$sl(2,\mathbb{R})$-CONNECTIONS ON CIRCLE BUNDLES OVER SPACE-TIME

DISTORTION OF GAUGE FIELDS AND ORDER PARAMETERS

Michael Freedman
Roman Lutchyn

August 19, 2014

Microsoft® Research

Outline

I will discuss the possibility of slightly generalizing $U(1)$-principal bundles familiar in

1. Electromagnetism
2. Superfluids
3. Superconductors

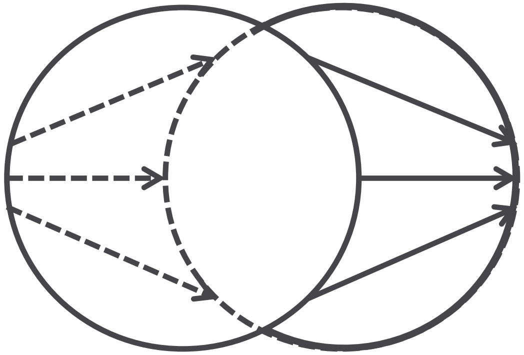
What we propose is looking at things "halfway" between

- $U(1)$-principal bundle with $u(1)$-connection and
- $PSL(2,\mathbb{R})$-principal bundle with $sl(2,\mathbb{R})$-connection.[1]

1. Going this far is problematic: the $(+)$ definite Killing directions yield, in quantum theories, excitations of negative energy.

Namely

1. $U(1)$-principal bundle with local $U(1)$ gauge symmetry of the Lagrangian *but* with an $sl(2,\mathbb{R})$-connection (i.e., not "left invariant")



2. Or an S^1 *associated* bundle to $PSL(2,\mathbb{R})$ built from the quotient action: $PSL(2,\mathbb{R}) \to {}^{PSL(2,\mathbb{R})}\Big/_{\left\{\begin{vmatrix} a & b \\ 0 & a^{-1} \end{vmatrix}\right\}} \cong S^1$
   - This bundle has an $sl(2,\mathbb{R})$ connection.
   - There is no canonical $U(1)$-symmetry. But imposing one leads back to case 1.

Energy Penalty

- Possible to measure distortion of the Mobius structure on S^1 and charge some cost:

$$\int d\text{area}\ \|A^{\perp}\|^2,$$

"nematic distortion energy" or "Beltrami energy", where $A^{\perp}$ are the (+) Killing direction of $\tilde{A} \in sl(2, \mathbb{R})$.

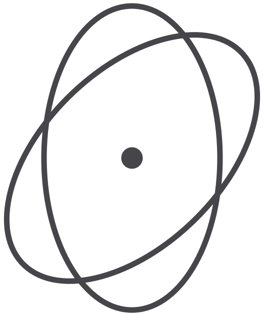
- **Idea:** Consider models which tolerate a little (elliptical) distortion (at a price). Study the limit: distortion $\to$ 0.
- This new flexibility has some surprising consequences.

Literature

- Witten: $SL(2,\mathbb{R})$ – gravity (Nucl. Phys. B311 (1988), 46–78, 0712.0155, 1001.2933)
- Haldane: $SL(2,\mathbb{R})$ anisotropic model for FQHE (1201.1983, 1202.5586)

Effective mass tensor g_{ij}

compared to Coulomb



1. $E(p) = g_{ij}\frac{p_i p_j}{2m}$ (kinetic energy of free electrons in crystal with B-field)
2. Coulomb interaction energy (inside lattice)

Mathematical Starting Point

A surprising flexibility of circle bundles over surfaces with flat $sl(2,\mathbb{R})$ connection

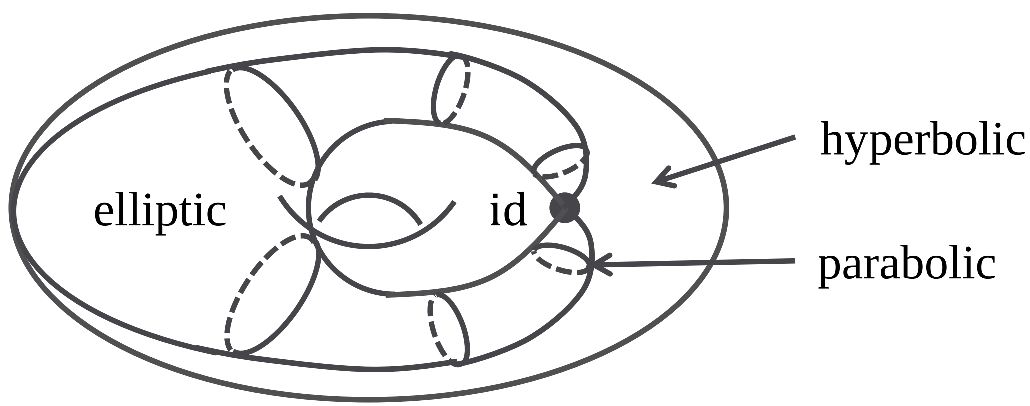
- $PSL(2,\mathbb{R}) = 2 \times 2$, real, $\det = 1$, $^{\text{matrices}}\Big/\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right)$
- $sl(2,\mathbb{R}) =$ Lie algebra, real, 2×2, $\mathrm{tr} = 0$
- $PSL(2,\mathbb{R}) \cong \tau_{\text{unit}}(\mathbb{H}^2)$ via polarization $M = \sqrt{MM^T}\mathrm{Rot} \cong \mathrm{isom}^+(\mathbb{H}^2)$, $z \mapsto \frac{az+b}{cz+b}$

