

# Integrability vs exact solvability in the quantum Rabi model

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in collaboration with Huan-Qiang Zhou, arXiv:1408.3816

# Outline of this talk

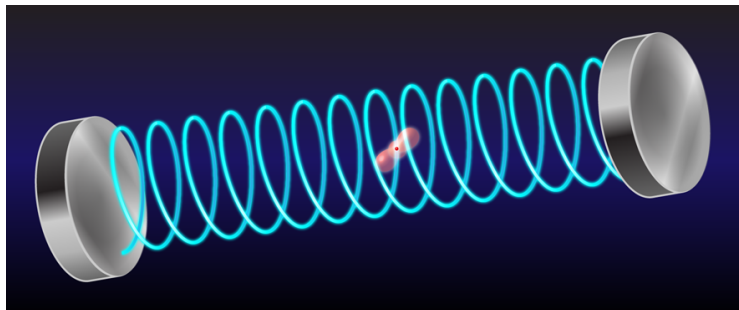
- 0) interaction between light and matter
- 1) Rabi model
- 2) Jaynes-Cummings model
- 3) solution of the quantum Rabi model
- 4) phenomenological criterion for quantum integrability
- 5) Yang-Baxter integrability
- 6) Dicke model

# Interaction between quantum light and matter

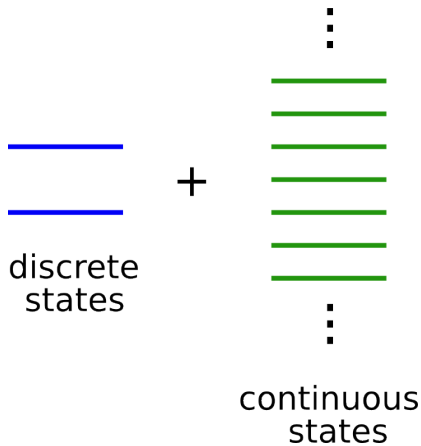
The Rabi model describes a two-level atom coupled to a quantised, single mode harmonic oscillator (bosonic field).

Applicable to a wide range of physical systems:

- interaction between light and trapped ions or quantum dots
- interaction between microwaves and superconducting qubits
- cavity QED
- circuit QED



## different class of models



# The quantum Rabi model

I Rabi, Phys. Rev. **49**, 324 (1936); **51**, 652 (1937)

The Hamiltonian ( $\hbar = 1$ ) reads

$$H_R = \Delta \sigma_z + \omega a^\dagger a + g \sigma_x (a + a^\dagger)$$

where

- $\sigma_x$  and  $\sigma_z$  are Pauli matrices for the two-level system with level splitting  $2\Delta$ , and
- $a^\dagger$  ( $a$ ) denote creation (destruction) operators for a single bosonic mode with  $[a, a^\dagger] = 1$  and frequency  $\omega$ .
- $g$  is the coupling between the two systems.

The Rabi model has  $Z_2$  symmetry (parity).

$\implies$  simplest system of quantum light interacting with matter.

# The Jaynes-Cummings model

E T Jaynes & F W Cummings, Proc. IEEE **51**, 89 (1963)

The JC model is the rotating-wave approximation to the Rabi model, with Hamiltonian

$$H_{JC} = \Delta \sigma_z + \omega a^\dagger a + g (\sigma^+ a + \sigma^- a^\dagger)$$

with  $\sigma^\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$ .

⇒ described as a theoretical and experimental milestone in the history of quantum physics.

- applicable because the conditions of near resonance,  $2\Delta \approx \omega$  and weak coupling  $g \ll \omega$ , for the rotating-wave approximation apply to many experiments.
- the JC model is integrable in the Yang-Baxter sense.

nice review in N M Bogoliubov & P P Kulish, J. Mathematical Sciences **192**, 14 (2013)

# Solution of the quantum Rabi model

D Braak, PRL **107**, 100401 (2011) & Ann. Phys. (Berlin) **525**, L23 (2013)

The Rabi model has both regular and exceptional parts of the eigenspectrum.

