Topological Superconductor-Luttinger Liquid Junctions

Ian Affleck (and D. Giuliano and Y. Komijani)

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Outline

1) Theoretical & experimental background
2) A simple model from Y-junction
3) Renormalization group analysis and non-trivial critical point
4) Conductance
5) Further support for phase diagram
6) Realizing model with single quantum wire

1) Theoretical & Experimental Background

-as shown by Kitaev, a 1 dimensional p-wave "superconductor" can be in a topological phase characterized by a Majorana mode at each end, weakly coupled [exp (-L/ξ)] to each other

$$H = \sum_{j=0}^{L-1} [-t c_j^+ c_{j+1} + \Delta c_j c_{j+1} + h.c.]$$

$$c_j = (\gamma_{ja} + i\gamma_{jb})/2$$

For Δ=t, $$H = -(t/2)\sum_{j=0}^{L-1} \gamma_{jb}\gamma_{j+1,a}$$

γ_{0a}, γ_{Lb} don't appear in H!

$(\gamma_{0a}+i\gamma_{Lb})/2$ annihilates a zero mode that lives at both ends of system. This topological phase persists for a range of Δ and μ (chemical potential).

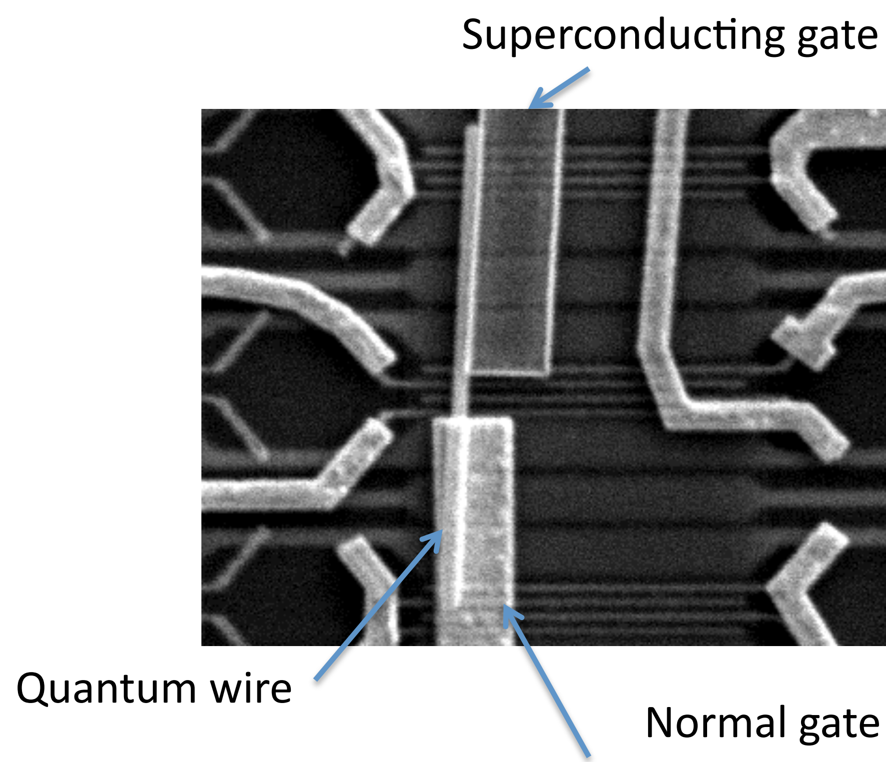
Kitaev model is simplified description of a quantum wire with spin-orbit coupling in magnetic field, proximate to an ordinary s-wave superconductor.

$$H = \int dx \left[\psi^{+} \left(-\frac{1}{2m}\frac{d^2}{dx^2} + V(x) - i\alpha \frac{d}{dx}\sigma^{y} + B(x)\sigma^{x} \right) \psi + \left(\Delta(x)\psi_{\uparrow}\psi_{\downarrow} + h.c. \right) \right]$$

($g\mu_B/2$ set to 1, sum over spin indices implied in $\psi^{+}\psi$ term.)

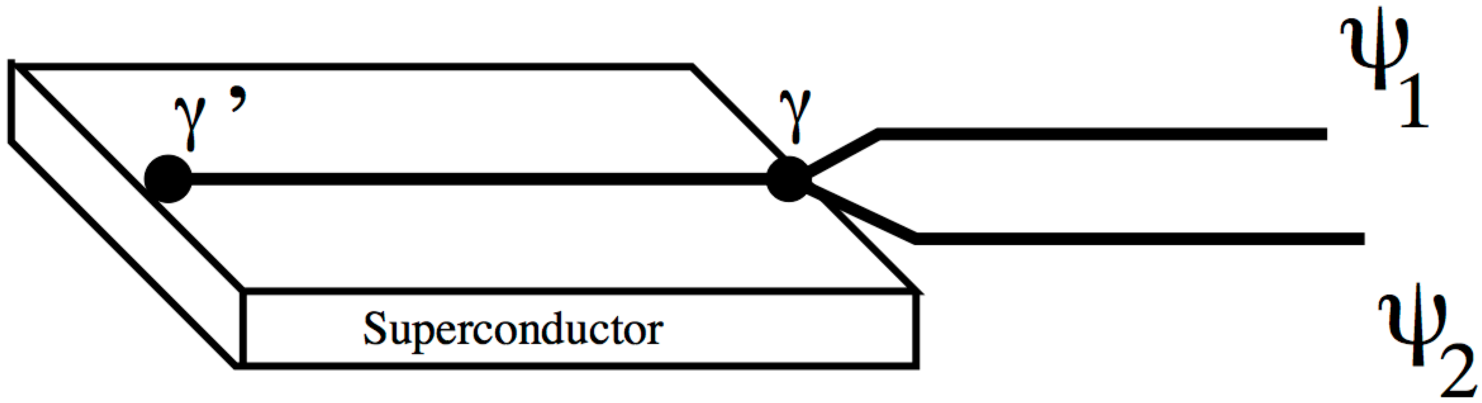
Now gapless Majorana modes occur at edges of region where pairing term ~Δ, exists.