Mathematical Starting Point

$$\begin{array}{ccl}
SL(2,\mathbb{R}) & & \text{commutes with } z \to \bar{z} \\
\mathrm{Con}\begin{pmatrix}1 & -i \\ 1 & i\end{pmatrix} \uparrow & \subset & \\
 & SL(2,\mathbb{C}) & \\
 & \cup & \\
SU(1,1) & & \text{commutes with } z \to \frac{1}{\bar{z}} \\
\parallel & &
\end{array}$$

$$\left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \middle| \; |a|^2 - |b|^2 = 1 \right\} \to \left({}^{a}/_{|a|}, {}^{b}/_{a} \right) \in S^1 \times \mathring{D}^2$$

$PSL(2,\mathbb{R})$:

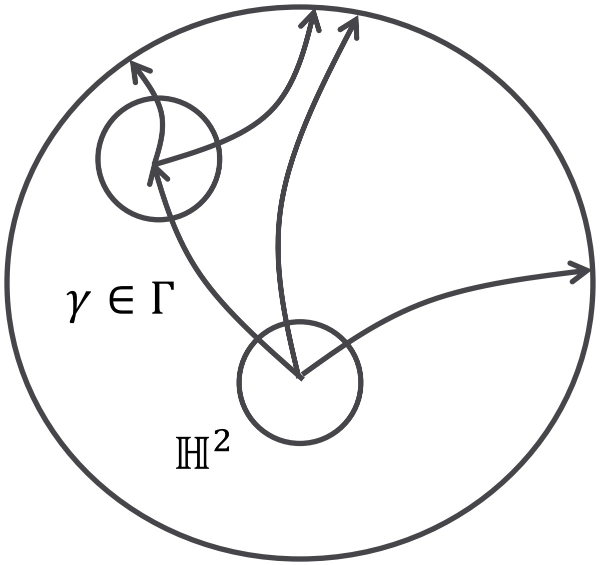


Fact

For Σ_g a closed surface of genus $g > 1$ $\tau_{\text{unit}}(\Sigma_g)$ has a flat $sl(2, \mathbb{R})$ connection (acting projectively between fibers).

Proof

Unwrap Σ_g to H^2, ${}^{H^2}/_{\Gamma} = \Sigma_g$.
Geodesic flow canonically identifies all unit tangent circles to H^2 with the circle at infinity S^1_∞. This integrates the connection $\tilde{A}$.



Fact

- This gives a (actually, many) irreps

$$\rho\colon \pi_1(\Sigma_g) \to PSL(2,\mathbb{R}).$$

(Such geometric reps and their Galois conjugates are the chief source of examples.)

- **"Chern number":** $\chi(\rho) = 2 - 2g < 0$ for $g > 1$
- Although $\chi(\rho) \neq 0$, ρ defines a flat S^1-bundle with structure group $PSL(2,\mathbb{R})$:

