Phase-junctions in one-dimensional topological superconductors

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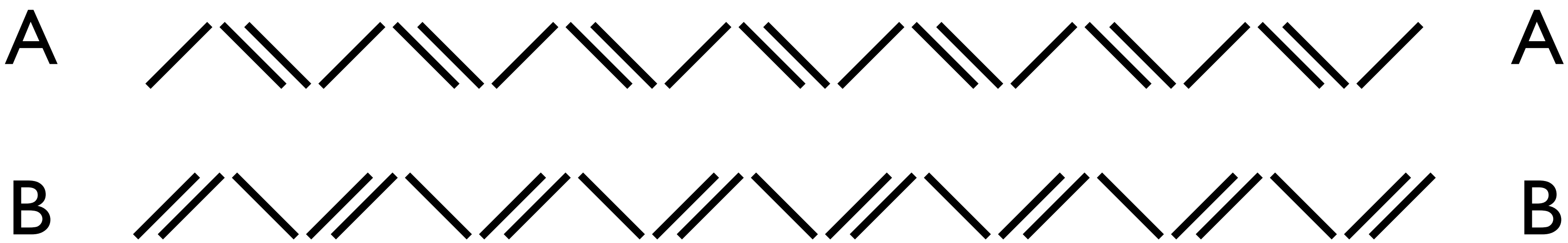
Some history: polyacetylene

The solitons in the Su-Schrieffer-Heeger model



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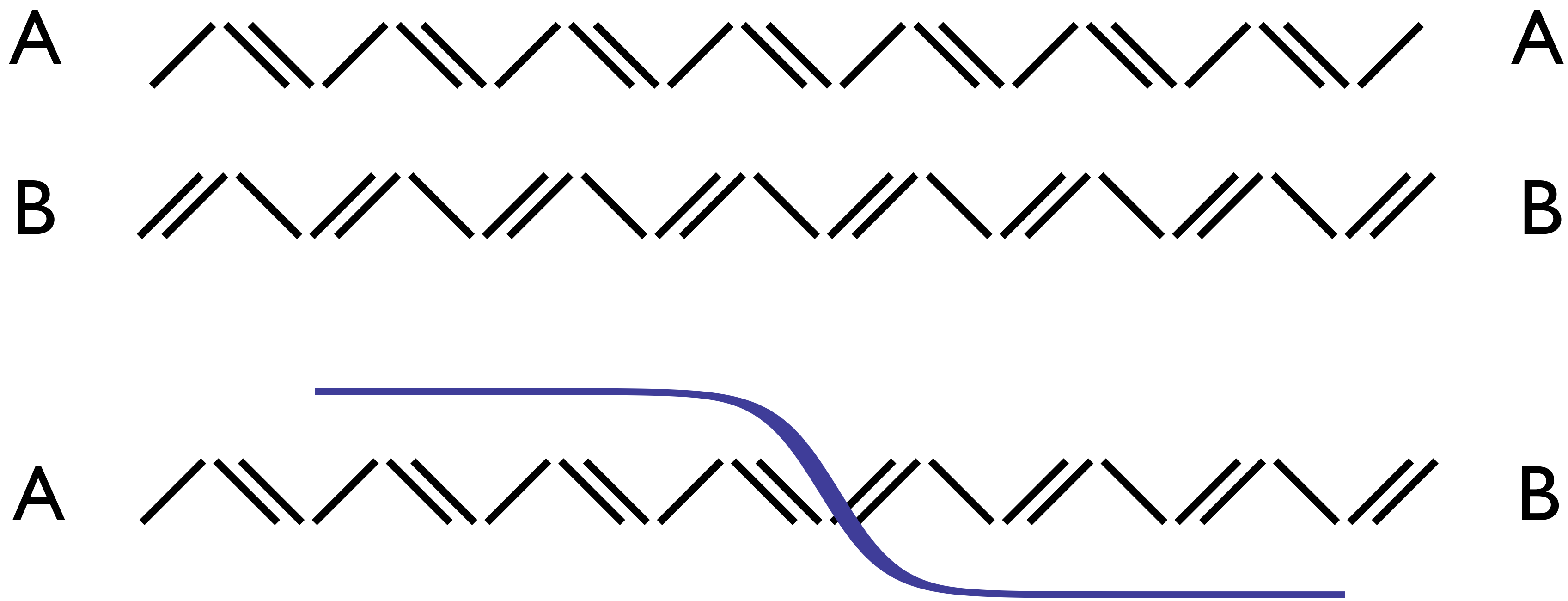

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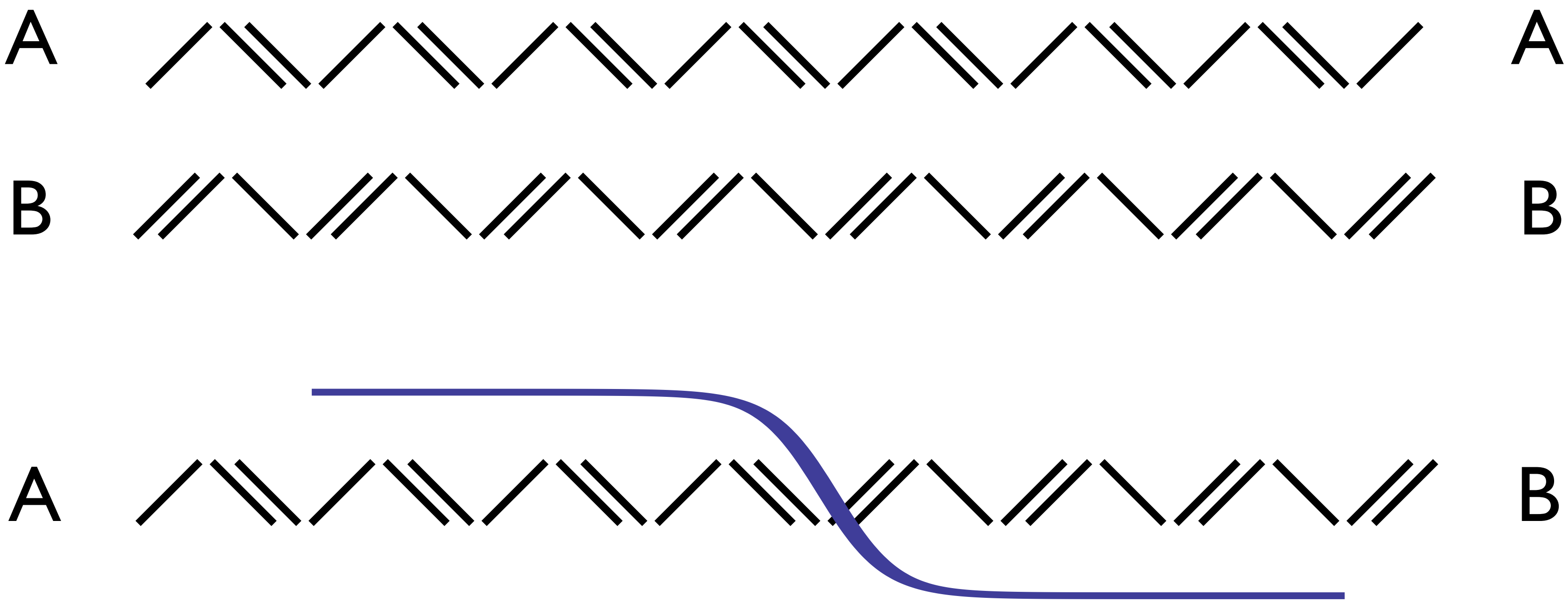


Takayama, Lin-Liu & Maki (1980) provided an exact solution for the zero-energy bound state in the junction (linearized model).