This phase may have been seen in InSb quantum wires proximate at one end to a Nb superconductor.



Y Mourik et al
Science 336, 1003 (2012)

2. A Simple Model



We consider 2 (or more) Luttinger liquid channels corresponding to a T-junction or multi-channel wire. Interesting critical behavior occurs when both channels couple to the Majorana mode.

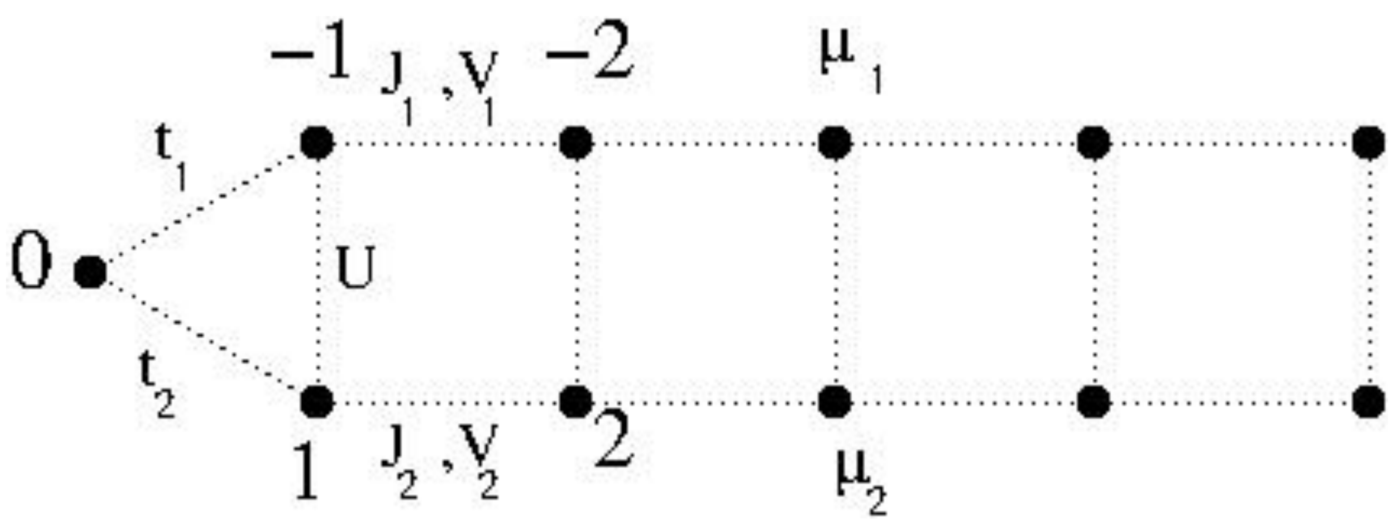
Following L. Fidkowski et al. we consider a low energy model of a long interacting normal region coupled to a long superconducting region. At energy scales below gap of superconducting portion of wire we integrate out all degrees of freedom of superconductor except for 1 Majorana mode.

In a tight binding model for the system we represent the topological superconductor by a single site, 0, at the end of the chain, with $c_0=(\gamma+i\gamma')/2$. Only $(c_0^+ + c_0)$ appears in Hamiltonian, not $(c_0^+ - c_0)$. We, in general, include interactions between the two channels, but are interested in cases where they both remain gapless, due to different chemical potentials:

$$H_0 = -\sum_{j=-\infty}^{-1}\left[(J_1/2)c_j^+ c_{j-1} + \mu_1 n_j\right] - \sum_{j=1}^{\infty}\left[(J_2/2)c_j^+ c_{j-1} + \mu_2 n_j\right] + h.c.$$

$$H_{\text{int}} = V_1 \sum_{j=-\infty}^{-1}\left[n_j n_{j-1}\right] + V_2 \sum_{j=1}^{\infty}\left[n_j n_{j+1}\right] + U \sum_{j=1}^{\infty}\left[n_j n_{-j}\right]$$

$$H_b = -t_1(c_0^+ + c_0)(c_{-1}^+ - c_{-1})/2 - t_2(c_0^+ + c_0)(c_1^+ - c_1)/2$$



We can write a low energy effective Hamiltonian in terms of Dirac fermions coupled to the Majorana mode:

$$H_0 = \sum_{j=1}^{2} i v_{jF} \int_0^\infty \left[\psi_{Rj}^+ \partial_x \psi_{Rj} - \psi_{Lj}^+ \partial_x \psi_{Lj} \right]$$

$$H_{\text{intra}} = \sum_{j=1}^{2} \int_0^\infty dx [V_{j\varphi} (:\psi_{Rj}^+ \psi_{Rj}: + :\psi_{Lj}^+ \psi_{Lj}:)^2 + V_{j\vartheta} (:\psi_{Rj}^+ \psi_{Rj}: - :\psi_{Lj}^+ \psi_{Lj}:)^2]$$

$$H_{\text{inter}} = \int_0^\infty dx [U_\varphi (:\psi_{R1}^+ \psi_{R1}: + :\psi_{L1}^+ \psi_{L1}:)(:\psi_{R2}^+ \psi_{R2}: + :\psi_{L2}^+ \psi_{L2}:) +$$

$$U_\vartheta (:\psi_{R1}^+ \psi_{R1}: - :\psi_{L1}^+ \psi_{L1}:)(:\psi_{R2}^+ \psi_{R2}: - :\psi_{L2}^+ \psi_{L2}:)]$$

$$H_b = \gamma \sum_{j=1}^{2} t_j [\psi_{Lj}(0) - \psi_{Lj}^+(0)], \qquad [\psi_{Lj}(0) = \psi_{Rj}(0)]$$