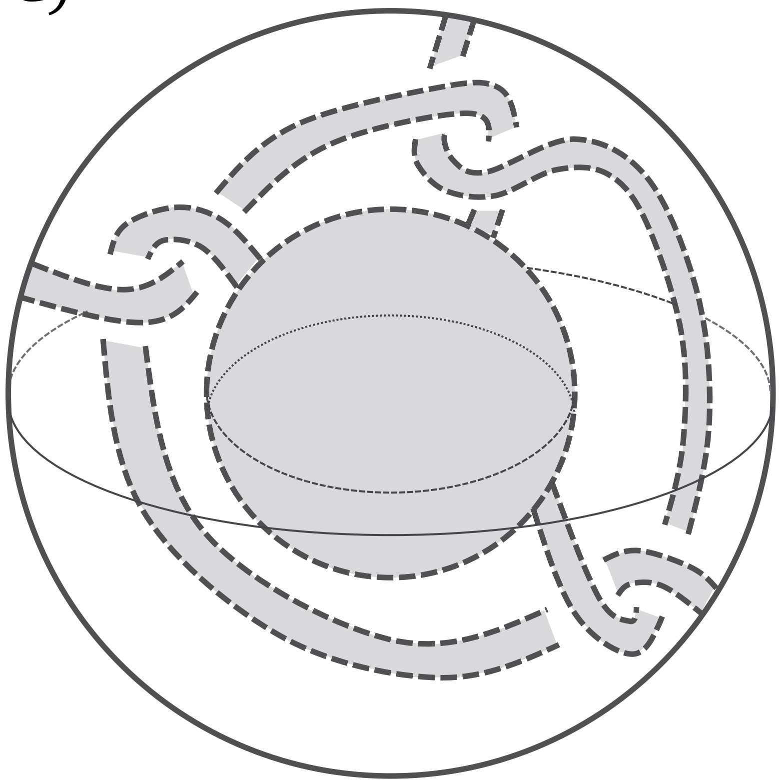
$$\mathbb{H}^2 \times S^1_\infty \big/_{(p,\alpha)\equiv(\rho(\gamma)(p),\rho(\gamma)(\alpha))} \to \mathbb{H}^2 \big/_{\Gamma} \cong \Sigma_g,\ \gamma \in \Gamma$$

Fact

- For $g > 1$, ρ *cannot* extend as $PSL(2,\mathbb{R})$ rep over any bounding 3-manifold M^3, $\partial M^3 = \Sigma_g$ but by Thurston's orbifold theorem $\exists$ extensions over $PSL(2,\mathbb{C})$
  - $\pi_1(\Sigma_g) \hookrightarrow PSL(2,\mathbb{R})$

    $\quad\quad\quad \downarrow \quad\quad\quad\quad\quad \downarrow$
  - $\pi_1(M_g^3) \hookrightarrow PSL(2,\mathbb{C})$
- For $g = 2$,

  $M_g^3 =$ "tripus" $\subset \mathbb{R}^3$



Fact

- This can be used to make pairs of $\frac{2\pi}{e}$-monopoles if one allows $PSL(2,\mathbb{C})$ not $U(1)$ to act near a point.
- **Recall:** In EM $\mathcal{F}^2 = \mathrm{E}^1 \wedge dt + c^{-1}\mathcal{B}^2$

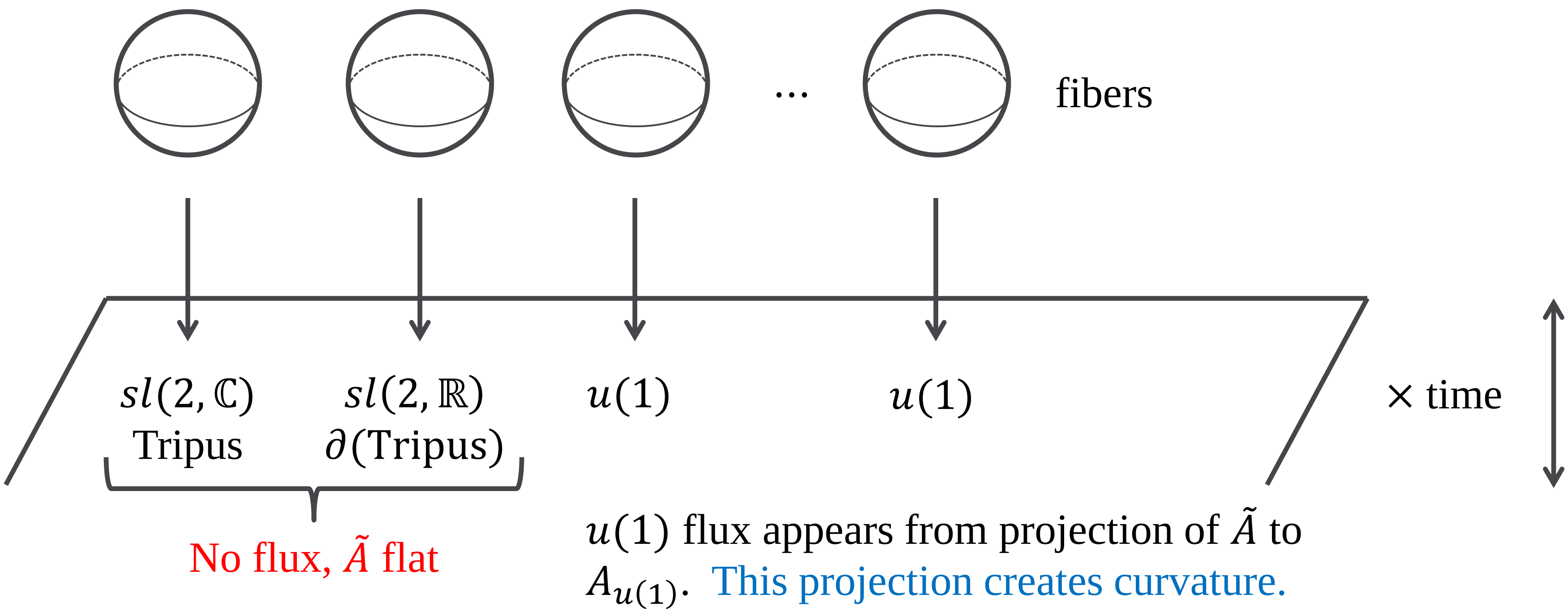
$$F_{ij} = \begin{vmatrix} 0 & -E_2 & -E_2 & -E_5 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & B_3 & 0 & B_1 \\ E_3 & -B_2 & B_1 & 0 \end{vmatrix}$$

