Comments on the q-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring

Sergey Frolov

with Gleb Arutyunov and Riccardo Borsato: 1312.3542 and in progress

Hamilton Mathematics Institute and School of Mathematics, Trinity College Dublin
and
Institut für Mathematik und Institut für Physik, Humboldt-Universität zu Berlin
and
Steklov Mathematical Institute, Moscow

Supersymmetric Field Theories, Nordita, August 15, 2014

Outline

1. Introduction
2. $(AdS_5 \times S^5)_q$
3. Small $\varkappa$ limit
4. Large $\varkappa$ limit
5. Folded string
6. RR-fields
7. L.c. gauge and S-matrix
8. Summary

Motivation

"Harmonic oscillator" of AdS/CFT: Maldacena '97

$$\text{IIB strings in } \mathrm{AdS}_5 \times \mathrm{S}^5 \quad \Longleftrightarrow \quad \mathcal{N} = 4\ SU(N)\ \text{SYM}$$

- In planar limit one gets *integrable* quantum string sigma model
- Energies of string states $\equiv$ Scaling dimensions of SYM operators
- Integrable deformations
  - Orbifolds
  - γ-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ (TsT-transformed)
  - Both can be described by the *same* $\mathrm{AdS}_5 \times \mathrm{S}^5$ action but with twisted boundary conditions for the world-sheet fields
- New type of deformation with real parameter Delduc, Magro, Vicedo '13
- Uses solutions of classical YBE Klimcik '08
- We show that the (bosonic part of the) world-sheet S-matrix coincides with the $\mathfrak{psu}_q(2|2) \oplus \mathfrak{psu}_q(2|2)$-invariant S-matrix
- $\mathfrak{psu}(2,2|4) \longmapsto \mathfrak{psu}_q(2,2|4) \supset \mathfrak{psu}_q(2|2) \oplus \mathfrak{psu}_q(2|2)$ Delduc, Magro, Vicedo '14
- Dual field theory is not Lorentz invariant.

  New type of noncommutative field theory?

Motivation

"Harmonic oscillator" of AdS/CFT: Maldacena '97

$$\text{IIB strings in } AdS_5 \times S^5 \iff \mathcal{N} = 4\ SU(N)\ \text{SYM}$$

- In planar limit one gets *integrable* quantum string sigma model
- Energies of string states $\equiv$ Scaling dimensions of SYM operators
- Integrable deformations
  - Orbifolds
  - γ-deformed $AdS_5 \times S^5$ (TsT-transformed)
  - Both can be described by the *same* $AdS_5 \times S^5$ action but with twisted boundary conditions for the world-sheet fields
- New type of deformation with real parameter Delduc, Magro, Vicedo '13
- Uses solutions of classical YBE Klimcik '08
- We show that the (bosonic part of the) world-sheet S-matrix coincides with the $\mathfrak{psu}_q(2|2) \oplus \mathfrak{psu}_q(2|2)$-invariant S-matrix
- $\mathfrak{psu}(2,2|4) \longmapsto \mathfrak{psu}_q(2,2|4) \supset \mathfrak{psu}_q(2|2) \oplus \mathfrak{psu}_q(2|2)$ Delduc, Magro, Vicedo '14
- Dual field theory is not Lorentz invariant.

  New type of noncommutative field theory?

Motivation

"Harmonic oscillator" of AdS/CFT: Maldacena '97

$$\text{IIB strings in } \mathrm{AdS}_5 \times \mathrm{S}^5 \quad \Longleftrightarrow \quad \mathcal{N} = 4\ SU(N)\ \text{SYM}$$

- In planar limit one gets *integrable* quantum string sigma model
- Energies of string states $\equiv$ Scaling dimensions of SYM operators
- Integrable deformations
  - Orbifolds
  - γ-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ (TsT-transformed)
  - Both can be described by the *same* $\mathrm{AdS}_5 \times \mathrm{S}^5$ action but with twisted boundary conditions for the world-sheet fields
- New type of deformation with real parameter Delduc, Magro, Vicedo '13
- Uses solutions of classical YBE Klimcik '08
- We show that the (bosonic part of the) world-sheet S-matrix coincides with the $\mathfrak{psu}_q(2|2) \oplus \mathfrak{psu}_q(2|2)$-invariant S-matrix
- $\mathfrak{psu}(2,2|4) \longmapsto \mathfrak{psu}_q(2,2|4) \supset \mathfrak{psu}_q(2|2) \oplus \mathfrak{psu}_q(2|2)$ Delduc, Magro, Vicedo '14
- Dual field theory is not Lorentz invariant.