For 2 channels, after bosonizing:

$$H_0 = \frac{1}{2}\sum_{\lambda=\rho,\sigma} u_j \int_0^\infty dx \left[K_\lambda \left(\frac{\partial \phi_\lambda}{\partial x} \right)^2 + K_\lambda^{-1} \left(\frac{\partial \theta_\lambda}{\partial x} \right)^2 \right]$$

$$H_b = i\gamma \sum_j t_j \Gamma_j \cos[\sqrt{\pi}\ \phi_j(0)]$$

after bosonizing. The Majorana mode, γ, couples linearly to the fermion field in each channel. Γ_j are Klein factors. Boundary conditions $\theta_j(0)=0$ are imposed. ϕ_ρ, ϕ_σ are linear combinations of ϕ_1, ϕ_2. For free fermions, $K_\lambda=1$, t_j have RG scaling dimension $d_j=1/2$. d_j increase with repulsive interactions.

Diagonalization of general bulk model:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} r^{-1}\cos\alpha & r^{-1}\sin\alpha \\ -r\sin\alpha & r\cos\alpha \end{pmatrix} \begin{pmatrix} \phi_\sigma \\ \phi_\rho \end{pmatrix}$$

$$H_0 = \sum_{\lambda=\rho,\sigma} \frac{u_\lambda}{2} \int_0^\infty dx \left[K_\lambda \left(\frac{\partial \phi_\lambda}{\partial x} \right)^2 + K_\lambda^{-1} \left(\frac{\partial \theta_\lambda}{\partial x} \right)^2 \right]$$

4 universal parameters classifying bulk model are 2 Luttinger parameters, $K_{\rho,}$ K_σ and 2 anisotropy parameters, α and r (plus 2 velocities, u_ρ and u_σ).

3) Renormalization Group analysis and non-trivial critical point

If t_2=0, we expect t_1 flows to ∞ under renormalization. γ and Γ_1 couple together to form a Dirac fermion: $\psi_0=(\gamma+i\Gamma_1)/2$,

$$H_b = 2t_1(\psi_0^+\psi_0 - 1/2)\cos[\sqrt{\pi}\ \phi_1(0)] \qquad i\gamma\Gamma = 2(\psi_0^+\psi_0 - 1)$$

At strong coupling fixed point, $\phi_1(0)$ is pinned at either 0 or , $\sqrt{\pi}$ depending on whether ψ_0 state is filled or empty. This is a Schroedinger's cat state: electron has equal amplitude to be in superconductor or normal wire.

Using $\psi_{L/R} \prec \exp i\sqrt{\pi}(\phi \pm \theta)$

weak coupling boundary condition, $\theta_j(0)=0$, corresponds to normal reflection boundary conditions on fermions: $\psi_L(0)=\psi_R(0)$. Strong coupling boundary condition, $\phi_j(0)=0$, is

$$\psi_L(0) = \psi_R^+(0)$$

corresponding to perfect Andreev reflection of electrons at SN junction, at low energies and $2e^2/h$ conductance from superconductor to normal lead at zero temperature. (Can be checked explicitly for non-interacting, K=1 case.)

What happens when Majorana mode couples to both normal channels, t_1, t_2>0?

$$\frac{dt_1}{dl} = \varepsilon t_1 + \cdots$$

where $d_j=1-\varepsilon_j$.

$$\frac{dt_2}{dl} = \varepsilon t_2 + \ldots$$

$$H_b = \gamma \sum_j t_j [\psi_j(0) - \psi_j^+(0)]$$

For non-interacting, or SU(2) symmetric case, we can change basis $\psi_1' \sim t_1\psi_1 + t_2\psi_2$ etc. and then bosonize. Expect perfect Andreev scattering in Ψ_1' channel, normal scattering in ψ_2' channel.