over a spatial surface Σ, $\int_\Sigma \mathcal{B}^2 = \chi$.

- If topology is standard $\Sigma = \partial M^3$ and $\chi = 0$ (*no* monopoles).

Fact

But expanding the nominal fiber from S^1 to S^2 and letting the $U(1)$ connection (potential) A of EM take $sl(2,\mathbb{C})$ values near a point creates a $\frac{4\pi}{e}$-monopole (charge $= 2$).



Chern-Weil Theory

- What is going on? How can you have a characteristic class without curvature?
- It all depends on the characteristic class:
- **char. class** $\leftrightarrow \int$ **func. (curvature)** for rational classes has an exception:
- Exception for $\chi =$ Euler Class, group non-compact.
- For $SO(2n)$ connections A, $\chi :: \mathrm{tr}\, \underset{\text{“Pfaffian”}}{\underset{\uparrow}{\Sigma}}\, \epsilon_{i_1,\ldots,i_{2n}}\, F_{i_1,i_2} \cdots F_{i_{2n-1},i_{2n}}$

  where $F_{\mu\nu} = \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu} + [A_\mu, A_\nu]$
- However, for $\tilde{A} \in sl(2,\mathbb{R})$, there is *no* such *formula* even though $SO(2n)$ and $SL(2n,\mathbb{R})$ have equivalent bundle theories.

Chern-Weil Theory

- **Fact:** There exist $\mathbb{R}^{2n}$-bundles with $\chi \neq 0$ which admit flat $sl(2n, \mathbb{R})$ connections.
- Milnor (1958) proved a sharp threshold for surfaces Σ_g:

Given
$$\begin{array}{ccc} S^1 & \to & E, \\ & & \downarrow \\ & & \Sigma_g \end{array}$$

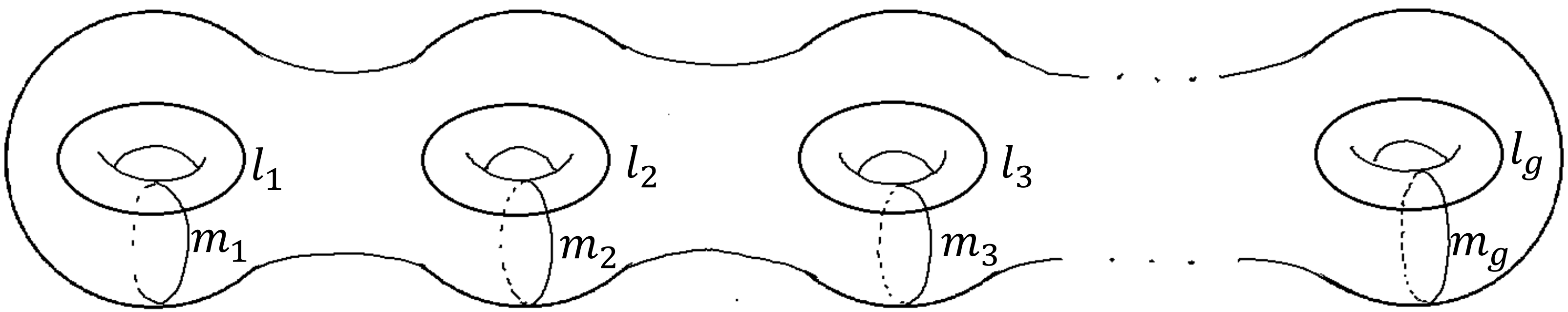
$\left|\chi(E)\left[\Sigma_g\right]\right| < g \Leftrightarrow \exists$ a flat linear $sl(2, \mathbb{R})$ connection, and

Wood (1970) showed

$\left|\chi(E)\left[\Sigma_g\right]\right| < 2g \Leftrightarrow \exists$ a flat projective $sl(2, \mathbb{R})$ connection.