Not so long ago: Majorana modes?

Kitaev's topological p-wave superconductor has Majorana bound states

Kitaev (2001)

Model can be realized:

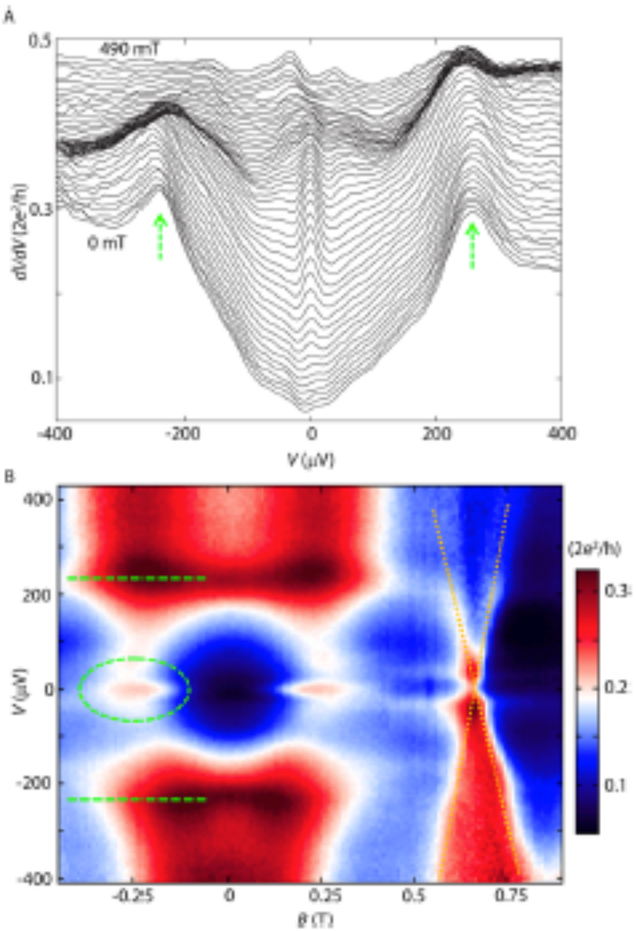
- strong spin-orbit coupled wire
- superconductor
- magnetic field

Lutchyn et al.,
Oreg et al., (2010)

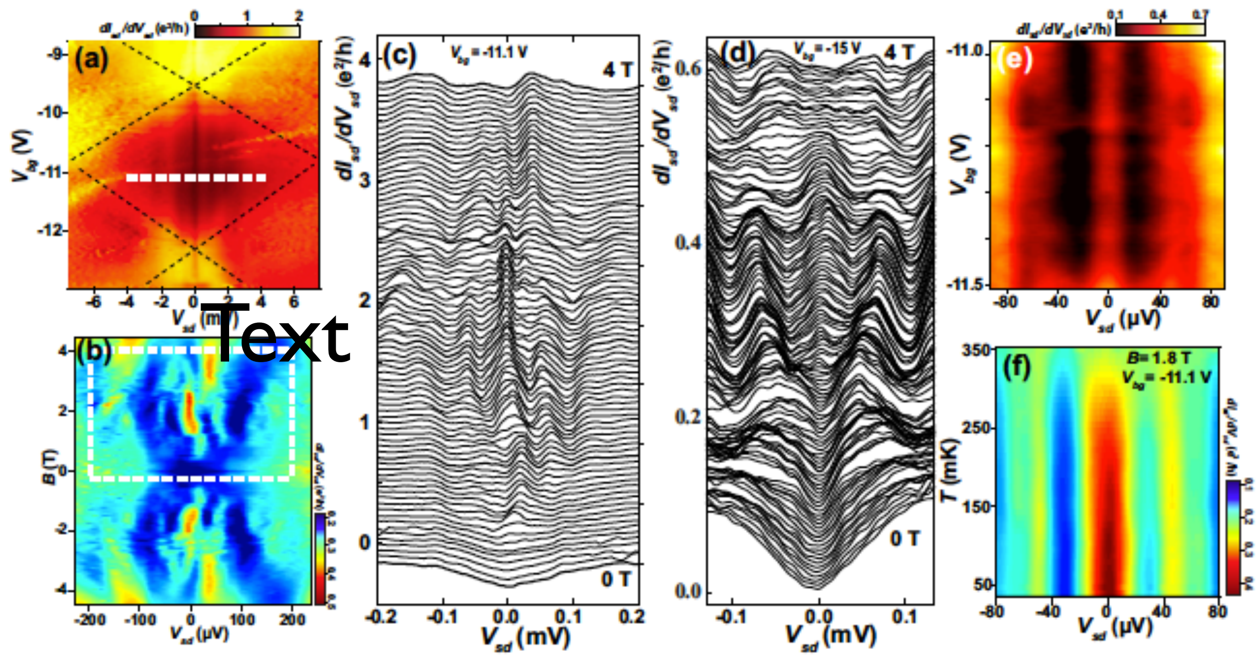
Detecting the zero energy Majorana bound state: tunnel electrons in the wire!

Observation of zero modes

Several groups (Delft, Lund, Weizmann, ...) have observed zero modes with promising properties



Mourik et al. (2012)