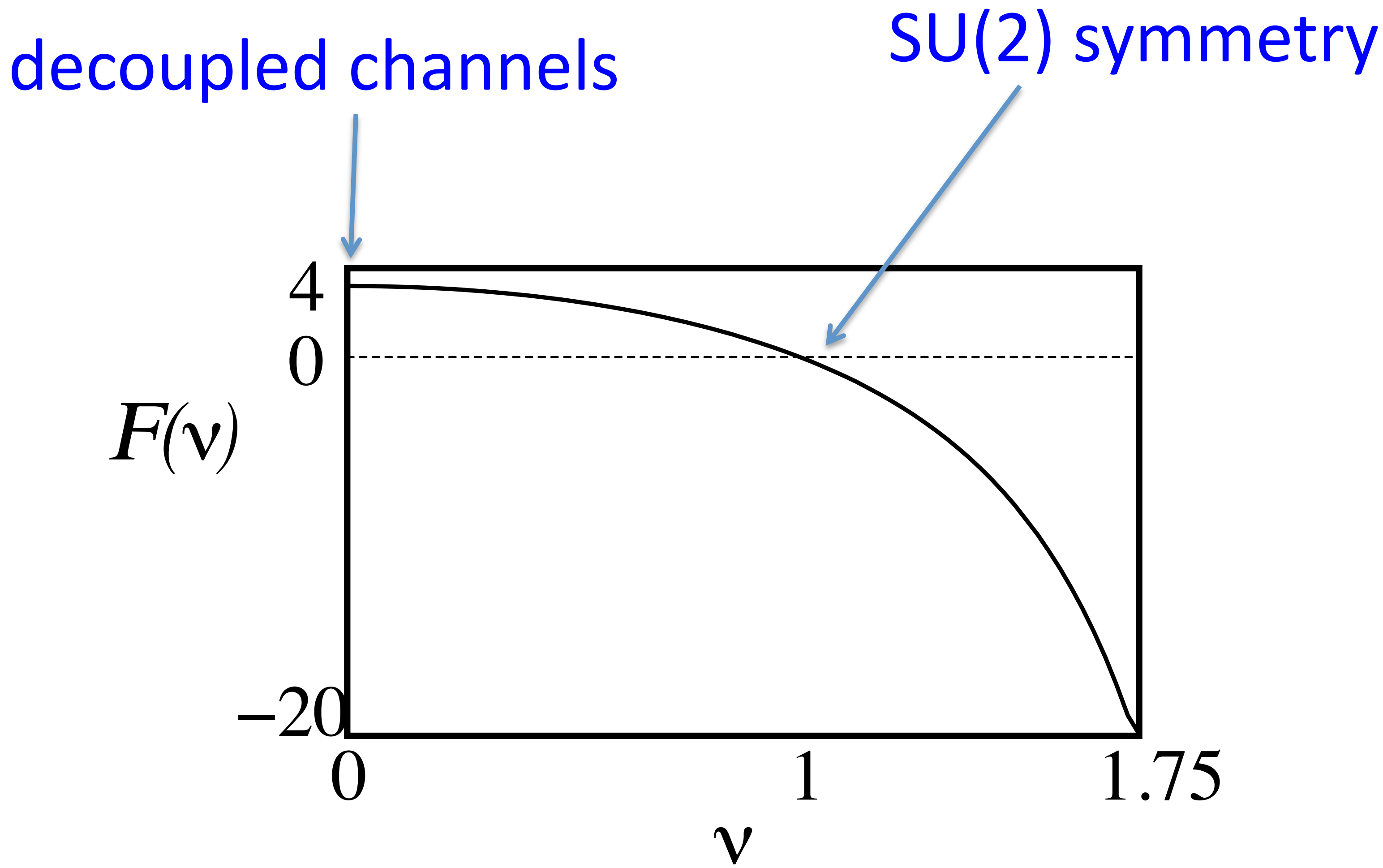
For general bulk interactions we can't make this transformation. Can Majorana mode couple strongly to two channels? This would violate principle of "Majorana monogamy". 2 Majorana modes make a Dirac mode, not 3. We can study what happens, for barely relevant tunnelling, small ε_j, by calculating next order term in β functions:

$$\frac{dt_1}{dl} = \varepsilon_1 t_1 - F t_1 t_2^2 + \ldots$$

$$\frac{dt_2}{dl} = \varepsilon_2 t_2 - F t_2 t_1^2 + \ldots$$

F depends on details of bulk interactions, vanishes with SU(2) symmetry

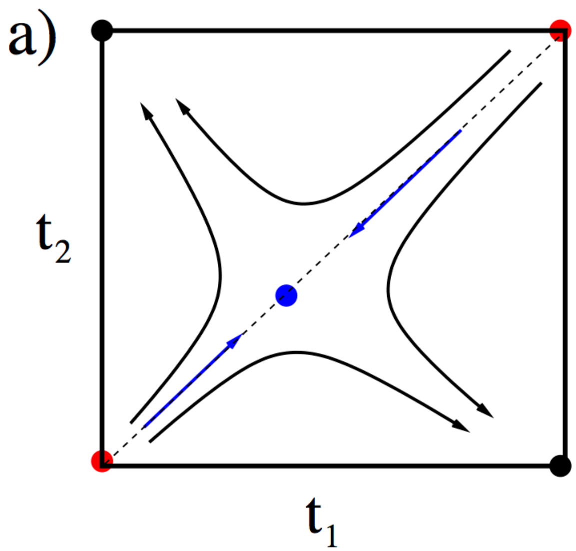
$$\upsilon \equiv \frac{\sin 2\alpha}{2}\left(\frac{1}{K_\rho} - \frac{1}{K_\sigma}\right)$$



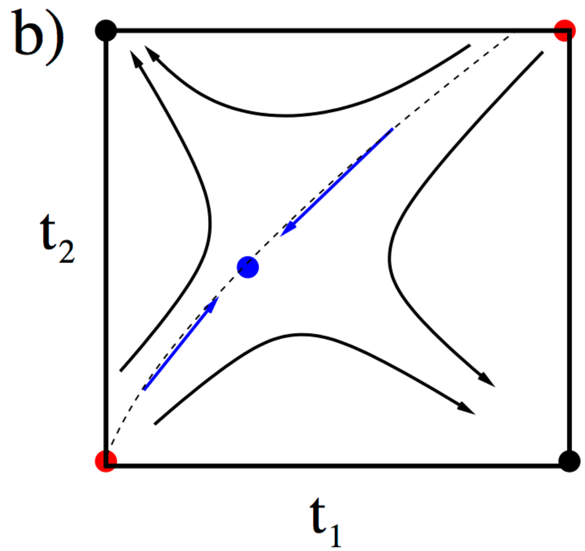
These RG equations predict a fixed point:

$$t_{2c} = \sqrt{\varepsilon_1 / F}, \qquad t_{1c} = \sqrt{\varepsilon_2 / F}$$

For small ε_j, higher order terms in β-functions are negligible near fixed point.



$\varepsilon_1 = \varepsilon_2$



$\varepsilon_1 > \varepsilon_2$

The non-trivial critical point (NTCP) is unstable. Any imbalance of t_1 and t_2 leads to a flow to "AxN" fixed point, $t_1=\infty$, $t_2=0$ or vice versa. So a Majorana mode acts as a switch. Slightly increasing one of tunnel couplings leads to $2e^2/h$ conductance to 1 channel and 0 to the other.