Infinitesimal Milnor

For genus g $\approx \frac{\pi n}{2\epsilon^2}$, Σ_g has an $sl(2,\mathbb{R})$-flat bundle E with $\chi(E) = n$ and holonomies $h(m_i) \approx_{\exp} \begin{vmatrix} 0 & \epsilon \\ \epsilon & 0 \end{vmatrix}$, $h(l_i) =_{\exp} \begin{vmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{vmatrix}$, $i = 1, \ldots, g$



$[m_i, l_i] \approx 2\epsilon^2$ radian rotation

(Commutating boots yield a rotation.)

Infinitesimal Milnor

To second order:

$$h(m_i)h(l_i)h^{-1}(l_i) \approx \begin{pmatrix} \cos 2\epsilon^2 & -\sin 2\epsilon^2 \\ \sin 2\epsilon^2 & \cos 2\epsilon^2 \end{pmatrix}$$

But this is not exact.

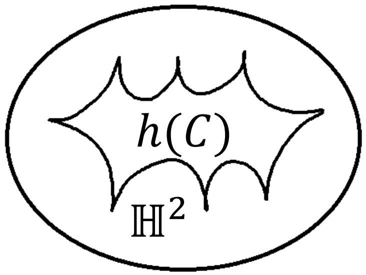
However, topology implies an exact solution.

$$(S^2 \times S^2 \backslash \pm\Delta) \xrightarrow[(a,b)\mapsto \frac{a\times b}{\|a\times b\|}]{} S^2 \text{ is} \simeq \text{to } SO(3) \to S^2 \;\; g \mapsto g(*).$$

Since this is non-contractable, perturbations remain surjective. This defines a representation $\pi_1(T_-^2) \to PSL(2,\mathbb{R})$ near boost on meridian and longitude and pure rotation around puncture. Band summing copies ⇒ "infinitesimal Milnor"

There is a converse

A $U(1)$ principal bundle with flat $\tilde{A} \in sl(2,\mathbb{R})$ satisfies $|\chi(E)| < \text{const.} \underbrace{\int_\Sigma d\text{ area } \left\|\tilde{A}^\perp\right\|^2}_{\text{“Beltrami energy”} := E_\beta}$



Proof: Use flat $\tilde{A}$ to trivialize E over top cell $C \subset \Sigma_g$. Comparing this $SL(2,\mathbb{R})$ trivialization with “round” $U(1)$ structure on each S^1 fiber gives a map $h: C \to \mathbb{H}^2$.

$$h_*\left(d\text{area}_{\Sigma_g}(x)\right) = \det\begin{pmatrix}\tilde{A}_{11}(x) & \tilde{A}_{21}(x)\\ \tilde{A}_{21}(x) & \tilde{A}_{22}(x)\end{pmatrix} d\text{area}_{\mathbb{H}^2} < \text{const.} E_\beta,$$

where $\tilde{A}_{ij}$ is the jth component of $\tilde{A}_i$, $j = 1 \Leftrightarrow \begin{pmatrix}0 & 1/2\\ 1/2 & 0\end{pmatrix}$, $j = 2 \Leftrightarrow \begin{pmatrix}1/2 & 0\\ 0 & -1/2\end{pmatrix}$. But $\chi(E) = \int_{\Sigma_g} h_*(d\text{area}_{\Sigma_g}(x))$. $\square$

Application of hyperbolic geometry in condensed matter physics

- Quantum Hall effect: Haldane et al., 2011–12, Maciejko et al., 2013)
- Superfluids and superconductors: Freedman and Lutchyn, 2014 (this talk ☺)

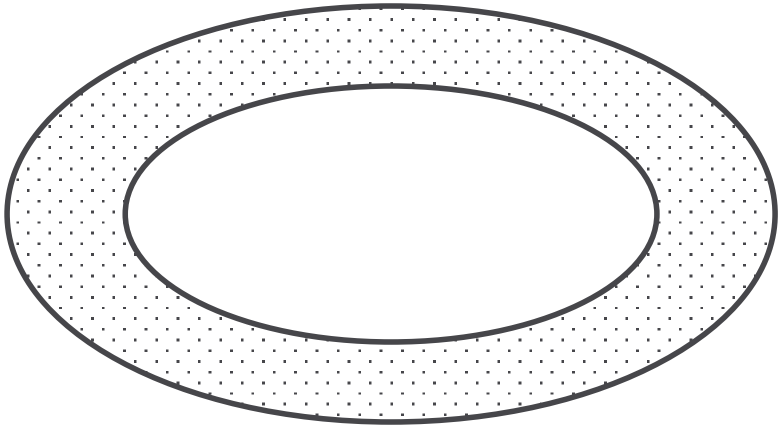
Properties of two dimensional models on a space with negative curvature are very different!