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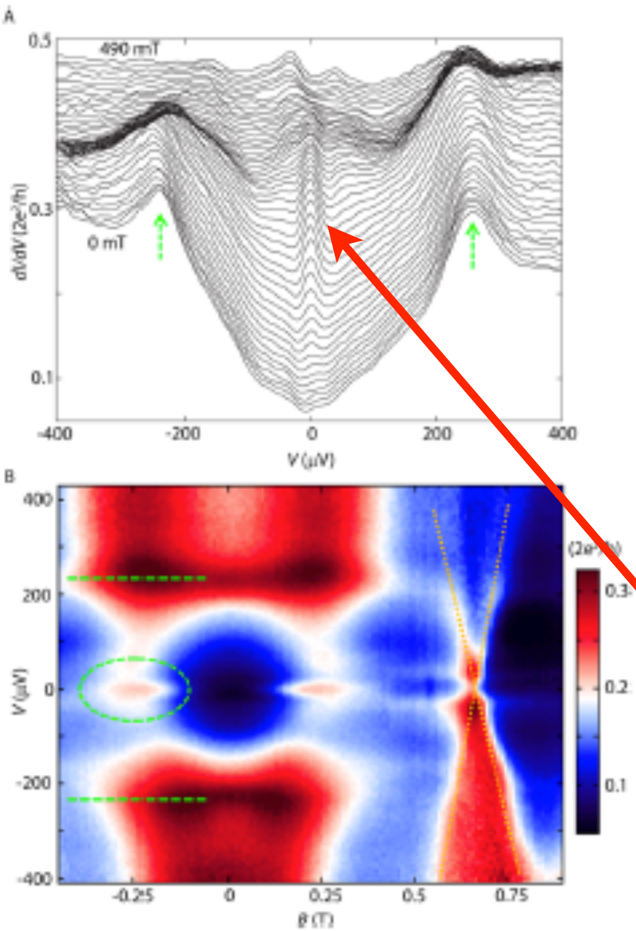

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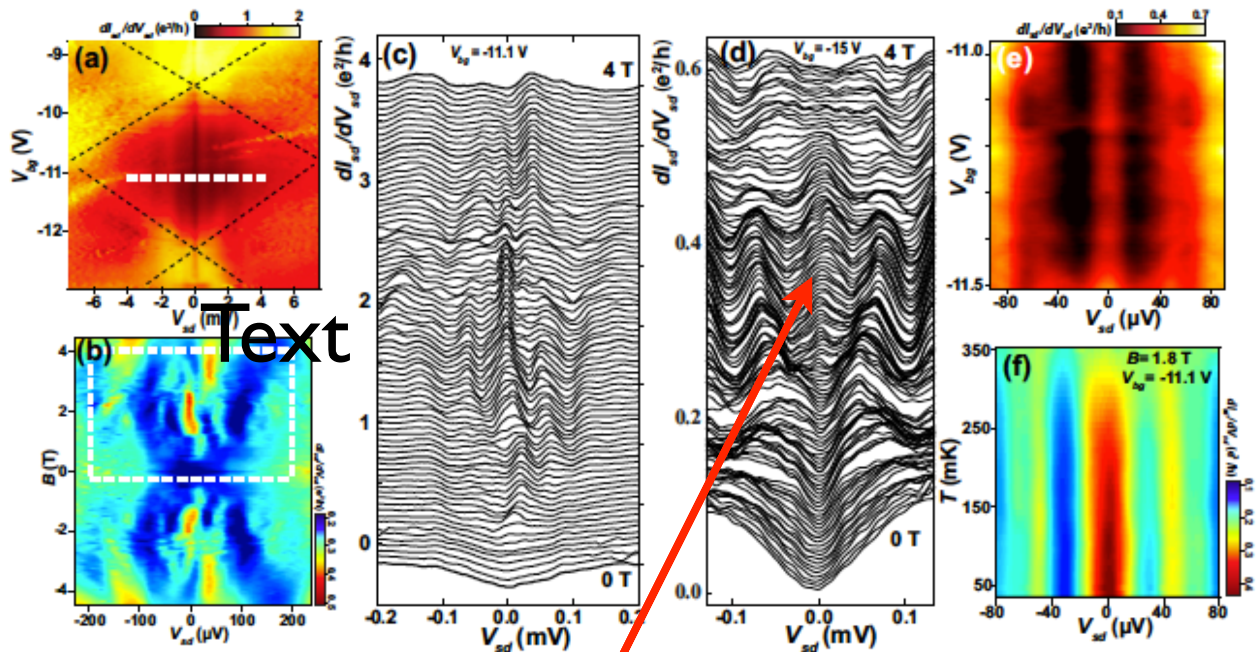

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Zero mode!

Smoking gun?

Maybe not...

Build up evidence in small steps?

Build up evidence in small steps?

Outline

- Kitaev's Majorana chain
- Symmetries of the model & topological invariants
- Real junctions: comparison to the SSH model
- Phase winding junctions: sub-gap states

Kitaev's Majorana chain

In terms of the (polarized) fermions c_i, Kitaev's model reads

$$H_{\text{Kitaev}} = \sum_i -\mu\left(c_i^\dagger c_i - c_i c_i^\dagger\right) - t\left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i\right) + \left(\Delta_i c_i c_{i+1} + \Delta_i^* c_{i+1}^\dagger c_i^\dagger\right)$$

This model belongs to class D of the classification of free fermion systems

Kitaev; Ryu et al.; c.f. Altland & Zirnbauer

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$$H_{\text{Kitaev}}(k) = \big(-\mu - t\cos(k)\big)\tau_z - \Re(\Delta)\sin(k)\tau_y + \Im(\Delta)\sin(k)\tau_x$$

τ_α: Pauli matrices in particle-hole space

Anti-unitary particle-hole 'symmetry' $\mathcal{C} = \tau_x K$, $\mathcal{C}^2 = 1$
anti-commutes with the hamiltonian.

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$\mathcal{C}$ is not a real symmetry here, but a redundancy in the description, which can not be broken!

Periodic table of topological phases

Classification of topological phases of systems with time-reversal $\mathcal{T}$, particle-hole $\mathcal{C}$, and/or chiral symmetry $U_{CS} = \mathcal{T} \circ \mathcal{C}$

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Time reversal and particle-hole are anti-unitary, $\mathcal{T} = U_T K$, $\mathcal{C} = U_C K$

satisfying $[H, \mathcal{T}] = 0$, $\{H, \mathcal{C}\} = 0$ and $\mathcal{T}^2 = \pm 1$, $\mathcal{C}^2 = \pm 1$

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This gives ten different classes, with three possibilities for $\mathcal{T}$ and $\mathcal{C}$.
If both are absent, the system can be chiral or not.

Periodic table of topological phases

Class	$\mathcal{T}$	$\mathcal{C}$	U_{CS}	$d=1$	$d=2$	$d=3$	$d=4$
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0
BDI	+1	+1	1	$\mathbb{Z}$	0	0	0
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
DIII	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
AII	−1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CII	−1	−1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
C	0	−1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
CI	+1	−1	1	0	0	$\mathbb{Z}$	0
AI	+1	0	0	0	0	0	$\mathbb{Z}$

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AII	−1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CII	−1	−1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
C	0	−1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
CI	+1	−1	1	0	0	$\mathbb{Z}$	0
AI	+1	0	0	0	0	0	$\mathbb{Z}$