4) Conductance

We may calculate conductance at NTCP in lowest order perturbation theory in the t_j, setting them equal to t_{jc}. We can also calculate the crossover of the conductance versus source-drain voltage. In the case of small bare t_j:

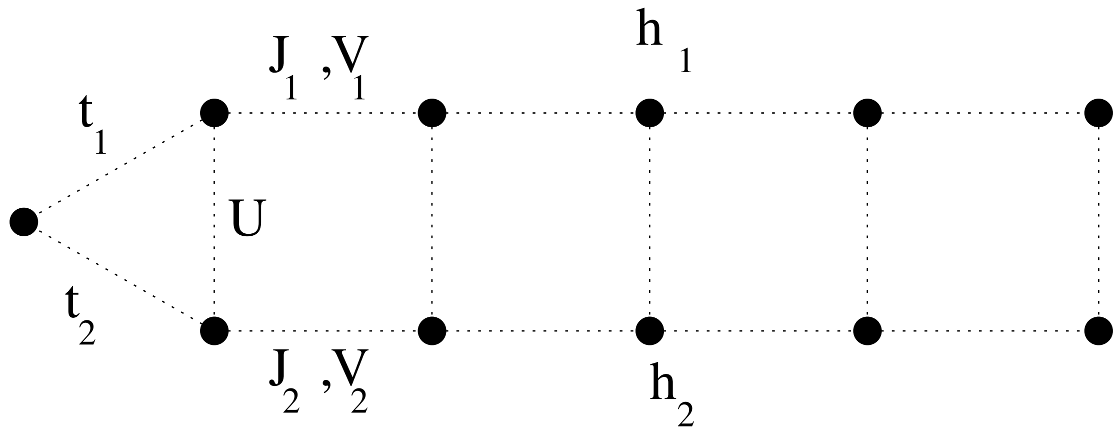
$$G = \frac{e^2}{h}(2\pi)^2 \frac{\varepsilon}{F} \frac{1}{1+(V/V^*)^{2\varepsilon}}$$

G takes fixed point value at V➔0 and vanishes For V>>V*, a cross-over scale. Similar scaling with T or ω.

With a small imbalance of the t_j, $t_1>t_2$, $G_{S2} \rightarrow 0$, $G_{S1} \rightarrow 2e^2/h$ at low V.
If the Luttinger liquids of length L are connected adiabatically to Fermi liquid leads, we expect this to produce a cross-over to the non-interacting conductance for V below a cross-over scale v_F/L.
Conductance can be shown to be robust against disorder near SN interface, by topological arguments.

5) Further Support for Phase Diagram

Starting with a tight-binding version of the Luttinger liquid-topological superconductor model and then making a Jordan-Wigner transformation gives a spin chain impurity model. A 2-leg xxz ladder coupled to an impurity spin at one end, arising from Majorana mode.



$$H_b = -2t_1 S_0^x S_{1,1}^x - 2t_2 S_0^y S_{1,2}^y$$

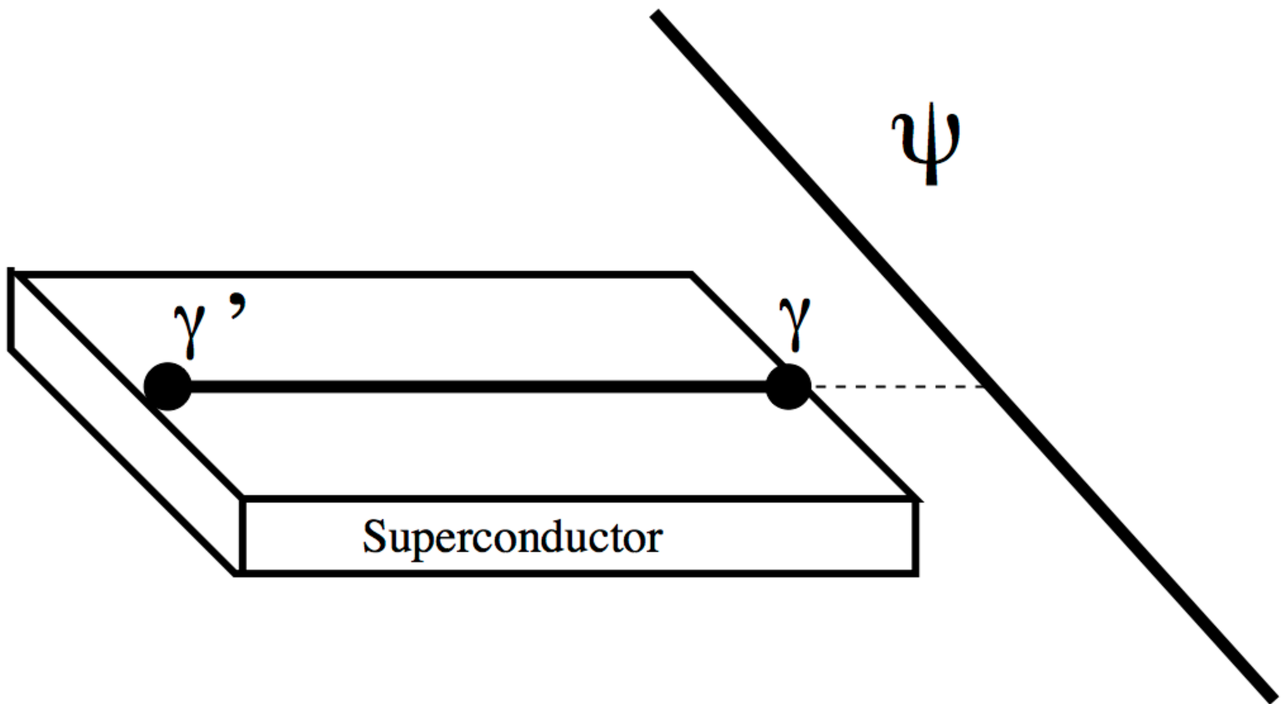
$$H_0 = \sum_{i=1}^{\infty} \sum_{j=1}^{2} \left[J_j (S_{i,j}^x S_{i+1,j}^x + S_{i,j}^y S_{i+1,j}^y) + V_j S_{i,j}^z S_{i+1,j}^z - h_j S_{i,j}^z \right]$$

