Dirac Equation

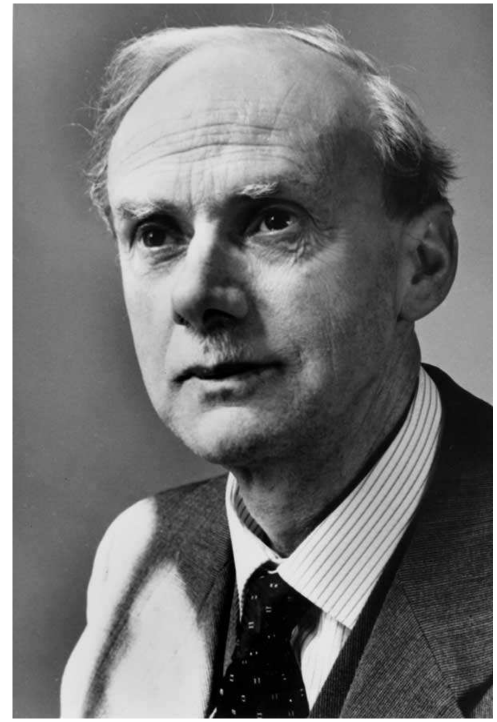
$$(\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)\Psi = i\,\frac{\partial}{\partial t}\,\Psi$$

$\boldsymbol{\alpha}, \beta \sim$ Dirac matrices, $\Psi \sim$ complex spinor, $\mathbf{p} = -i\nabla$

$$\Psi = e^{-iEt}\ \psi$$

$$(\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)\psi = E\,\psi$$

Dirac $\sim \pm\sqrt{-\nabla^2 + m^2}\ \Rightarrow E \lessgtr 0$



(diverse dimension d)

	particle physics:	condensed matter physics:
$E > 0$	particles	conduction electrons
$E < 0$	anti-particles	valence electrons
m	mass	gap

Extensions of Dirac equation (condensed matter)

(i) diverse dimensions ($d = 1, 2, 3$)

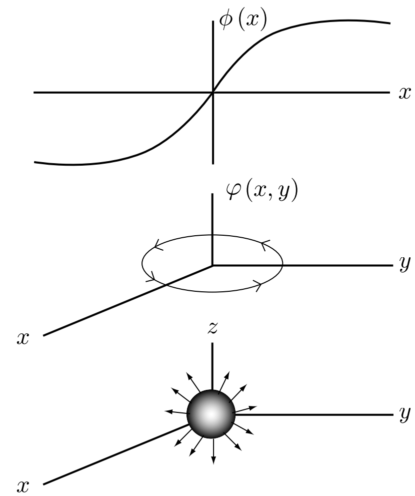
(ii) position dependent mass: $m \rightarrow \varphi(\mathbf{r})$

$\varphi(\mathbf{r})$ topologically non trivial

1-d kink

2-d vortex

3-d magnetic monopole



$\Rightarrow$ $(\boldsymbol{\alpha} \cdot \mathbf{p} + \beta\varphi(\mathbf{r}))\,\psi = E\psi$

$E \lesseqgtr$ as before

also $E = 0$ when $\varphi(\mathbf{r})$ is topologically non trivial

$Q_{E=0\text{state}} = (-/+)\frac{1}{2}$, empty/filled! [$d = 1, 3$ Rebbi & RJ (76)]

$Q = \pm\frac{1}{2}$ eigenvalue, not VEV!

EXPERIMENTAL USES OF FRACTIONAL CHARGE

$d = 1$ polyacetylene [Peierls' instability breaks Z_2 symmetry;

kink $\sim$ domain wall between phases;

Su, Schrieffer & Heeger (1979)]

$d = 2$ quantum Hall effect [not defect based]

$d = 2$ graphene [gapless Dirac equation: Semenoff (1985); manufacture and verification: Geim & Novoselov (2005); Kekulé distortion induces mass gap, breaks chiral symmetry, vortex interpolates; charge fractionalizes: Chamon *et al.* (2006), Jackiw & Pi (2007)].