  New type of noncommutative field theory?

Motivation

"Harmonic oscillator" of AdS/CFT: Maldacena '97

$$\text{IIB strings in } \mathrm{AdS}_5 \times \mathrm{S}^5 \quad \Longleftrightarrow \quad \mathcal{N} = 4 \; SU(N) \text{ SYM}$$

- In planar limit one gets *integrable* quantum string sigma model
- Energies of string states $\equiv$ Scaling dimensions of SYM operators
- Integrable deformations
  - Orbifolds
  - γ-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ (TsT-transformed)
  - Both can be described by the *same* $\mathrm{AdS}_5 \times \mathrm{S}^5$ action but with twisted boundary conditions for the world-sheet fields
- New type of deformation with real parameter Delduc, Magro, Vicedo '13
- Uses solutions of classical YBE Klimcik '08
- We show that the (bosonic part of the) world-sheet S-matrix coincides with the $\mathfrak{psu}_q(2|2) \oplus \mathfrak{psu}_q(2|2)$-invariant S-matrix
- $\mathfrak{psu}(2,2|4) \longmapsto \mathfrak{psu}_q(2,2|4) \supset \mathfrak{psu}_q(2|2) \oplus \mathfrak{psu}_q(2|2)$ Delduc, Magro, Vicedo '14
- Dual field theory is not Lorentz invariant.

  New type of noncommutative field theory?

Matrix realisation of $\mathfrak{su}(2,2|4)$

$$M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix}$$

1. $\mathrm{str}\, M \equiv \mathrm{tr}\, m - \mathrm{tr}\, n = 0$
2. $M^\dagger H + HM = 0\,, \quad H = \begin{pmatrix} \Sigma & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix}\,, \quad \Sigma = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$
3. $m^\dagger = -\Sigma\, m\, \Sigma\,, \quad n^\dagger = -n\,, \quad \eta^\dagger = -\Sigma\, \theta\,,$
   $m \in \mathfrak{u}(2,2)\,, \qquad n \in \mathfrak{u}(4)$
4. Bosonic subalgebra of $\mathfrak{su}(2,2|4)$: $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1)$
5. $\mathfrak{psu}(2,2|4)$ is a *quotient algebra* of $\mathfrak{su}(2,2|4)$ over this $\mathfrak{u}(1)$

Z_4-grading

$$M = M^{(0)} + M^{(1)} + M^{(2)} + M^{(3)}$$

- $$M^{(0,2)} = \frac{1}{2}\begin{pmatrix} m \mp K m^t K^{-1} & 0 \\ 0 & n \mp K n^t K^{-1} \end{pmatrix}$$
  $$M^{(1,3)} = \frac{1}{2}\begin{pmatrix} 0 & \theta \mp iK\eta^t K^{-1} \\ \eta \pm iK\theta^t K^{-1} & 0 \end{pmatrix}$$
  $$K = \mathrm{diag}(-i\sigma_2, -i\sigma_2)$$
- $M^{(0)}$ and $M^{(2)}$ are even; $M^{(1)}$ and $M^{(3)}$ are odd
- $M^{(0)}$ form $\mathfrak{so}(4,1) \oplus \mathfrak{so}(5) \subset \mathfrak{su}(2,2) \oplus \mathfrak{su}(4)$.
- $M^{(2)}$ form the tangent space of the coset $\mathrm{SU}(2,2) \times \mathrm{SU}(4)/\mathrm{SO}(4,1) \times \mathrm{SO}(5) = \mathrm{AdS}_5 \times \mathrm{S}^5$
- $[M_1^{(k)}, M_2^{(m)}] = M_3^{(k+m)}$ modulo $\mathbb{Z}_4$
- Notation: P_i are projectors on $M^{(i)}$: $P_i(M) = M^{(i)}$