Kitaev chain, general couplings

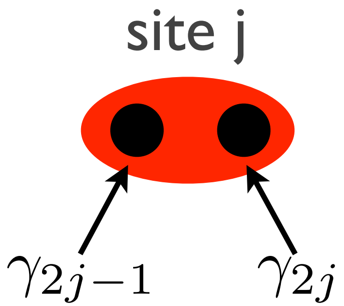
Kitaev chain, real couplings

Kitaev's Majorana chain

To reveal the topological nature of Kitaev's chain

$$H_{\text{Kitaev}} = \sum_j -\mu(c_j^\dagger c_j - c_j c_j^\dagger) - t(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + \Delta(c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger)$$

we introduce Majorana fermions:



$$\gamma_{2j-1} = c_j + c_j^\dagger \ , \ \gamma_{2j} = -i(c_j - c_j^\dagger)$$

$$\{\gamma_i, \gamma_j\} = 2\delta_{i,j} \ , \ \gamma_i^\dagger = \gamma_i$$

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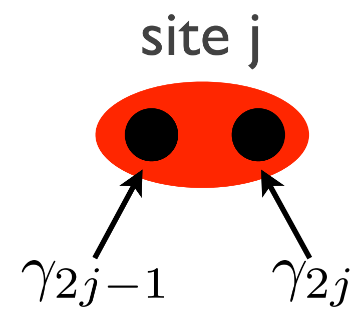

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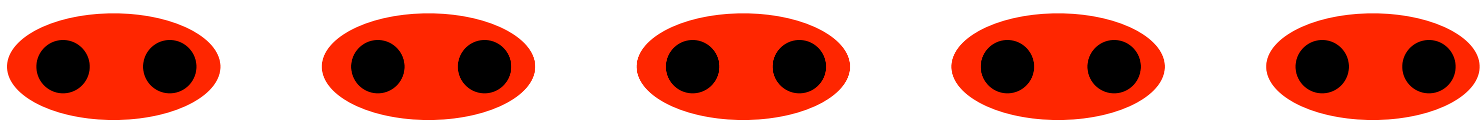

$$H_{\text{Kitaev}} = \frac{i}{2} \sum_j -(2\mu)\gamma_{2j-1}\gamma_{2j} + (t+\Delta)\gamma_{2j}\gamma_{2j+1} + (-t+\Delta)\gamma_{2j-1}\gamma_{2j+2}$$

Kitaev's Majorana chain: two phases

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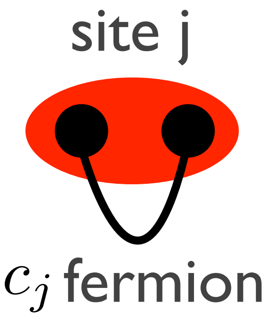
For $t = \Delta = 0$ and $\mu < 0$, one finds a trivial phase:

$$H_{\text{Kitaev}} = -i\mu\sum_j \gamma_{2j-1}\gamma_{2j} = -\mu\sum_j (2c_j^\dagger c_j - 1)$$



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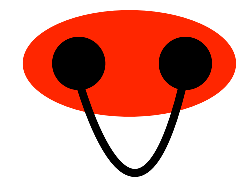
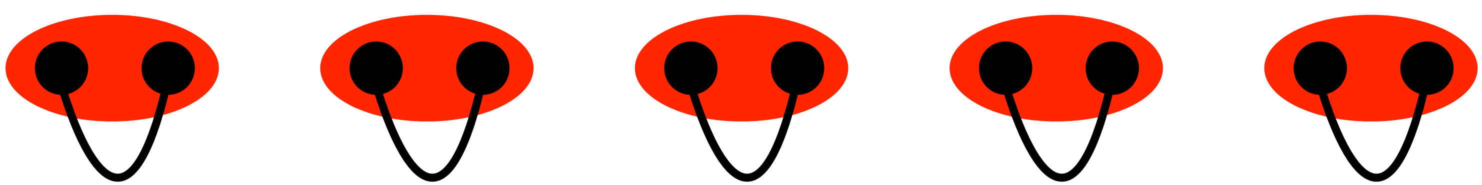


$$H_{\text{Kitaev}} = -i\mu\sum_j \gamma_{2j-1}\gamma_{2j} = -\mu\sum_j (2c_j^\dagger c_j - 1)$$

Majorana's from the same site are paired together!



The ground state is completely empty (filled for $\mu > 0$)

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For $t = \Delta > 0$ and $\mu = 0$, one finds a non-trivial phase:

$$H_{\text{Kitaev}} = it\sum_j \gamma_{2j}\gamma_{2j+1} = t\sum_j (2\tilde{c}_j^\dagger \tilde{c}_j - 1)$$

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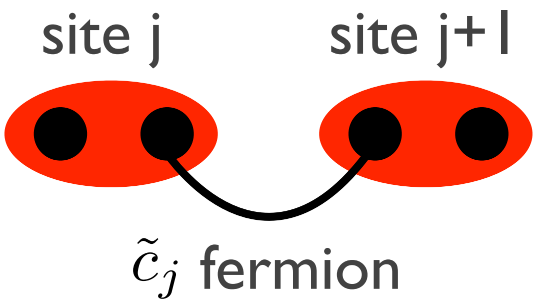

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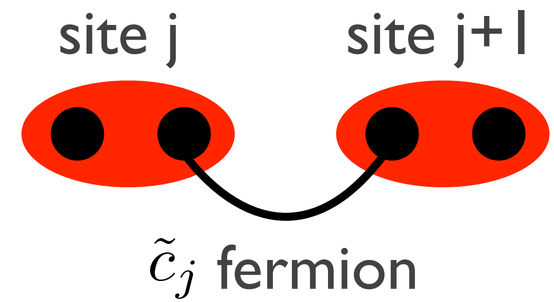

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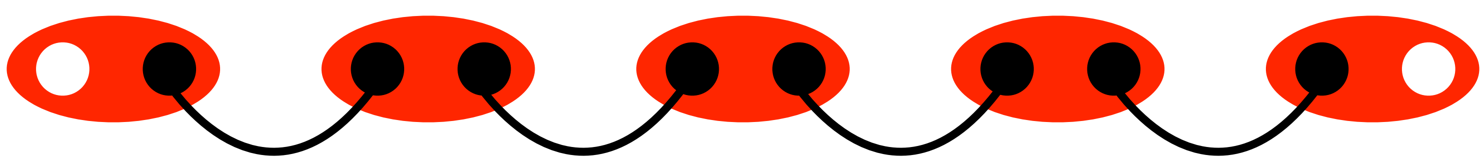
Majorana's from neighbouring sites are paired together!