$$+U \sum_{i=1}^{\infty} S_{i,1}^z S_{i,2}^z$$

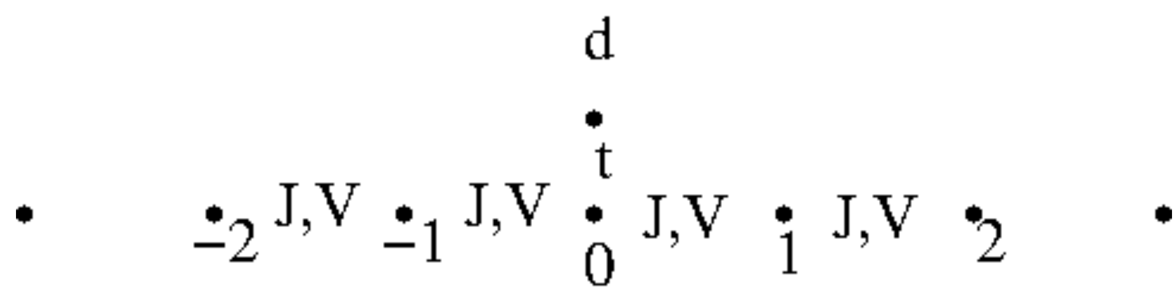
If t_2=0, $[S_0^x, H] = 0$ and this is equivalent to a spin ladder with a boundary magnetic field, acting on 1 leg only, in ± x direction.

If t_1, t_2>0, the spin at zero, corresponding to the Majorana mode, is in a non-trivial state and there is magnetic frustration.

What if Majorana mode couples to centre of single channel Luttinger liquid?



A tight-binding version of model is now:



$$H_0 + H_{\text{int}} = \sum_{j=-\infty}^{\infty} [-J(c_j^+ c_{j+1} + h.c.) + V n_j n_{j+1}]$$

$$H_b = t(d + d^+)(c_0^+ - c_0)$$

$$H_0 = \frac{u}{2}\int_0^\infty dx\left[K\left(\frac{\partial\phi}{\partial x}\right)^2 + K^{-1}\left(\frac{\partial\theta}{\partial x}\right)^2\right]$$

$$H_b = i\gamma\left[t_L\cos\left[\sqrt{\pi}\,(\phi(0)+\theta(0))\right] + t_R\cos\left[\sqrt{\pi}\,(\phi(0)-\theta(0))\right]\right]$$

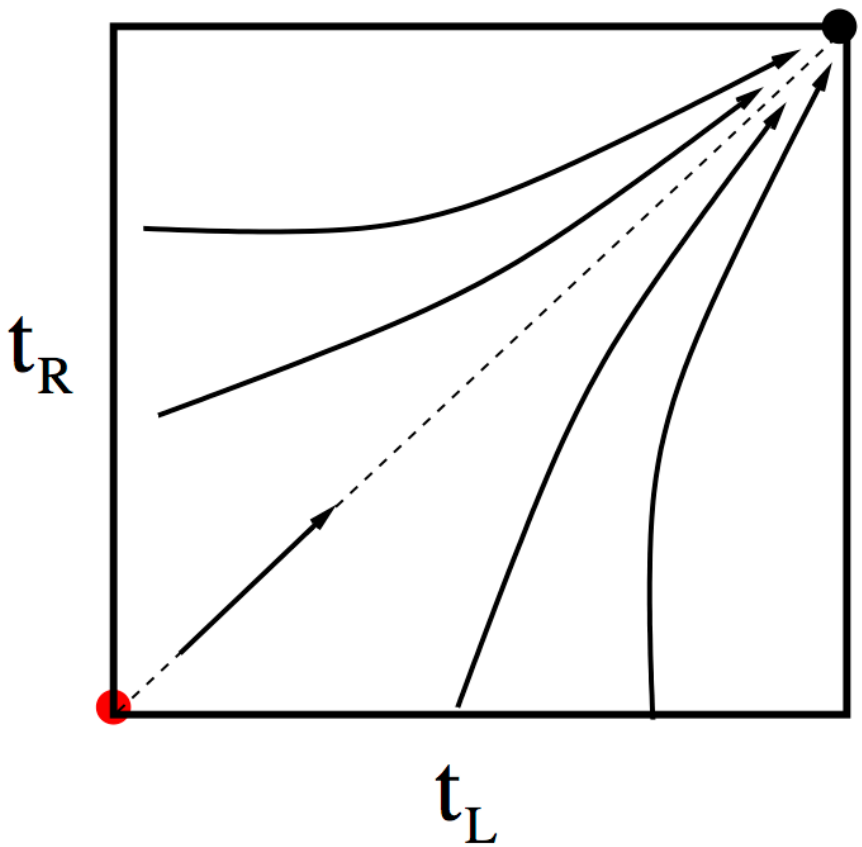
Essentially same model as before with 2 channels corresponding to right and left movers, $\phi\pm\theta$. We get same β–function:

$$\frac{dt_1}{dl} = \varepsilon_1 t_1 - F t_1 t_2^2 + \ldots$$

$$\frac{dt_2}{dl} = \varepsilon_2 t_2 - F t_2 t_1^2 + \ldots$$

but now F<0, for interactions not too large

NB: If F<0 stable fixed points would correspond to Andreev transmission for right movers and normal transmission for left movers. With F<0 flow is to non-trivial critical point, now occurring at strong coupling.