Majorana Equation

Dirac equation describes charged fermions and anti-fermions by a complex field Ψ

In nature one observes uncharged, self-conjugate bosons that are their own anti-particles and are described by a real field

e.g. neutral pion	$S = 0$
photon	$S = 1$
[hypothetical] graviton	$S = 2$

All are bosons! Are there self-conjugate fermions?
Not yet seen in nature, but used in theoretical and experimental speculation: e.g.

- neutrinos may be self-conjugate particles;
- super symmetry partners of self conjugate bosons should be self conjugate fermions;
- speculation about dark matter

question: how to describe self-conjugate fermions?

answer: by a Dirac equation for REAL Ψ!!

Majorana Equation

$$(\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)\Psi = i \frac{\partial}{\partial t} \Psi$$

Ψ real $= \Psi^*$

$$\begin{array}{lll} \boldsymbol{\alpha} & \text{real} & = & \boldsymbol{\alpha}^* \\ \beta & \text{imaginary} & = & -\beta^* \end{array}$$

Majorana Representation

$$\alpha_M^1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \ \alpha_M^2 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \ \alpha_M^3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

$$\beta_M = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \qquad \Psi_M^* = \Psi_M$$

Equivalently $C\boldsymbol{\alpha}^* C^{-1} = \boldsymbol{\alpha},\ C\beta^* C^{-1} = -\beta \quad C\Psi^* = \Psi$

$C = 1$ in Majorana representation
e.g. Weyl representation

$$\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \mathcal{C} = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}$$

Weyl-Majorana $\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}$

$$\begin{pmatrix} \boldsymbol{\sigma} \cdot \boldsymbol{p} & m \\ m & -\boldsymbol{\sigma} \cdot \boldsymbol{p} \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = i\frac{\partial}{\partial t} \begin{pmatrix} \psi \\ \chi \end{pmatrix}$$

$$C\psi^* = \psi \;\Rightarrow\; \chi = i\sigma_2\psi^*$$

$$\boldsymbol{\sigma} \cdot \mathbf{p}\,\psi + i\sigma_2\, m\, \psi^* = i\frac{\partial}{\partial t}\psi \qquad (2 \times 2)$$

NB: Majorana mass term does not preserve any quantum numbers; there is no distinction between particle and anti-particle since there are no quantum numbers to tell them apart.

Field expansion for charged Dirac field

$$\Psi = \sum_{E>0} \left(a_E\, e^{-iEt}\, \Psi_E + b_E^\dagger\, e^{iEt}\, C\, \Psi_E^* \right)$$

with a annihilating and $b^\dagger$ creating particles and anti-particles respectively

Field expansion for self-conjugate Majorana field

$$\Psi = \sum_{E>0} \left(a_E\, e^{-iEt}\, \Psi_E + a_E^\dagger\, e^{iEt}\, C\, \Psi_E^* \right)$$

anti-particle operators $(b, b^\dagger)$ have disappeared!

Remarkable Fact:

Condensed Matter theorist have encountered essentially same equation in a description of a superconductor in contact with a topological insulator. The relevant 2-dimensional Hamiltonian reads

$$H = \psi^* \ (\boldsymbol{\sigma} \cdot \frac{1}{i}\ \boldsymbol{\nabla} - \mu)\ \psi + \frac{1}{2}\ (\triangle \psi^*\, i\, \sigma_2\, \psi^* + h.c.)$$

$\psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}, \boldsymbol{\sigma} = (\sigma_1, \sigma_2), \mu$ is chemical potential and $\triangle$ is the order parameter that may be constant: $\triangle = \triangle_0$ or takes vortex profile, $\triangle = v(r)e^{i\theta}, v(0) = 0, v(\infty) = \triangle_0$

Equation of motion

$$(\boldsymbol{\sigma} \cdot \mathbf{p} - \mu)\ \psi + \triangle\, i\, \sigma_2\, \psi^* = i\frac{\partial}{\partial t}\ \psi$$

$2 - d$ Majorana equation (with chemical potential)