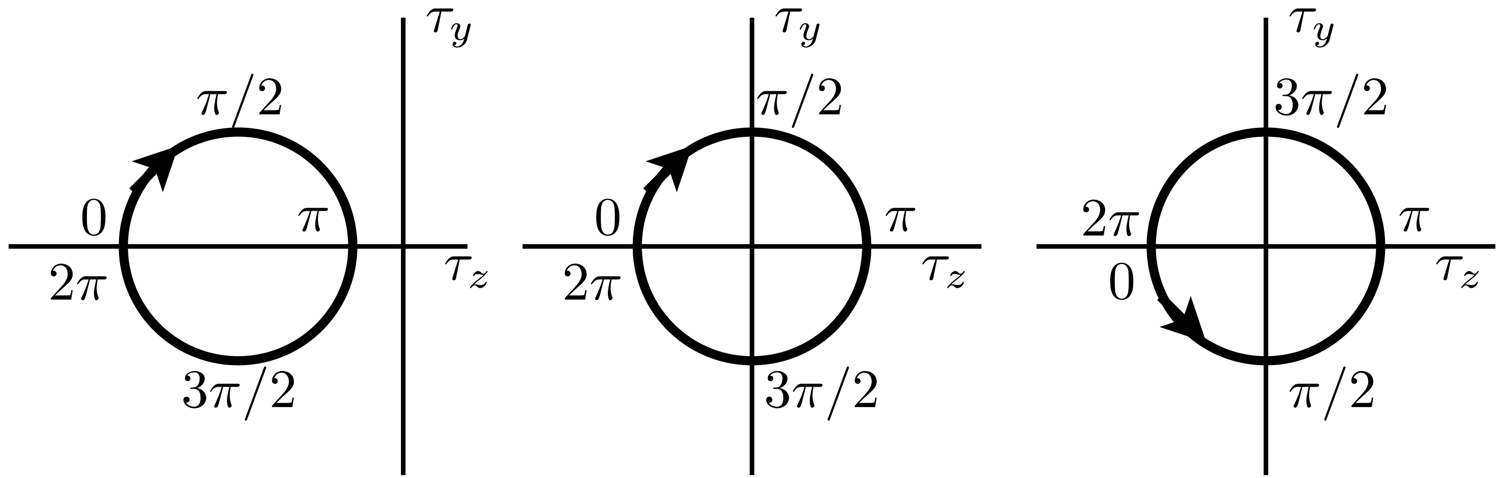
The Majorana's at the end are left unpaired. They combine into one, non-local fermionic mode **with zero energy**, which can be either filled or empty.

Invariant: real order parameter

$$\mathcal{H}(k) = \vec{d}(k) \cdot \vec{\tau}$$

$$\vec{d}(k) = (0, -\Re(\Delta)\sin(k), -\mu - t\cos(k))$$

Topological invariant is given by the winding number around the origin:



trivial, ν=0

topological, ν= -1 Δ>0

topological, ν= -1 Δ<0

The other possible values of ν are not realized in Kitaev’s model

Invariant: general order parameter

$$\mathcal{H}(k) = \vec{d}(k) \cdot \vec{\tau}$$

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Winding around the origin is no-longer defined...

The Hamiltonian at the real momenta determine the invariant.

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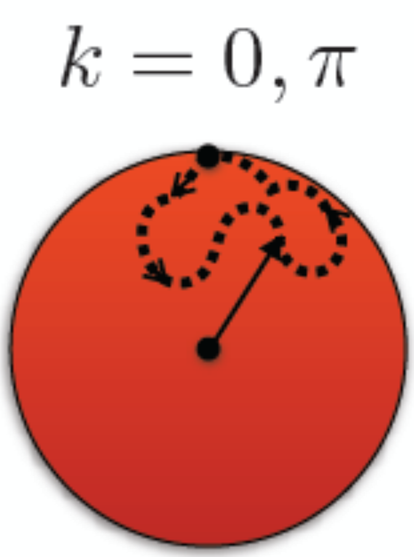
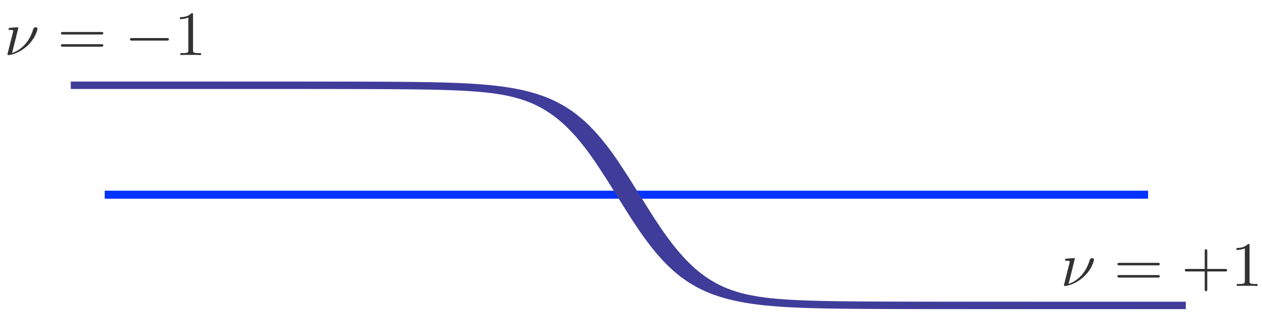


trivial

topological

Real junction: zero mode (Dirac)

Real junction separates two different topological phases:



$$\Delta(x) = -\Delta_0 \tanh(x/\xi)$$

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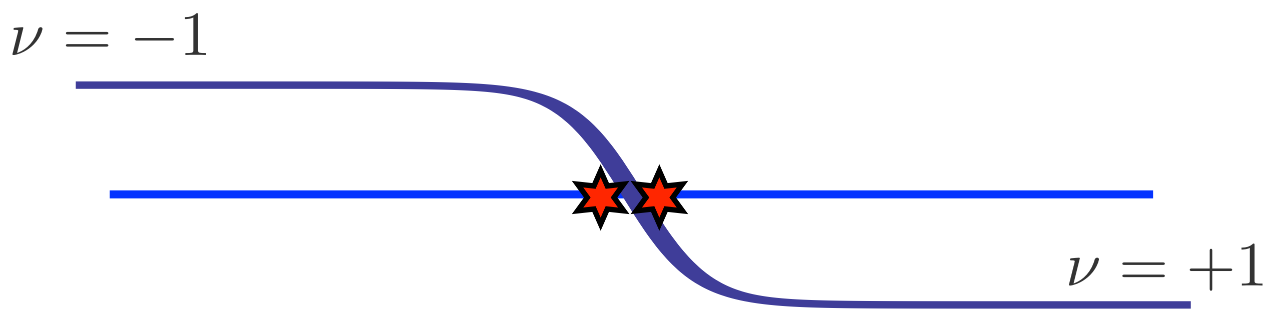

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Two zero modes (1 Dirac fermion) are localized at the junction