Using the representation of the bosonic operators in the Bargmann space of analytic functions, the regular eigenvalues are given in terms of the zeros  $x_n^\pm$  of the function

$$G_\pm(x) = \sum_{n=0}^{\infty} K_n(x) \left[ 1 \mp \frac{\Delta}{x - n\omega} \right] \left( \frac{g}{\omega} \right)^n$$

where  $K_n(x)$  are defined recursively by  $nK_n = f_{n-1}(x) K_{n-1} - K_{n-2}$  with initial conditions  $K_0 = 1$ ,  $K_1(x) = f_0(x)$ , and

$$f_n(x) = \frac{2g}{\omega} + \frac{1}{2g} \left( n\omega - x + \frac{\Delta^2}{x - n\omega} \right).$$

The eigenvalues follow from  $E_n^\pm = x_n^\pm - g^2/\omega$ .

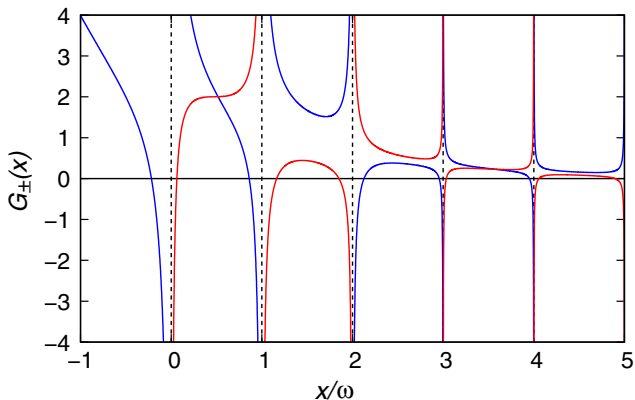


Figure from Braak for  $\omega = 1$ ,  $g = 0.7$ ,  $\Delta = 0.4$

Red lines: + parity, blue lines: - parity.

Simple poles at  $x = 0, \omega, 2\omega, \dots$  correspond to the eigenvalues of the uncoupled bosonic modes.

# Rabi spectrum

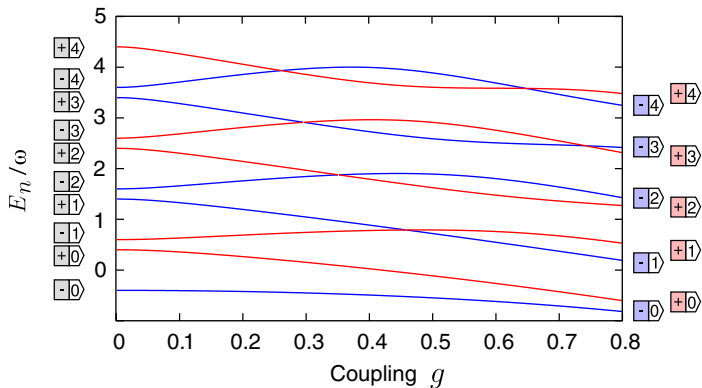


Figure from Braak for  $\omega = 1, \Delta = 0.4$

# Eigenstates of the quantum Rabi model

H Zhong, Q Xie, MTB & C Lee, J. Phys. A **46**, 415302 (2013)

- Symmetric, anti-symmetric and asymmetric solutions for the eigenstates given in terms of confluent Heun functions.

See also A J Maciejewskia, M Przybylskab & T Stachowiakc, Phys. Lett. A **378**, 16 (2014)

The Heun functions satisfy a 2nd order linear ODE (Heun 1889). See "The Heun Project" [theheunproject.org](http://theheunproject.org)

- The conditions proposed by Braak are a type of sufficiency condition for determining the regular solutions.
- The exceptional parts of the eigenspectrum – known as Judd isolated exact solutions – appear naturally as truncations of the confluent Heun functions.
- The Rabi model should be called a *quasi-exactly solved model*?