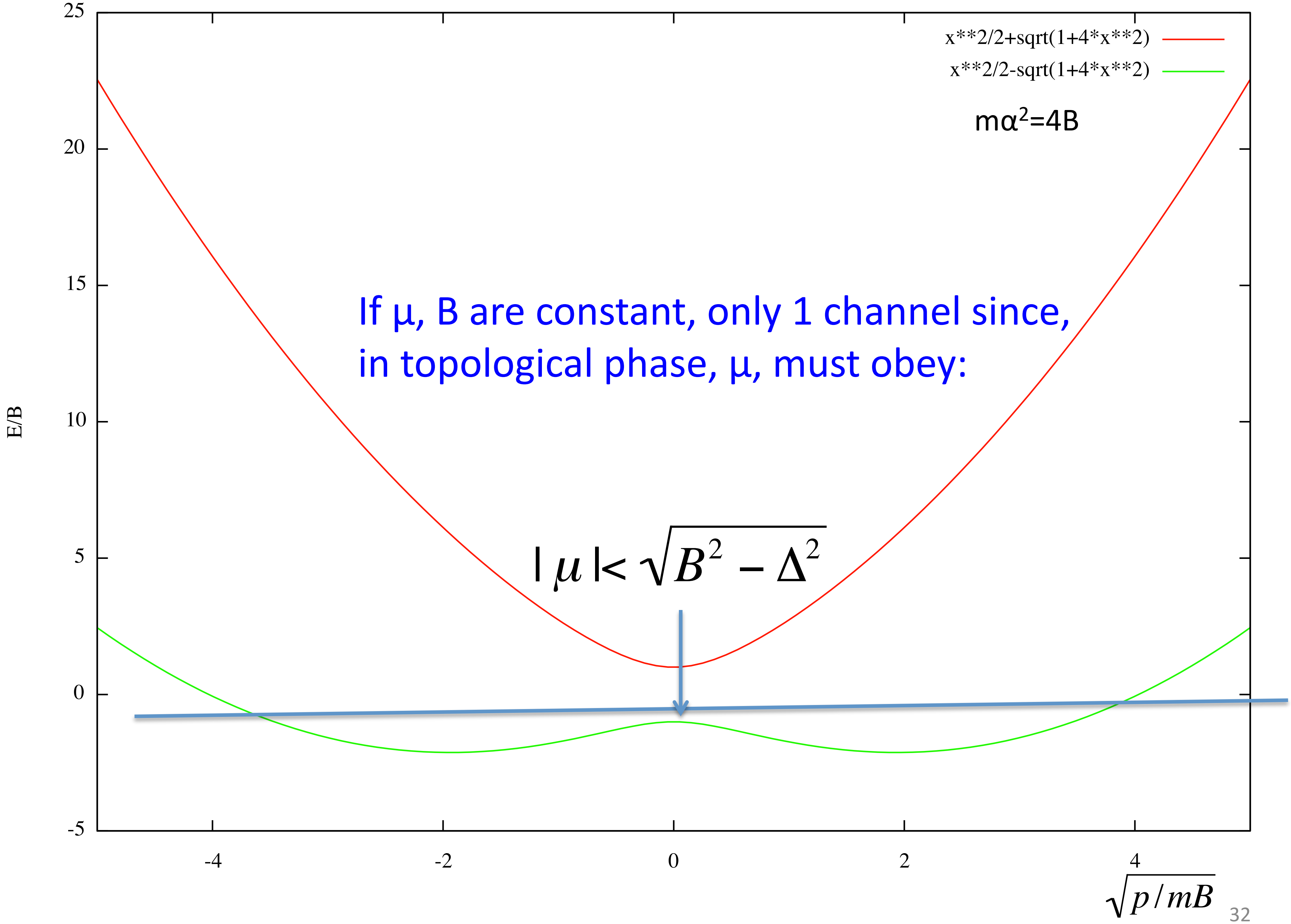
Or, if PT symmetry (parity-time reversal) Is broken, flow is to fixed point with chain broken into 2 parts at origin, with perfect Andreev reflection on 1 side, normal on other.

PT symmetry complex conjugates c-numbers and takes: $c_j \rightarrow c_{-j}$
This symmetry can be broken in tight-binding model, for instance, by:

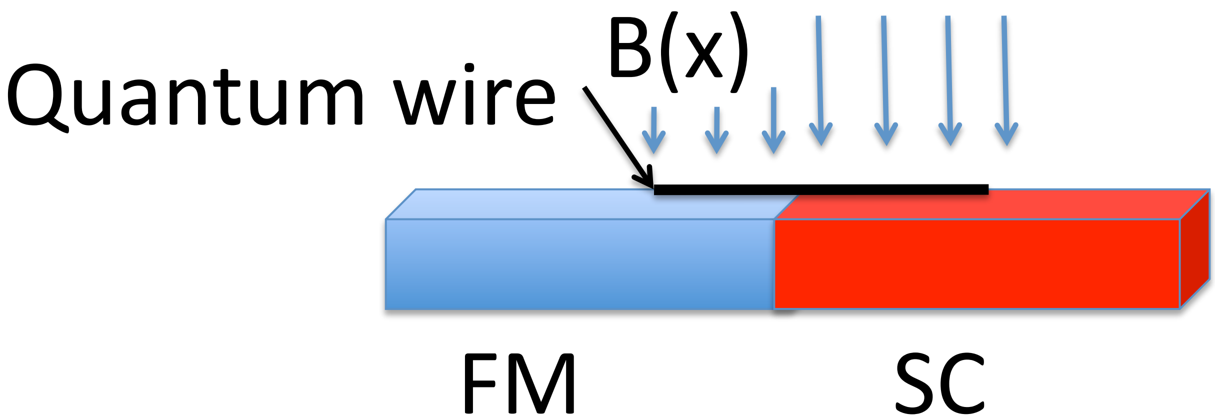
$$\delta H = J' c_0^{+}(c_1 - c_{-1}) + h.c.$$