- No long range interaction between vortices in XY model: Callan and Wilczek, Nucl. Phys. B340 (1990), 366–386
- In contrast, the last bit of *this* talk is about curvature the *target space*
- Connections may boost as well as rotate
- Exotic quantization condition

Synthetic Gauge Fields in Cold Atoms

- The cold atoms community is now proficient at simulating $u(1)$ and $su(2)$ gauge fields.
- We suspect that a similar technique would permit simulation of $sl(2,\mathbb{R})$ gauge field-gravity in the lab. Specifically, a rotationally symmetric, pure boost $A_\phi^B \in sl(2,\mathbb{R})$ might be imposed on a ring of cold atoms.

$$A_\phi^B = \frac{1}{2}\sinh\tau \begin{pmatrix} -\sin\phi & \cos\phi \\ \cos\phi & \sin\phi \end{pmatrix}$$

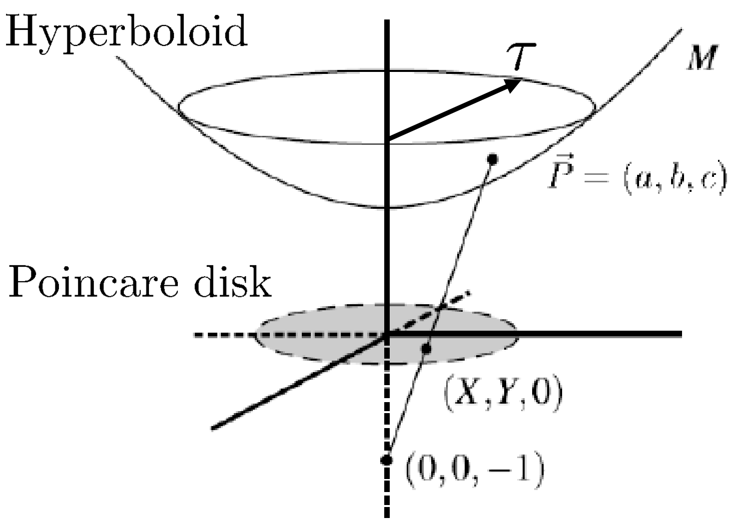


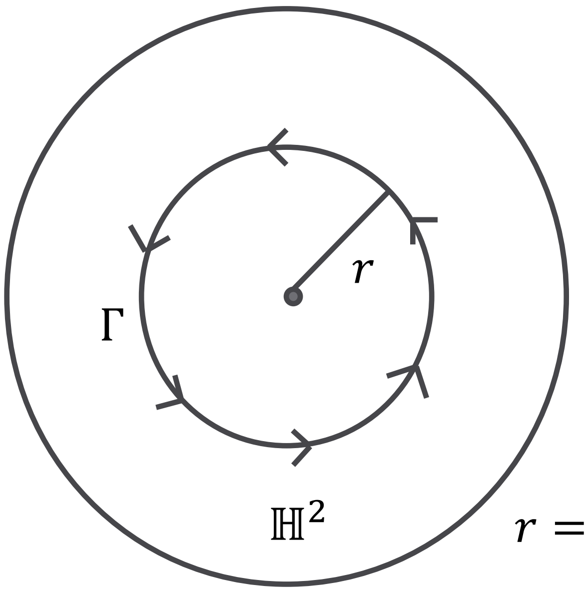
Cold Atoms

- A_ϕ^B has the interpretation of the tangents to the circle Γ of radius $= \tau$ in the hyperbolic plane $\mathbb{H}^2$.
- Poincaré disk model:

Stereographic projection:

Metric: $ds_{\mathbb{H}^2} = \frac{4}{(1-r^2)^2} ds_{E^2}$



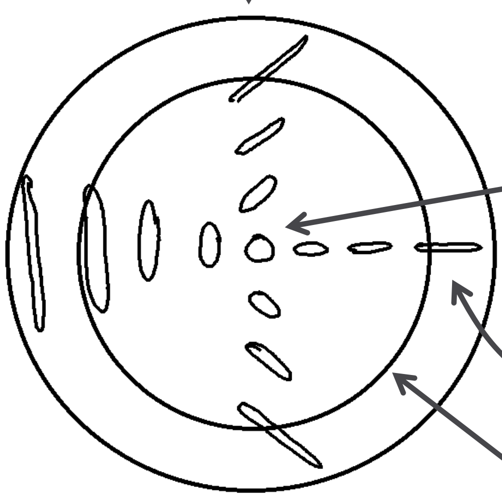


$r = \tanh\left(\frac{\tau}{2}\right)$

Cold Atoms

Hyperbolic geometry arises as ${}^{SL(2,\mathbb{R})}/_{U(1)} \cong \mathbb{H}^2$ when $PSL(2,\mathbb{R})$ has the bi-invariant Killing metric.

