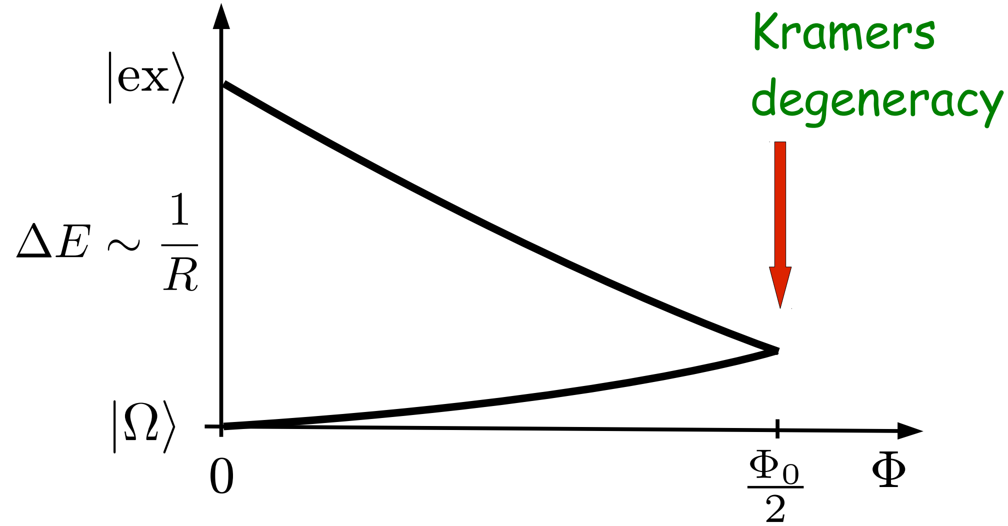


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Question: Can we extend this argument to interacting fermions?

# Answer: Yes

## Strategy:

- Study partition functions of TI (& QSHE) using known general results for QHE

(AC, Zemba '97; AC, Georgiev, Todorov '01; AC, Viola '10)

- Use them to analyze flux insertions and repeat stability argument

→ $\mathbb{Z}_2$ classification extends to interacting & non-Abelian edges

$$(-1)^{2\Delta S} = +1 \qquad \text{unstable}$$
$$-1 \qquad \text{stable}$$

$$2\Delta S = \frac{\sigma_{sH}}{e^*} = \frac{\nu^{\uparrow}}{e^*}$$

$\frac{\text{spin-Hall conduct.} = \text{chiral Hall conduct.}}{\text{minimal fractional charge}}$

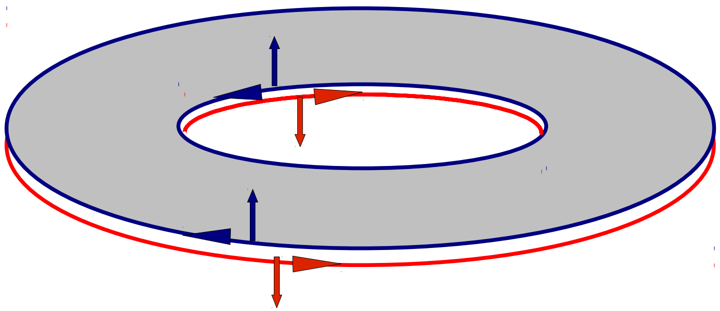
(Levin, Stern, '09, '12)

- Stability, i.e. $\mathbb{Z}_2$ anomaly, is associated to a discrete gravitational anomaly, i.e. to the lack of modular invariance of partition function (S. Ryu, S.-C. Zhang '12)

- As a backup: study time-reversal invariant edge interactions → same result

(Neupert et al. '11; AC, Randellini '14)

# Partition Function of Topological Insulators



- Grand-canonical partition function of a single edge, combining the two chiralities

- Four sectors of fermionic systems

$$NS,\ \widetilde{NS},\ R,\ \widetilde{R},\quad \text{risp.}\quad (AA),\quad (AP),\quad (PA),\quad (PP)$$

- Neveu-Schwarz sector describes ground state and integer flux insertions:

$$Z^{NS}(\tau,\zeta) = Z^{NS}(\tau,\zeta+\tau),\qquad V:\zeta\to\zeta+\tau \quad \text{adds a flux} \quad \Phi\to\Phi+\Phi_0,$$