6. Realizing the Model with a Single Quantum Wire

Even for a single quantum wire, two channels are naturally present due to spin or dispersion curve due to Rashba coupling but:


x**2/2+sqrt(1+4*x**2)
x**2/2-sqrt(1+4*x**2)
$m\alpha^2=4B$
If μ, B are constant, only 1 channel since,
in topological phase, μ, must obey:
$|\mu| < \sqrt{B^2 - \Delta^2}$
E/B
$\sqrt{p/mB}$
25
20
15
10
5
0
-5
-4
-2
0
2
4

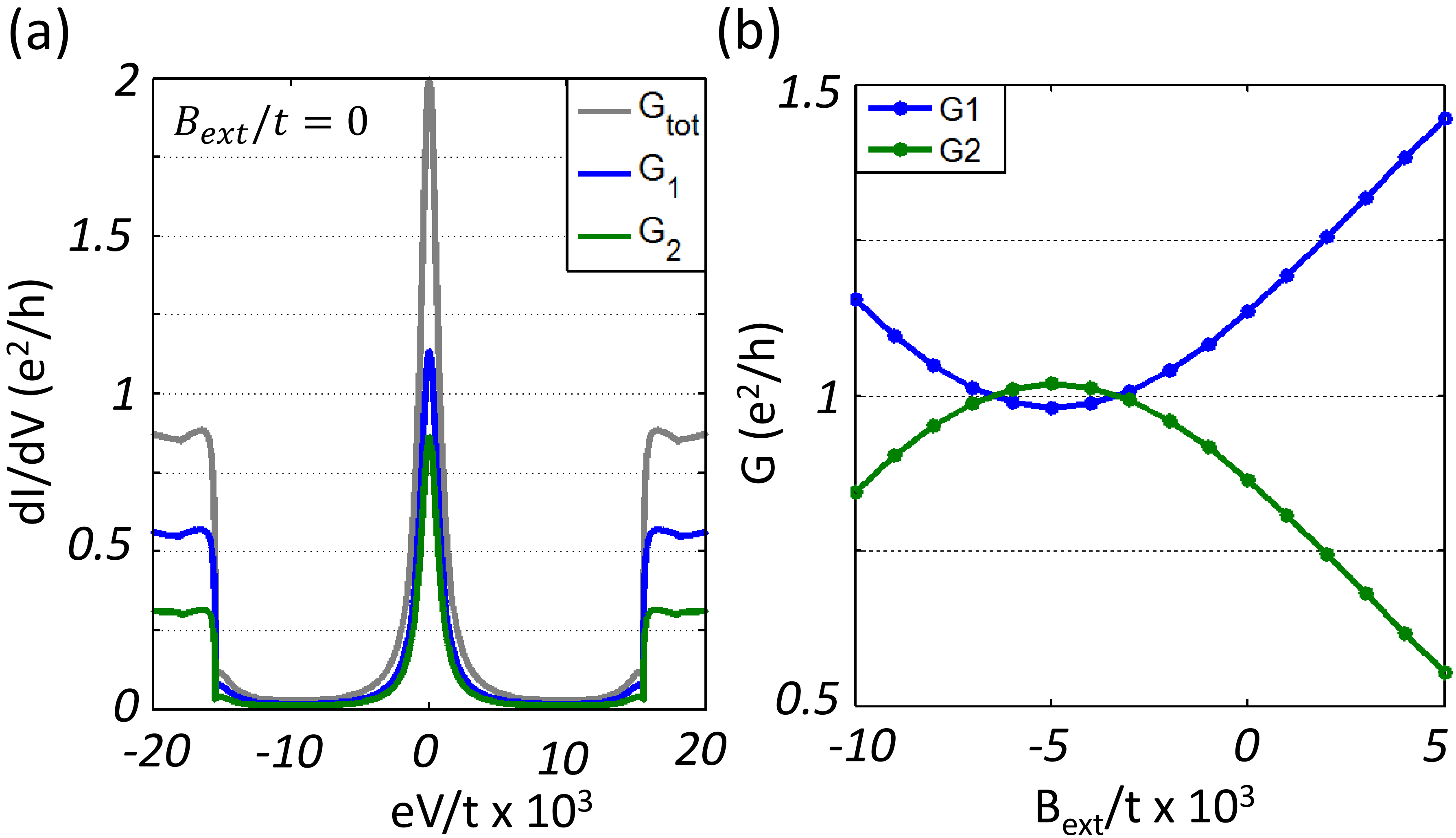
By letting B(x) and/or a gate voltage, μ(x) vary rapidly along wire near edge of superconductor, we can have 2 channels in the normal region.



We are calculating effective couplings of Majorana mode to 2 channels from conductance to each channel, in non-interacting case, obtained from BTK formula

$$G_i = \frac{e^2}{h}\left[1 + \sum_{j=1}^{2} \frac{v_j}{v_i}\left(\left|r_{ij}^{eh}\right|^2 - \left|r_{ij}^{ee}\right|^2\right)\right]$$

Here r_{ij}'s are reflection matrices for ee (normal) Andreev (Andreev) reflection, at Fermi energy. v_i are Fermi velocities in each channel. By tuning B, μ, we expect to be able to make $G_1=G_2$, corresponding to equal couplings to Majorana



(a)
$B_{ext}/t = 0$
G_tot
G_1
G_2
dI/dV (e²/h)
eV/t x 10³
(b)
G1
G2
G (e²/h)
B_ext/t x 10³

These considerations, and others, support universality of our phase diagram. A Majorana mode coupled to 2 Luttinger liquid channels acts as a switch, producing perfect Andreev reflection In 1 channel, perfect normal reflection in the other. Frustration, due to Majorana monogamy produces non-trivial critical point when both channels are coupled to the Majorana mode with equal strength.

1) I.A. & D. Giuliano, J. Stat.Mech. (2013) P06011
2) … arxiv 1404.0047 (to appear in J Stat. Phys.)
3) Y. Komijani and I.A., arxiv1408.3804.