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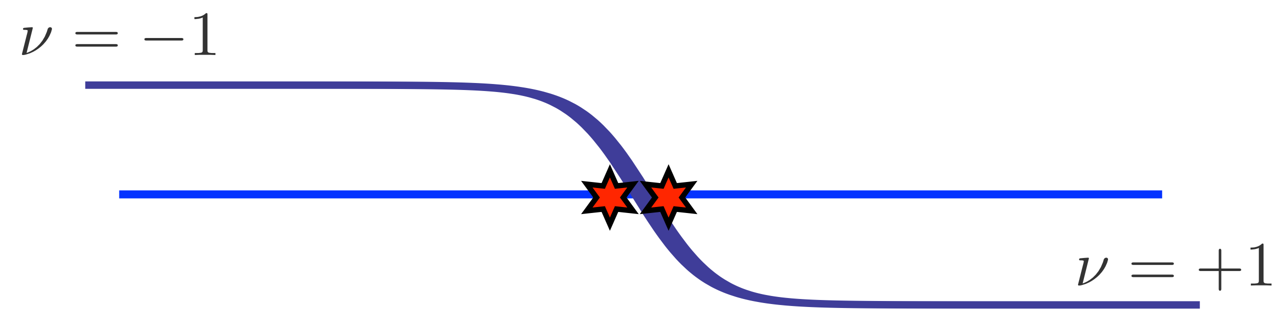

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Analytic solution in the linearized SSH model:

$$\mathcal{H} = -iv_F \tau_z \partial_x + \Delta(x)\tau_x$$

$$-iv_F \partial_x u(x) + \Delta(x) v(x) = \epsilon u(x)$$
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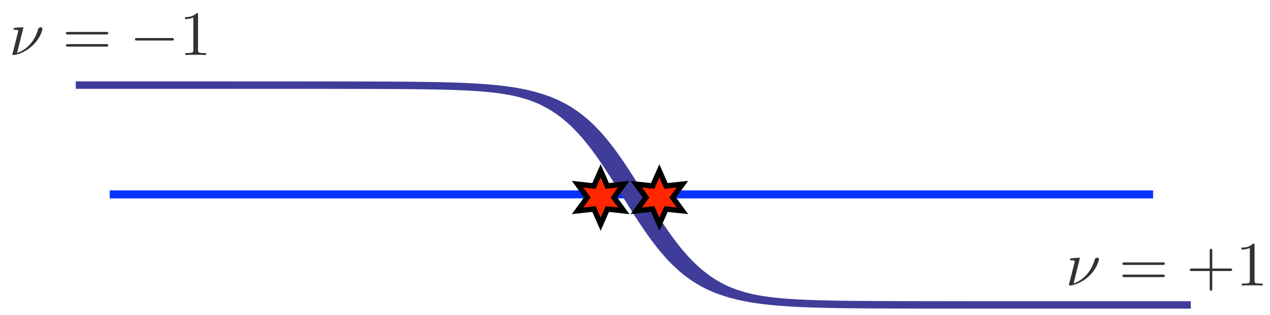

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Zero energy solution:

$$|\psi_0(x)|^2 = 1/\xi_0 \cosh^{-2}(x/\xi_0) \quad \xi_0 = v_F/\Delta_0$$

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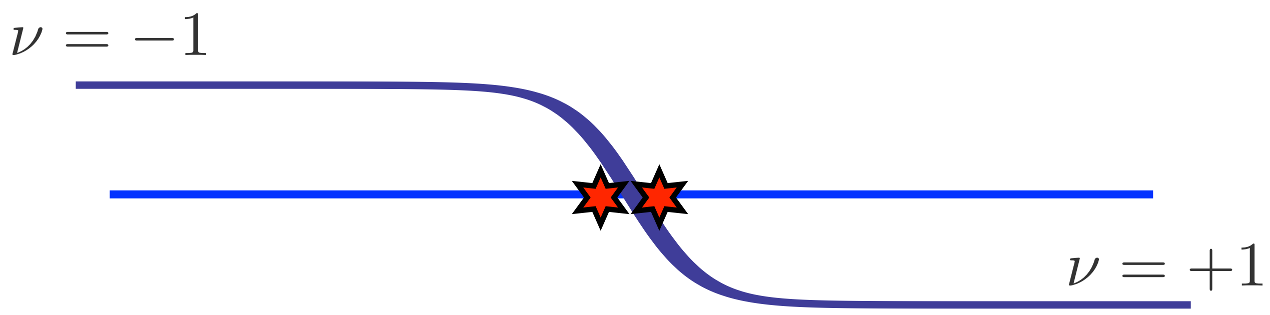

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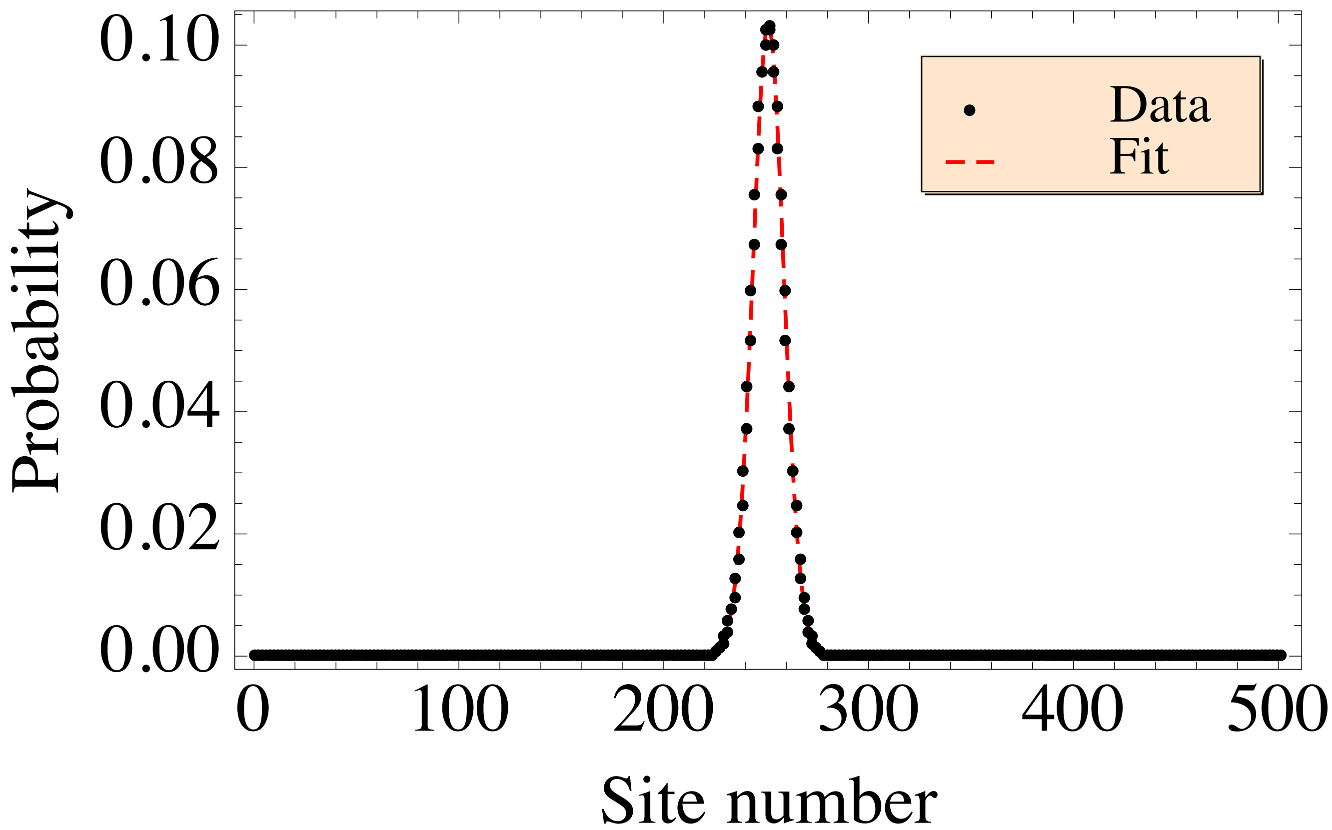
General profile: $|\psi_0(x)|^2 \propto e^{-1/v_F \int^x \Delta(x')dx'}$