Z_4-grading

$$M = M^{(0)} + M^{(1)} + M^{(2)} + M^{(3)}$$

- $$M^{(0,2)} = \frac{1}{2}\begin{pmatrix} m \mp K m^t K^{-1} & 0 \\ 0 & n \mp K n^t K^{-1} \end{pmatrix}$$
$$M^{(1,3)} = \frac{1}{2}\begin{pmatrix} 0 & \theta \mp iK\eta^t K^{-1} \\ \eta \pm iK\theta^t K^{-1} & 0 \end{pmatrix}$$
$$K = \mathrm{diag}(-i\sigma_2, -i\sigma_2)$$
- $M^{(0)}$ and $M^{(2)}$ are even; $M^{(1)}$ and $M^{(3)}$ are odd
- $M^{(0)}$ form $\mathfrak{so}(4,1) \oplus \mathfrak{so}(5) \subset \mathfrak{su}(2,2) \oplus \mathfrak{su}(4)$.
- $M^{(2)}$ form the tangent space of the coset $\mathrm{SU}(2,2) \times \mathrm{SU}(4)/\mathrm{SO}(4,1) \times \mathrm{SO}(5) = \mathrm{AdS}_5 \times \mathrm{S}^5$
- $[M_1^{(k)}, M_2^{(m)}] = M_3^{(k+m)}$ modulo $\mathbb{Z}_4$
- Notation: P_i are projectors on $M^{(i)}$: $P_i(M) = M^{(i)}$

Strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$

Metsaev, Tseytlin '98

Let $\mathfrak{g} \in \mathrm{SU}(2,2|4)$. Introduce the one-form $A \in \mathfrak{su}(2,2|4)$

$$A = -\mathfrak{g}^{-1}\mathrm{d}\mathfrak{g} = A^{(0)} + A^{(2)} + A^{(1)} + A^{(3)}$$

Action $S = \int \mathrm{d}\sigma\mathrm{d}\tau\mathscr{L}$ for superstrings on $\mathrm{AdS}_5 \times \mathrm{S}^5$

$$\mathscr{L} = -\frac{g}{2}\left[\gamma^{\alpha\beta}\mathrm{str}\left(A^{(2)}_\alpha A^{(2)}_\beta\right) + \epsilon^{\alpha\beta}\mathrm{str}\left(A^{(1)}_\alpha A^{(3)}_\beta\right)\right]$$

$g = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$, $\epsilon^{\tau\sigma} = 1$ and $\gamma^{\alpha\beta} = h^{\alpha\beta}\sqrt{-h}$, $\det\gamma = -1$

$$\mathscr{L} = -\frac{g}{4}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta})\,\mathrm{str}\Big[(P_3 + 2P_2 - P_1)(A_\alpha).A_\beta\Big]$$

q-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$: $\mathscr{L}$ and R-operator

Delduc, Magro, Vicedo '13

$$\mathscr{L} = -\frac{g}{4}(1+\eta^2)\left(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}\right) \mathrm{str}\left[\tilde{d}(A_\alpha)\frac{1}{1-\eta R_{\mathfrak{g}} \circ d}(A_\beta)\right],$$

- $q = e^{-\nu/g}\,,\ \nu = \frac{2\eta}{1+\eta^2}$ — Arutyunov, Borsato, SF '13; Delduc, Magro, Vicedo '14
- P_i are projections on $M^{(i)}$: $P_i(M) = M^{(i)}$

$$d = P_1 + \frac{2}{1-\eta^2}P_2 - P_3\,, \quad \tilde{d} = P_3 + \frac{2}{1-\eta^2}P_2 - P_1\,, \quad \mathrm{str}\big(A\,d(B)\big) = \mathrm{str}\big(\tilde{d}(A)\,B\big)$$

- $R_{\mathfrak{g}}(M) = \mathfrak{g}^{-1}R(\mathfrak{g}M\mathfrak{g}^{-1})\mathfrak{g} = \mathrm{adj}_{\mathfrak{g}^{-1}} \circ R \circ \mathrm{adj}_{\mathfrak{g}}(M)$
- R satisfies the modified classical Yang-Baxter equation

$$[R(M),R(N)] - R([R(M),N]+[M,R(N)]) = [M,N]\,, \quad \mathrm{str}\big(R(M)\,N\big) = -\mathrm{str}\big(M\,R(N)\big)$$

- We choose

$$R(M)_{jk} = \begin{cases} -i\,M_{jk} & \text{if} \quad j<k \\ 0 & \text{if} \quad j=k \\ i\,M_{jk} & \text{if} \quad j>k \end{cases}$$

- $\mathscr{L}$ is invariant under $\mathfrak{g} \to \mathfrak{g}\mathfrak{h}$, $\mathfrak{h} \in \mathrm{SO}(4,1) \times \mathrm{SO}(5) \times \mathrm{U}(1)$
- $\mathscr{L}$ is κ-symmetric, and a Lax connection is known

q-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$: $\mathscr{L}$ and R-operator