Zero Mode

In the presence of a single vortex order parameter $\Delta(\mathbf{r}) = v(r)e^{i\theta}$ there exists a zero-energy (static) isolated mode
$\big[$Fu & Kane, *PRL* **100**, 096407 (08); Rossi & RJ *NPB* **190**, 681 (81)$\big]$

$$\psi_0 = \# \begin{pmatrix} J_0(\mu r)\,\mathrm{exp}\,\{-i\pi/4 - V(r)\} \\ J_1(\mu r)\,\mathrm{exp}\,\{i(\theta + \pi/4) - V(r)\} \end{pmatrix}$$

real constant, $V'(r) = v(r)$

Majorana field expansion:

$$\Psi = \underset{E\neq 0 \text{ modes}}{\ldots\ldots\ldots\ldots\ldots\ldots} + a\,\psi_0$$

where zero mode operator a satisfies

$$\{a, a^\dagger\} = 1, a^\dagger = a \Rightarrow a^2 = 1/2$$

How to realize a on states? Two Possibilities!

[Chamon, Nishida, Pi, Santos & RJ; *PRB* **81**, 224515 (10)]

(i) Two 1-dimensional realizations: take vacuum state to be eigenstate of a, with possible eigenvalue $\pm 1/\sqrt{2}$.

$$a\,|0\pm\rangle = \pm\,\frac{1}{\sqrt{2}}\,|0\pm\rangle$$

There are two ground states $|0+\rangle$ and $|0-\rangle$. Two towers of states are constructed by repeated application of $a_E^\dagger$. No operator connects the two towers.

Fermion parity is broken because a is a fermionic operator. Like in spontaneous breaking, a vacuum $|0+\rangle$ or $|0-\rangle$ must be chosen, and no tunneling connects to the other ground state.

(ii) One 2-dimensional realization: vacuum doubly degenerate $|1\rangle, |2\rangle$, and a connects the two vacua.

$$a\,|1\rangle = \tfrac{1}{\sqrt{2}}\,|2\rangle$$

$$a\,|2\rangle = \tfrac{1}{\sqrt{2}}\,|1\rangle$$

Two towers of states are constructed by repeated application of $a_E^\dagger$. a connects the towers. Fermion parity is preserved.

We shall assume that fermion parity is preserved, and adopt second possibility

- Curious fact in (1-d)

  total $\mathcal{L}$ for scalar kink $\oplus$ fermions

  $\mathcal{L} = \frac{1}{2}\,\partial_\mu \Phi\, \partial^\mu \Phi + \frac{\mu}{2}^2 \Phi^2 - \frac{\lambda}{8}^2 \Phi^4 \quad + i\bar{\Psi}\,\gamma^\mu\,\partial_\mu \Psi - g\,\Phi\,\bar{\Psi}\Psi$

  $\mathcal{L}$ possesses SUSY for $g = \lambda, \Psi$ Majorana

  Center anomaly in SUSY algebra $\Rightarrow$ fermion parity can be absent.

[Losev, Shifman & Vainshtein, *PLB* **522**, 327 (01)

Any relevance for condensed matter?

Semenoff & Sodano, *EJTP* **10**, 57 (08)]

Multiple Vortices

With N vortices, governed by operators $a_1, a_2, \ldots, a_N$ that satisfy

$$\{a_i, a_j\} = 2\,\delta_{ij} \qquad \text{(Clifford algebra)}$$

one can show that one needs

$$\mathcal{N} = 2^{\frac{N}{2}} \quad \text{states for even } N$$

$$\text{and} \quad \mathcal{N} = 2^{\frac{N+1}{2}} \quad \text{states for odd } N$$

$N = 1$	$\mathcal{N} = 2$	σ_1 or σ_2 (not σ_3)	(2×2)
$N = 2$	$\mathcal{N} = 2$	σ_1 and σ_2 (not σ_3)	(2×2)
$N = 3$	$\mathcal{N} = 4$	α_1 α_2 α_3 or β (not diagonal)	(4×4)
$N = 4$	$\mathcal{N} = 4$	α_1 α_2 α_3 and β (not diagonal)	(4×4)
	etc.		

Clifford algebra, with a restriction: use for a_i Pauli, Dirac, . . . , matrices excluding diagonal one since it would correspond to diagonalizing a mode operator and would produce fermion parity violation [Pi & RJ, *PRB* **85**, 033102 (12)]