A. Moroz, Ann. Phys. (N.Y.) **338**, 319 (2013)

only "a finite number of their eigenvalues and corresponding eigenfunctions can be determined algebraically"

# Integrability of the Rabi model??

PRL **107**, 100401 (2011)

 Selected for a **Viewpoint** in *Physics*  
PHYSICAL REVIEW LETTERS

week ending  
2 SEPTEMBER 2011

## Integrability of the Rabi Model

D. Braak

*EP VI and Center for Electronic Correlations and Magnetism, University of Augsburg, 86135 Augsburg, Germany*

(Received 22 April 2011; published 29 August 2011)

The Rabi model is a paradigm for interacting quantum systems. It couples a bosonic mode to the smallest possible quantum model, a two-level system. I present the analytical solution which allows us to consider the question of integrability for quantum systems that do not possess a classical limit. A criterion for quantum integrability is proposed which shows that the Rabi model is integrable due to the presence of a discrete symmetry. Moreover, I introduce a generalization with no symmetries; the generalized Rabi model is the first example of a nonintegrable but exactly solvable system.

# Phenomenological criterion for quantum integrability

D Braak, PRL **107**, 100401 (2011)

Integrability is stated to be equivalent to the existence of  $f$  “quantum numbers” to classify eigenstates uniquely.

“If each eigenstate of a quantum system with  $f_1$  discrete and  $f_2$  continuous degrees of freedom can be uniquely labelled by  $f_1 + f_2 = f$  quantum numbers  $\{d_1, \dots, d_{f_1}, c_1, \dots, c_{f_2}\}$ , such that the  $d_j$  can take on  $\dim(\mathcal{H}_j)$  different values, where  $\mathcal{H}_j$  is the state space of the  $j$ th discrete degree of freedom and the  $c_k$  range from 0 to infinity, then this system is quantum integrable.”

“The Rabi model has  $f_1 = f_2 = 1$  and degeneracies take place between levels of states with different parity, whereas within the parity subspaces no level crossings occur. ... The global label (valid for all values of  $g$ ) is two dimensional as  $f = f_1 + f_2 = 2$ ; the Rabi model belongs therefore to the class of integrable systems.”



Centre for Modern Physics Director and Teamster Prof. Huan-Qiang Zhou

29 April 2014

## But what about Yang-Baxter integrability?

*If the Rabi model is integrable, is it Yang-Baxter integrable?*

The concept of Yang-Baxter integrability is ideally suited to (1+1)-dimensional quantum systems.

Variables can either be discrete or continuous.

### The master key to integrability!



Mural at Simons Centre, Stony Brook

# Yang-Baxter integrability in the Rabi model

$$H_R = 2\Delta s_z + \omega a^\dagger a + g(s^+ + s^-)(a + a^\dagger)$$

## SUMMARY

We find the model is Yang-Baxter integrable for two cases:

- 1)  $\Delta = 0 \Rightarrow$  known as the degenerate atomic limit.
- 2)  $\omega = 0 \Rightarrow$  not included in Braak's solution.

In both cases, there is an extra conserved quantity  $C$ , i.e.,  $[H_R, C] = 0$ , where:

- 1) for  $\Delta = 0$ ,  $C = s^+ + s^-$ .
- 2) for  $\omega = 0$ ,  $C = a^\dagger + a$ .

The key idea is to introduce an operator-valued twist, which yields a solution to the Yang-Baxter relation.

# 1) $\Delta = 0$

We construct  $\tau(u) = \text{tr } T(u)$ , where the monodromy matrix

$$T(u) = \begin{bmatrix} 1 & s^+ + s^- \\ s^+ + s^- & -1 \end{bmatrix} \begin{bmatrix} 1 + \eta u + \eta^2 N & \eta a \\ \eta a^\dagger & 1 \end{bmatrix}$$

satisfies the intertwining relation

$$R_{12}(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u - v)$$

with the  $R$ -matrix

$$R_{12}(u) = \begin{bmatrix} u + \eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u + \eta \end{bmatrix}$$

satisfying the Yang-Baxter relation

$$R_{12}(u - v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u - v).$$

Thus by construction  $[\tau(u), \tau(v)] = 0$ , with

$$\tau(u) = \eta[u + \eta N + (s^+ + s^-)(a^\dagger + a)] = \eta[u + g^{-1} H_R],$$