Real junction in Kitaev's chain

Zero energy bound state, compared to the analytic solution in the SSH model



t=5.0, Δ=0.7, μ=0.0

Phase winding junctions

Alternative way to connect Δ=+1 to Δ=-1 region:

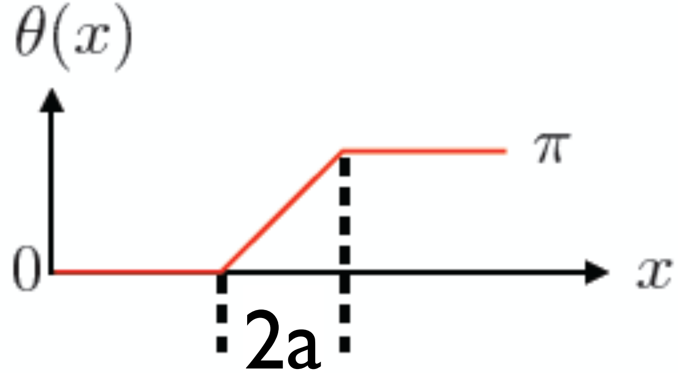
Wind the phase of the order parameter, with constant amplitude:

$$\Delta(x) = |\Delta_0| e^{i\theta(x)}$$

Pseudo-time reversal symmetry is broken, so no protected zero modes!

However, one still expect localized sub-gap modes

BdG equations do not decouple, so consider simple profile:



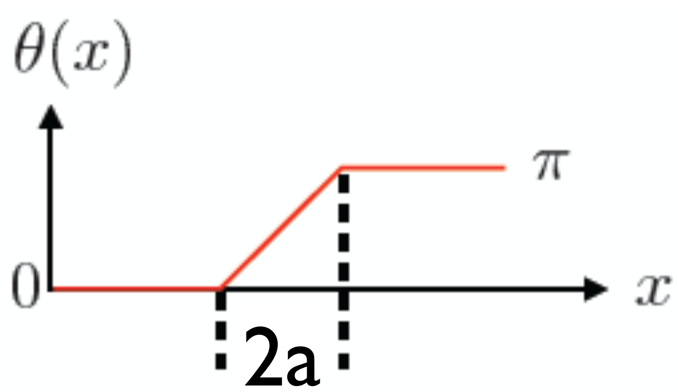
In the short junction limit, one reduces to a real π-junction!

Subgap states in phase junctions

BdG equations reduce to

$$\left(\partial_x^2 + i(\partial_x\theta(x))\partial_x + \tilde{\epsilon}\partial_x\theta(x) + (\tilde{\epsilon}^2 - 1/\xi_0^2)\right)u(x) = 0$$

Matching boundary conditions give a ‘particle in a box’ type equation.



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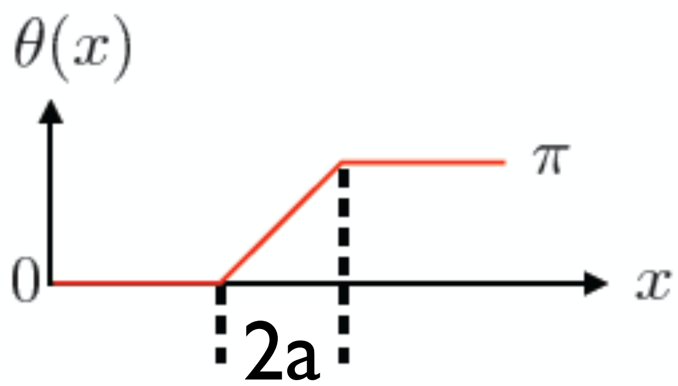
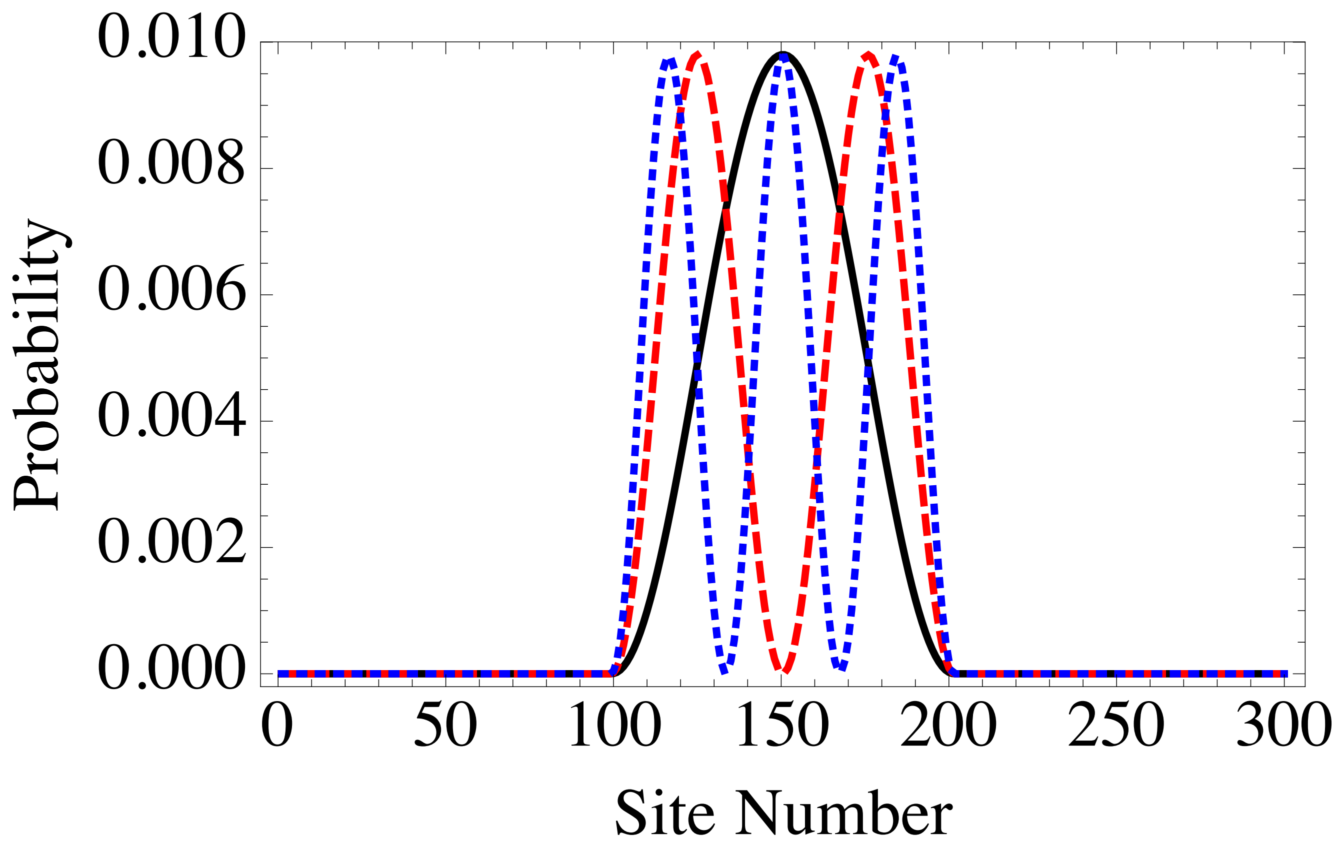
Properties of the sub-gap modes:

- Always at least one sub-gap states; longer junctions: more modes
- short junction limit: ε=0 mode (π-junction), $\varepsilon=\Delta$ (2π-junction)



$$\Delta(x) = |\Delta_0|e^{i\theta(x)}$$

Subgap states: some examples

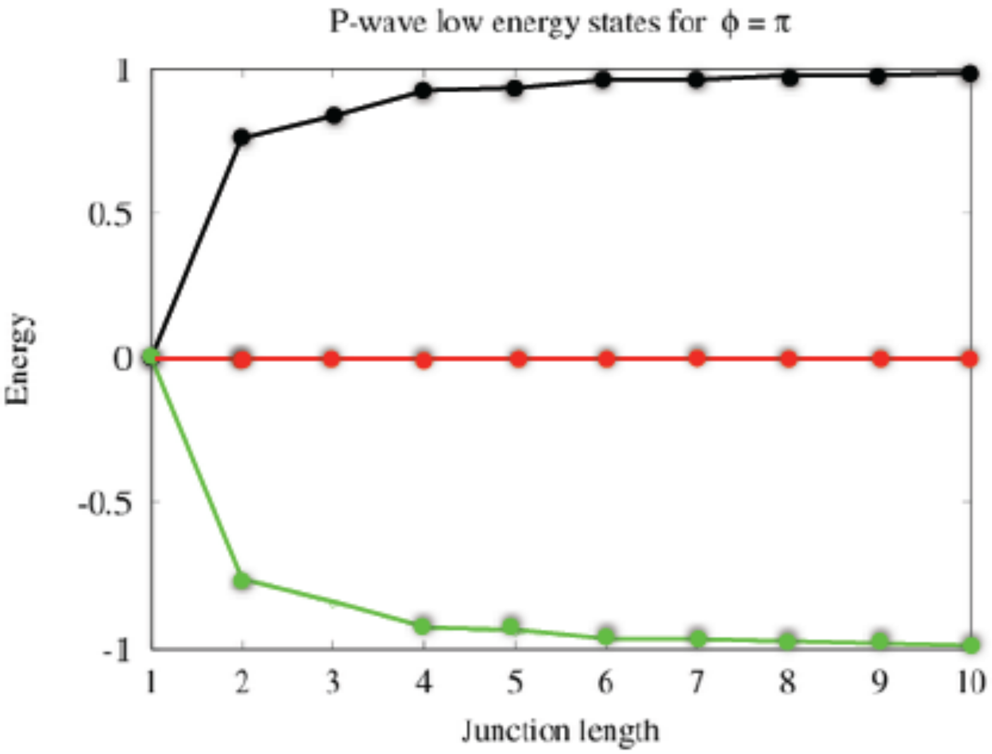


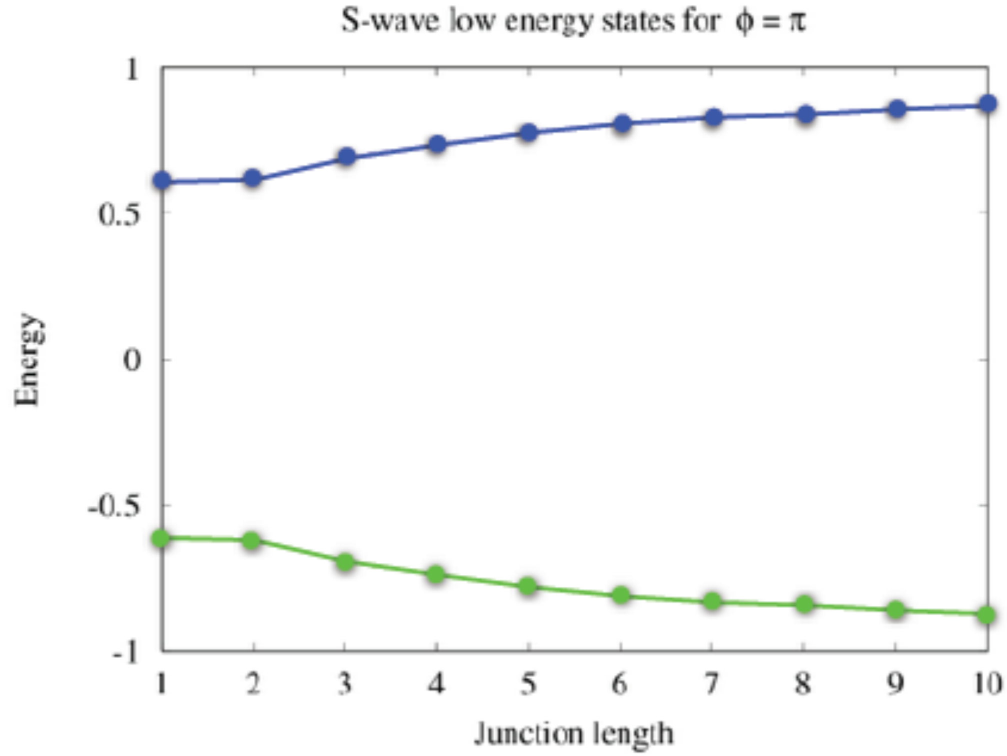
t=1.0, Δ=1.0, μ=0.0

Topological vs. trivial superconductor

Can we distinguish topological from trivial superconductors?

Energy of the sub-gap state behaves differently in the limit of a short junction:





Conclusion

- We considered real and phase-winding junctions in the Kitaev wire
- Studied the sub-gap states in the phase-winding junctions
- Sub-gap states behave differently in topological and trivial superconductors
- Constructed effective field theory, cf. Jackiw-Rebbi & Goldstone-Wilczek (not discussed here)
- For the future: study realistic setups

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Thank you for your attention!