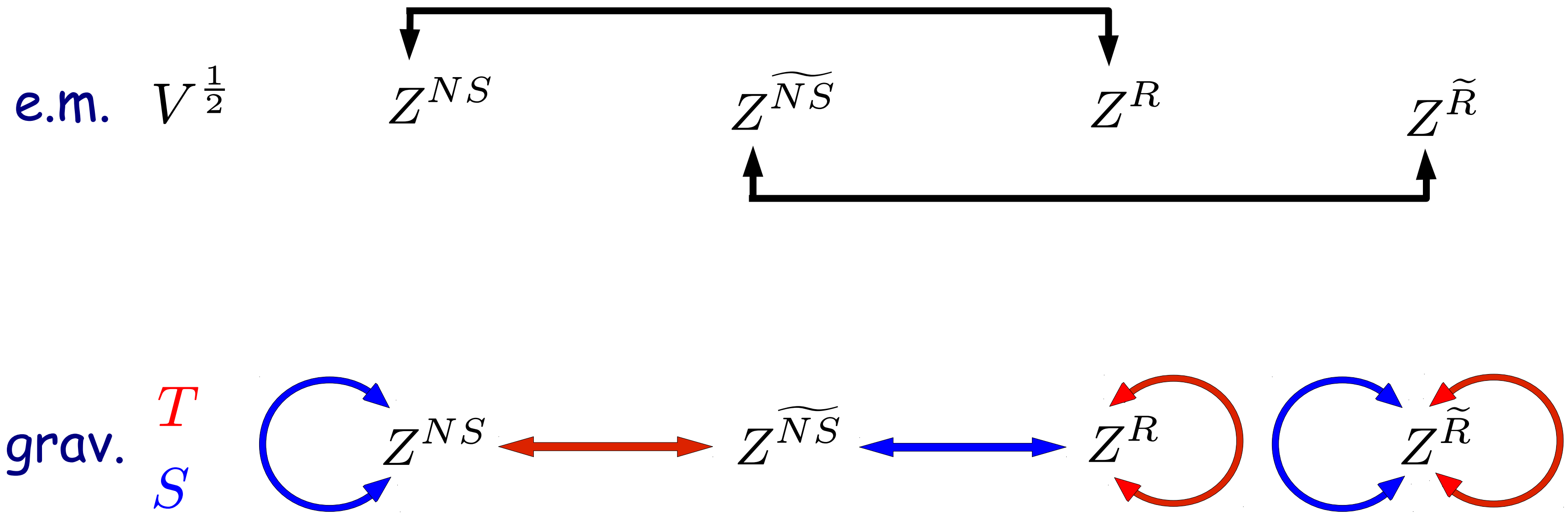
$$\tau = i\beta/L,\qquad \zeta=\beta(iV_o+\mu)$$

- Ramond sector describes half-flux insertions: $\frac{\Phi_0}{2},\frac{3\Phi_0}{2},\ldots$

$$V^{\frac{1}{2}}: Z^{NS}(\tau,\zeta)\ \to\ Z^{NS}\left(\tau,\zeta+\frac{\tau}{2}\right) = Z^{R}(\zeta,\tau)$$

(Remark: each sector contains fractional charges, that are not relevant for the argument)

# E.m. & gravitational responses




$$V : \zeta \to \zeta + \tau \qquad T : \tau \to \tau + 1 \qquad S : \tau \to -\frac{1}{\tau}$$

Flux addition Modular transformations

# Stability and modular non-invariance

- Flux argument: add half fluxes and check $(-1)^{2\Delta S} = \pm 1$

$$(-1)^{2S} |\Omega\rangle_{NS} = |\Omega\rangle_{NS} \quad \longrightarrow \quad (-1)^{2S} |\Omega\rangle_{R} = - |\Omega\rangle_{R}$$ ➡ Kramers pair

- Spin parities of Neveu-Schwarz and Ramond ground states are different

➡ $\mathbb{Z}_2$ spin-parity anomaly

$$Z = \left( Z^{NS}, Z^{\widetilde{NS}}, Z^{R}, Z^{\widetilde{R}} \right)$$ stable Topological Insulator

- Conversely, if $(-1)^{2\Delta S} = 1$ , i.e. no anomaly:

$$Z = Z^{NS} + Z^{\widetilde{NS}} + Z^{R} + Z^{\widetilde{R}}$$ unstable

TR symmetry + anomaly ➡ no modular invariance ➡ stable insulator

TR symmetry + modular invariance ➡ no anomaly ➡ trivial insulator

# General stability index

- Edge theory involves neutral excitations (possibly non-Abelian)
- fractional charge sectors always parametrized by two integers $(k, p)$
- minimal charge: $e^* = 1/p$
- Hall current: $\Delta Q = \nu^\uparrow = k/p$
- Add fluxes to create an electron excitation in the same charge sector:

$$V^{\frac{p}{2}} : \quad \Delta S = \Delta Q^\uparrow = \frac{p}{2}\,\nu^\uparrow = \frac{k}{2}$$