$\mathbb{H}^2 = SL(2,\mathbb{R})/U(1) = \{$ellipses of area $= 1$ modulo translation$\}$



$$ds_{\mathbb{H}^2} = \frac{4}{(1-r^2)^2} ds_{E^2}$$

$$\mathrm{dist}_{\mathbb{H}^2}\left(\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}, \begin{vmatrix}n & 0\\ 0 & {}^{1}/_{n}\end{vmatrix}\right) = \log n^2$$

Γ = loop of all ellipses of eccentricity $= n^2 = e^\tau$

Cold Atoms

- An order parameter $\mathbf{\Delta} = \Delta(\cos\theta, \sin\theta)$ coupled to A_ϕ^B *(not $U(1)$!)* will have energy

$$\mathcal{F} = \int d^d \times \sqrt{g}\, g_{\mu\nu} g^{\lambda\delta} \nabla_\lambda \mathbf{\Delta}^\mu \nabla_\delta \mathbf{\Delta}^\nu$$

- All zero-energy solutions:

$$\mathbf{\Delta}(\tau, \phi) = \Delta(0) \cosh^2 \frac{\tau}{2} \exp i2 \sinh^2 \frac{\tau}{2} (\phi)$$

- The quantization condition

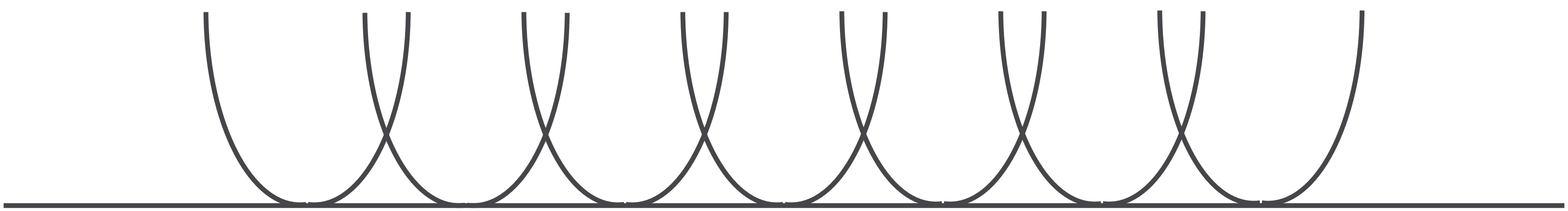
$$\mathbf{\Delta}(\tau, 0) = \mathbf{\Delta}(\tau, 2\pi): \quad \text{Area}_{\text{Hyp}}(\Gamma) = 2 \sinh^2 \frac{\tau}{2}$$

  is integral.

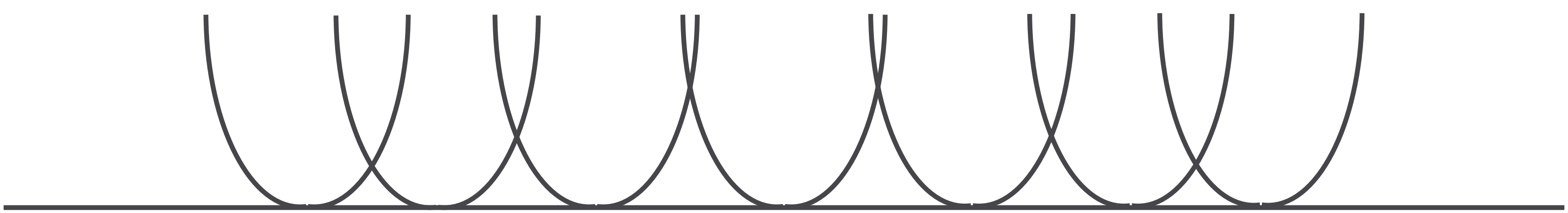
- **Note:** A_ϕ^B is precisely the angular component of the Levi-Civita connection.

The uninitiated would have an experimental surprise

1. A pure boost *integrates* to purely rotational holonomy
2. Hyperbolic quantization conditions
3. Energy vs. $||A^B||$, not: $\frac{(2\pi)^2\Delta_0^2}{L}\left(N - \frac{\Phi}{2\pi}\right)^2 + V(\Delta_0)L$



but



$sl(2,\mathbb{R})$ superconductivity?

- Let us assume EM is pure $U(1)$. Is there a role for $\tilde{A} \in sl(2,\mathbb{R})$ in an effective theory of superconductivity?
- The simplest opportunity is a spin polarized, two-dimensional, $p_x \pm ip_y$ superconductor.
- Consider the GL Lagrangian density:

$$\mathcal{L}_{GL} = -\frac{1}{4}F_{A\mu\nu}F_A^{\mu\nu} + \rho\|D_A\phi\|^2 + \mu^2\|\phi\|^2 - \lambda\|\phi\|^4$$