2)  $\omega = 0$

We construct  $\tau(u) = \text{tr} T(u)$ , with now the monodromy matrix

$$T(u) = \begin{bmatrix} 1 + \lambda & a + a^+ \\ a + a^+ & 1 - \lambda \end{bmatrix} \begin{bmatrix} u + \eta s^z & \eta s^- \\ \eta s^+ & u - \eta s^z \end{bmatrix}$$

where  $\lambda = \frac{\Delta}{g}$ , which satisfies the intertwining relation

$$R_{12}(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u - v)$$

with the same  $R$ -matrix

$$R_{12}(u) = \begin{bmatrix} u + \eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u + \eta \end{bmatrix}$$

satisfying the Yang-Baxter relation. Here

$$\tau(u) = 2u + \eta[2\lambda s^z + (a^+ + a)(s^+ + s^-)] = 2u + \eta g^{-1} H_R.$$

# Yang-Baxter integrability in general?

- Although integrable at the two parameter values  $\Delta = 0$  and  $\omega = 0$ , the fully quantised Rabi model does *not* appear to be Yang-Baxter integrable in general.
- Others have tried in the past to look for integrable extensions of the Rabi model beyond the random-wave approximation, i.e., beyond the Jaynes-Cummings model.

L. Amico, H. Frahm, A. Osterloh & G. A. P. Ribeiro, Nucl. Phys. B **787**, 283 (2007)

L. Amico, H. Frahm, A. Osterloh & T. Wirth, Nucl. Phys. B **839**, 604 (2010)

They could not construct the fully quantised Rabi model in this way.

- We contend that the Rabi model is not Yang-Baxter integrable in general.

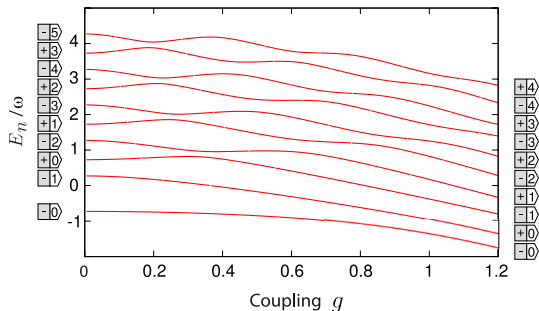
## “First example of a nonintegrable but exactly solvable system”??

Braak also considers the generalised Rabi model

$$H_\epsilon = \Delta \sigma_z + \omega a^\dagger a + g \sigma_x (a + a^\dagger) + \epsilon \sigma_x$$

- the term  $\epsilon \sigma_x$  breaks the parity symmetry.
- Braak solves this model in the same way.
- Eigenstates also obtained in terms of confluent Heun functions.

H Zhong, Q Xie, X W Guan, MTB, K Gao & C Lee, J Phys A **47**, 045301 (2014)



Braak figure for  $\Delta = 0.7$ ,  $\epsilon = 0.2$

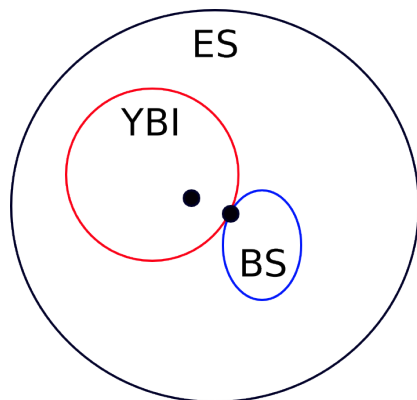
## Contradiction

- Braak concludes that  $H_\epsilon$  has no discrete symmetry. There is only one quantum number (energy) corresponding to the sole conserved quantity. But the number of degrees of freedom exceeds one  
 $\Rightarrow$  “this model must be considered nonintegrable”.  
 $\Rightarrow$  “First example of a nonintegrable but exactly solvable system”.
- **However**, also following Braak one may embed  $H_\epsilon$  into a larger system possessing a  $Z_2$ -invariance by adding a second spin degree of freedom, thus the extended Hamiltonian reads

$$H_{\text{ext}} = \Delta \sigma_z + \omega a^\dagger a + g \sigma_x (a + a^\dagger) + \epsilon \sigma_x \tau_x$$

This Hamiltonian admits  $\tau_x$  and  $e^{i\pi a^\dagger a} \sigma_z \tau_z$  as conserved quantities. These two operators yield two copies of  $Z_2$  symmetry, which are anti-commutative with each other. As such,  $H_\epsilon$  is actually “integrable” according to Braak’s criterion, in contrast to the claim!