Delduc, Magro, Vicedo '13

$$\mathscr{L} = -\frac{g}{4}(1+\eta^2)(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta})\,\mathrm{str}\Big[\tilde{d}(A_\alpha)\frac{1}{1-\eta R_{\mathfrak{g}}\circ d}(A_\beta)\Big]\,,$$

- $q = e^{-\nu/g}\,,\ \nu = \frac{2\eta}{1+\eta^2}$ — Arutyunov, Borsato, SF '13; Delduc, Magro, Vicedo '14
- P_i are projections on $M^{(i)}$: $P_i(M) = M^{(i)}$

$$d = P_1 + \frac{2}{1-\eta^2}P_2 - P_3\,, \quad \tilde{d} = P_3 + \frac{2}{1-\eta^2}P_2 - P_1\,, \quad \mathrm{str}\big(A\,d(B)\big) = \mathrm{str}\big(\tilde{d}(A)\,B\big)$$

- $R_{\mathfrak{g}}(M) = \mathfrak{g}^{-1}R(\mathfrak{g}M\mathfrak{g}^{-1})\mathfrak{g} = \mathrm{adj}_{\mathfrak{g}^{-1}}\circ R\circ \mathrm{adj}_{\mathfrak{g}}(M)$
- R satisfies the modified classical Yang-Baxter equation

$$[R(M),R(N)] - R([R(M),N]+[M,R(N)]) = [M,N]\,, \quad \mathrm{str}\big(R(M)\,N\big) = -\mathrm{str}\big(M\,R(N)\big)$$

- We choose

$$R(M)_{jk} = \begin{cases} -i\,M_{jk} & \text{if} \quad j<k \\ 0 & \text{if} \quad j=k \\ i\,M_{jk} & \text{if} \quad j>k \end{cases}$$

- $\mathscr{L}$ is invariant under $\mathfrak{g}\to\mathfrak{g}\mathfrak{h}$, $\mathfrak{h}\in \mathrm{SO}(4,1)\times\mathrm{SO}(5)\times\mathrm{U}(1)$
- $\mathscr{L}$ is κ-symmetric, and a Lax connection is known

q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$: $\mathscr{L}$ and R-operator

Delduc, Magro, Vicedo '13

$$\mathscr{L} = -\frac{g}{4}(1+\eta^2)\big(\gamma^{\alpha\beta}-\epsilon^{\alpha\beta}\big)\,\mathrm{str}\Big[\tilde{d}(A_\alpha)\frac{1}{1-\eta R_{\mathfrak{g}}\circ d}(A_\beta)\Big]\,,$$

- $q = e^{-\nu/g}\,,\ \nu = \frac{2\eta}{1+\eta^2}$ Arutyunov, Borsato, SF '13; Delduc, Magro, Vicedo '14
- P_i are projections on $M^{(i)}$: $P_i(M) = M^{(i)}$

$$d = P_1 + \frac{2}{1-\eta^2}P_2 - P_3\,,\quad \tilde{d} = P_3 + \frac{2}{1-\eta^2}P_2 - P_1\,,\quad \mathrm{str}\big(A\,d(B)\big) = \mathrm{str}\big(\tilde{d}(A)\,B\big)$$

- $R_{\mathfrak{g}}(M) = \mathfrak{g}^{-1}R(\mathfrak{g}M\mathfrak{g}^{-1})\mathfrak{g} = \mathrm{adj}_{\mathfrak{g}^{-1}}\circ R\circ \mathrm{adj}_{\mathfrak{g}}(M)$
- R satisfies the modified classical Yang-Baxter equation

$$[R(M),R(N)] - R([R(M),N]+[M,R(N)]) = [M,N]\,,\quad \mathrm{str}\big(R(M)\,N\big) = -\mathrm{str}\big(M\,R(N)\big)$$

- We choose

$$R(M)_{jk} = \begin{cases} -i\,M_{jk} & \text{if} \quad j<k \\ 0 & \text{if} \quad j=k \\ i\,M_{jk} & \text{if} \quad j>k \end{cases}$$

- $\mathscr{L}$ is invariant under $\mathfrak{g}\to\mathfrak{g}\mathfrak{h}$, $\mathfrak{h}\in \mathrm{SO}(4,1)\times\mathrm{SO}(5)\times\mathrm{U}(1)$
- $\mathscr{L}$ is κ-symmetric, and a Lax connection is known