Understanding the Order Parameter ϕ

- **First**, in what complex plane should a spin-polarized (fixed d-vector) p-wave superconducting order parameter ϕ take its values?
  - **It is a section of $\boldsymbol{\tau} \otimes \boldsymbol{L}^{-2}$**, where τ is the tangent bundle to the superconducting space-time. The exponent$(\tau) = 1$ comes from j: $j = 0 \Leftrightarrow s$, $j = 1 \Leftrightarrow p$, $j = 2 \Leftrightarrow d$, etc., and L is the $U(1)$ bundle of electromagnetism, EM.
- GL-Hamiltonian has symmetry $G = U(1)_{EM} \times SO(2)_\tau$
- <u>p-wave:</u> ground state symmetry

$$H = \{(\theta, -2\theta)\} \subset U(1) \times SO(2),$$

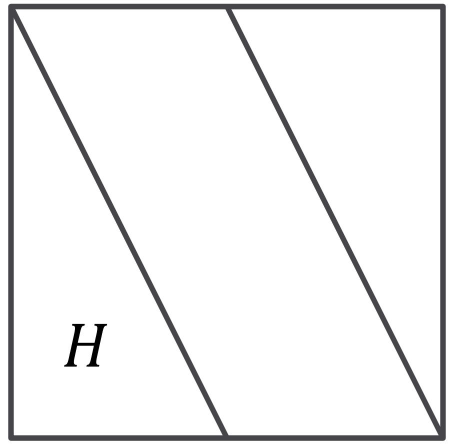
since $\phi_k = \langle c_{k\uparrow} c_{-k\uparrow} \rangle = |\Delta(k)| \arg(k)$ and if U_g implements

$$c_{k\uparrow} \rightarrow e^{i\theta} c_{k\uparrow}$$

and U_r implements spatial rot(θ),

then, $\langle U_r^2 U_g^{-1} c_{k\uparrow} c_{-k\uparrow} U_g U_r^{-2} \rangle = \langle c_{k\uparrow} c_{-k\uparrow} \rangle$

Thus ϕ_k lies in ${}^{G}/_{H}$.



Considering the effect of x, y-shifts, one calculates that ϕ_k is a section of $\tau \otimes L^{-2}$.

- Since the tangent bundle to a sample is only an abstraction coarsely connected to the experimental reality, it does not seem essential to postulate $\nabla_A g_{\mu\nu} \equiv 0$.
- A small amount of "slop" in metric transport may be modeled at lowest order by elliptic distortion ($\tilde{A} \in sl(2, \mathbb{R})$).

One may write a Ginzburg-Landau Lagrangian in the $sl(2,\mathbb{R})$ context

$$\mathcal{L}_{GL}^{sl(2,\mathbb{R})} = -\frac{1}{4}\|F_A\|^2 + \rho\|D_{\tilde{A}}\phi\|^2 + \mu^2\|\phi\|^2 - \lambda\|\phi\|^4 + \underbrace{\mathrm{func}(\|\phi\|)\|A^{\perp}\|^2}_{\text{Beltrami energy}}$$

1. A is, as in EM, a $u(1)$-connection. $\tilde{A}$ is an enhancement, an $sl(2,\mathbb{R})$-connection. Using orthogonal basis:

$$\left\{\underset{a}{\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}}, \underset{b}{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}}, \underset{c}{\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}}\right\},$$

project: $\tilde{A} = A + A^{\perp}$ ← $b + c$-part, boost

a-part rotation

2. All bundles have structure group $= U(1)$, *but* possess an $sl(2,\mathbb{R})$ connection, $\tilde{A}$.
3. Dynamic variables: $\tilde{A}, \phi$
4. $\mathcal{L}_{GL}^{sl(2,\mathbb{R})}$ is (local) $U(1)$-gauge invariant:

$$\tilde{A} \mapsto g\tilde{A}g^{-1} + dgg^{-1}, \qquad \phi \longmapsto g(\phi)$$

- Under a local $U(1)$-gauge transformation g, $\left\|A_\mu^\perp\right\|^2$ is invariant.
- Since conjugation by $g \in U(1)$ is an isometry of $u(1)^\perp$ and the dgg^{-1} contribution is parallel to $u(1)$ and thus projected out.

- In such a Lagrangian there are new ways to trade energy around, specifically between $\|\nabla_A \phi\|^2$, $\|F_A\|^2$, and the Beltrami term E_β.
- We expect some modification to Meissner physics, and
- Also new Josephson equations, if the Beltrami energy E_β were coupled to $\frac{d}{dt}$(phase) (as is charge density).
- We do not presently have a specific proposal, but there are a number of possibilities.

Summary

- $sl(2,\mathbb{R})$ connections on $U(1)$-principal bundles allow surprising flexibilities and may be useful in model building:
  - High energy (We discussed the $U(1)$-sector of the standard model only)
  - Low energy, superfluids, superconductors, and possibly QHE
- Non-compact forms of other Lie algebras can be considered in a similar vein.