Kramers pair if $k$ odd → stable TI

| | | | |
|---|---|---|---|
| Levin-Stern index | $2\Delta S = \dfrac{\nu^\uparrow}{e^*},$ | $(-1)^{2\Delta S} = (-1)^k$ | fully general |

Remark: (non-Abelian) neutral states are left invariant by flux insertions

# Examples

Levin-Stern index $2\Delta S = \frac{\nu^{\uparrow}}{e^*}, \quad (-1)^{2\Delta S} = (-1)^k$

- Jain-like TI $\quad \nu^{\uparrow} = \frac{k}{2nk+1}, \quad e^* = \frac{1}{2nk+1}, \quad 2\Delta S = \frac{\nu^{\uparrow}}{e^*} = k$ stable unstable
- (331) & Pfaffian TI $\quad \nu^{\uparrow} = \frac{1}{2}, \quad e^* = \frac{1}{4}, \quad 2\Delta S = 2$ unstable
- Abelian TI $\quad K = \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix} \quad \nu^{\uparrow} = \frac{3}{7}, \quad e^* = \frac{1}{7}, \quad 2\Delta S = 3$ stable
- Read-Rezayi TI $\quad \nu^{\uparrow} = \frac{k}{kM+2}, \quad e^* = \frac{1}{kM+2}, \quad 2\Delta S = k$ stable unstable
- NASS-like TI $\quad \nu^{\uparrow} = \frac{2k}{2kM+3}, \quad e^* = \frac{1}{2kM+3}, \quad 2\Delta S = 2k$ unstable

# Remarks

- general expression of partition function allows to extend Levin-Stern stability criterium to any TI with interacting fermions

$\mathbb{Z}_2$ classification of TI protected by TR invariance

- unprotected edge states do become fully gapped?

  - Abelian states: yes, by careful analysis of possible TR-invariant interactions

  (Levin, Stern; Neupert et al.; Y-M Lu, Vishwanath)

  - non-Abelian states: yes, use projection from "parent" Abelian states

  e.g. (331) -> Pfaffian because [projection, TR-symm.]=0 (A.C., Randellini '14)

# Conclusions

- $\mathbb{Z}_2$ spin parity anomaly characterizes Topological Insulators protected by Time Reversal symmetry (cf. Ringel, Stern; Koch-Janusz, Ringel)
- anomaly signalled by index $(-1)^{2\Delta S} = -1, \qquad 2\Delta S = \frac{\nu^{\uparrow}}{e^*}$
- Pfaffian TI is unstable
- To do:

  - stability of Topological Superconductors: $\mathbb{Z} \rightarrow \mathbb{Z}_8$

  - stability 3D systems and 2D-3D systems: $\mathbb{Z}_{16}$

# Gapping interactions for Pfaffian TI

(AC, Randellini, '14)

- Gapping interactions for Abelian states defined by $K$ matrix

$$U_\alpha = \exp\left( i\Lambda_\alpha^T K \Phi_\uparrow - i\overline{\Lambda}_\alpha^T K \overline{\Phi}_\downarrow \right) \; + \; \text{h.c.} \qquad \alpha = 1, \ldots, n = c$$

- For (331) state, they can be written in terms of Weyl fermions fields

$$U_1 = \Psi_{\uparrow 1}^\dagger \Psi_{\uparrow 2} \Psi_{\downarrow 1}^\dagger \Psi_{\downarrow 2} \; + \; \text{h.c.}$$

$$U_2 = \Psi_{\uparrow 1}^\dagger \Psi_{\uparrow 2}^\dagger \Psi_{\downarrow 1} \Psi_{\downarrow 2} \; + \; \text{h.c.}$$

- Projection (331) $\rightarrow$ Pfaffian, i.e. to identical layers $\Psi_{\uparrow 1}^\dagger \sim \Psi_{\uparrow 2}^\dagger \; \rightarrow \; V\chi$

$U_1 =: \chi \partial \chi : \, : \bar{\chi} \bar{\partial} \bar{\chi} :$ neutral Majorana

$U_2 =: V^2 \, \bar{V}^2 :, \qquad V = e^{i\sqrt{2}\phi}$ charged excit.

- $U_1$ couples to fermion field, $U_2$ to charges modes, giving both mass
- Analysis extends to Read-Rezayi and Ardonne-Schoutens NASS states