## Integrability vs exact solvability



- • Yang-Baxter integrable points of the quantum Rabi model
- ES:= Exactly Solvable  
BS:= Braak Solvable  
YBI:= Yang-Baxter Integrable

# The Dicke model

R H Dicke, Phys. Rev. **93**, 99 (1954)

Extension of the Rabi model to  $N$  two-level qubits:

$$H_D = 2\Delta S_z + \omega a^\dagger a + g(S^+ + S^-)(a + a^\dagger)$$

where now

$$S^z = \sum_{j=1}^N s_j^z, \quad S^x = \sum_{j=1}^N s_j^x, \quad S^\pm = \sum_{j=1}^N s_j^\pm.$$

- Of great interest for both small and large  $N$ :
  - a) For  $N = 2$  it constitutes the simplest model of the universal quantum gate for ion trap quantum computing.
  - b) Transition to super-radiant state for large  $N$ .
- Shown to be Braak solvable for  $N = 2$  and  $N = 3$ .

J Peng, Z-Z Ren, D Braak, G-J Guo, G-X Ju, X Zhang & X-Y Guo, J. Phys. A **47**, 265303 (2014)

H Wang, S He, L-W Duan, Y Zhao & Q-H Chen, EPL **106**, 54001 (2014)

D Braak, J. Phys. B **46**, 224007 (2013)

# Yang-Baxter integrability in the Dicke model

- Applying the RWA leads to the Tavis-Cummings model:

$$H_{TC} = 2\Delta S^z + \omega a^\dagger a + g(S^+ a + S^- a^\dagger).$$

M Tavis & F W Cummings, Phys. Rev. **170**, 379 (1968)

- The TC model reduces to the JC model for  $N = 1$ .
- The TC model is Yang-Baxter integrable for general  $N$  and can be solved by the algebraic Bethe Ansatz.

see N M Bogoliubov & P P Kulish, J. Mathematical Sciences **192**, 14 (2013)

⇒ so what about the Dicke model?

# 1) $\Delta = 0$

We construct  $\tau(u) = \text{tr} T(u)$ , where the monodromy matrix

$$T(u) = \begin{bmatrix} 1 & S^+ + S^- \\ S^+ + S^- & -1 \end{bmatrix} \begin{bmatrix} 1 + \eta u + \eta^2 N & \eta a \\ \eta a^+ & 1 \end{bmatrix}$$

satisfies the intertwining relation

$$R_{12}(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u - v)$$

with the  $R$ -matrix

$$R_{12}(u) = \begin{bmatrix} u + \eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u + \eta \end{bmatrix}$$

satisfying the Yang-Baxter relation

$$R_{12}(u - v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u - v).$$

Thus

$$\tau(u) = \eta [u + g^{-1} H_D],$$

2)  $\omega = 0$

We construct  $\tau(u) = \text{tr} T(u)$ , with the monodromy matrix

$$T(u) = \begin{bmatrix} 1 + \lambda & a + a^+ \\ a + a^+ & 1 - \lambda \end{bmatrix} \begin{bmatrix} u + \eta S^z & \eta S^- \\ \eta S^+ & u - \eta S^z \end{bmatrix}$$

where  $\lambda = \frac{\Delta}{g}$ , which satisfies the same intertwining and Yang-Baxter relations.

- In this case the spin part of the monodromy matrix can be factorised into  $N$  terms.
- Now the operator  $\tau(u)$  is a polynomial of degree  $N$ , with

$$\tau(u) = 2u^N + \eta g^{-1} u^{N-1} H_D + \dots$$

- The parameter value  $\omega = 0$  of the Dicke model has been identified as Yang-Baxter integrable using another approach. In particular, a Bethe Ansatz solution has been obtained from the elliptic Gaudin model through a limiting procedure.

# Conclusion

- 1) The fully quantised Rabi and Dicke models do not appear to be integrable in general.
- 2) They are however, Yang-Baxter integrable at two special parameter values. More work can be done at these points.
- 3) For systems of this kind the terms exact solved, integrable and Yang-Baxter integrable should all be synonymously interchangeable.
- 4) Braak's phenomenological criterion for quantum integrability is questionable.