Explicit bosonic Lagrangian: metric parts

Arutyunov, Borsato, SF '13

$$\mathscr{L} = -\frac{g}{2}(1+\varkappa^2)^{\frac{1}{2}}\left(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}\right)\mathrm{str}\left[A_\alpha^{(2)}\frac{1}{1-\varkappa R_{\mathfrak{g}}\circ P_2}(A_\beta)\right], \quad \varkappa = \frac{2\eta}{1-\eta^2}$$

- Choose a coset representative $\mathfrak{g}$
- Invert the operator $1-\varkappa R_{\mathfrak{g}}\circ P_2$
- Bosonic Lagrangian is given by the sum of the AdS and sphere parts

$$\mathscr{L} = \mathscr{L}_{\mathfrak{a}} + \mathscr{L}_{\mathfrak{s}} = \mathscr{L}_{\mathfrak{a}}^{G} + \mathscr{L}_{\mathfrak{a}}^{WZ} + \mathscr{L}_{\mathfrak{s}}^{G} + \mathscr{L}_{\mathfrak{s}}^{WZ}$$

- The metric pieces

$$\mathscr{L}_{\mathfrak{a}}^{G} = -\frac{g}{2}(1+\varkappa^2)^{\frac{1}{2}}\gamma^{\alpha\beta}\Big(-\frac{\partial_\alpha t\partial_\beta t\,(1+\rho^2)}{1-\varkappa^2\rho^2} + \frac{\partial_\alpha\rho\partial_\beta\rho}{(1+\rho^2)(1-\varkappa^2\rho^2)} + \frac{\partial_\alpha\zeta\partial_\beta\zeta\rho^2}{1+\varkappa^2\rho^4\sin^2\zeta} + \frac{\partial_\alpha\psi_1\partial_\beta\psi_1\rho^2\cos^2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta} + \partial_\alpha\psi_2\partial_\beta\psi_2\rho^2\sin^2\zeta\Big),$$

$$\mathscr{L}_{\mathfrak{s}}^{G} = -\frac{g}{2}(1+\varkappa^2)^{\frac{1}{2}}\gamma^{\alpha\beta}\Big(\frac{\partial_\alpha\phi\partial_\beta\phi\,(1-r^2)}{1+\varkappa^2 r^2} + \frac{\partial_\alpha r\partial_\beta r}{(1-r^2)(1+\varkappa^2 r^2)} + \frac{\partial_\alpha\xi\partial_\beta\xi r^2}{1+\varkappa^2 r^4\sin^2\xi} + \frac{\partial_\alpha\phi_1\partial_\beta\phi_1 r^2\cos^2\xi}{1+\varkappa^2 r^4\sin^2\xi} + \partial_\alpha\phi_2\partial_\beta\phi_2 r^2\sin^2\xi\Big),$$

Explicit bosonic Lagrangian: metric parts

Arutyunov, Borsato, SF '13

$$\mathscr{L} = -\frac{g}{2}(1+\varkappa^2)^{\frac{1}{2}}\left(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}\right)\mathrm{str}\left[A_\alpha^{(2)}\frac{1}{1-\varkappa R_\mathfrak{g}\circ P_2}(A_\beta)\right], \quad \varkappa = \frac{2\eta}{1-\eta^2}$$

- Choose a coset representative $\mathfrak{g}$
- Invert the operator $1 - \varkappa R_\mathfrak{g} \circ P_2$
- Bosonic Lagrangian is given by the sum of the AdS and sphere parts

$$\mathscr{L} = \mathscr{L}_\mathfrak{a} + \mathscr{L}_\mathfrak{s} = \mathscr{L}_\mathfrak{a}^G + \mathscr{L}_\mathfrak{a}^{WZ} + \mathscr{L}_\mathfrak{s}^G + \mathscr{L}_\mathfrak{s}^{WZ}$$

- The metric pieces

$$\mathscr{L}_\mathfrak{a}^G = -\frac{g}{2}(1+\varkappa^2)^{\frac{1}{2}}\gamma^{\alpha\beta}\Big(-\frac{\partial_\alpha t\partial_\beta t\,(1+\rho^2)}{1-\varkappa^2\rho^2} + \frac{\partial_\alpha\rho\partial_\beta\rho}{(1+\rho^2)(1-\varkappa^2\rho^2)} + \frac{\partial_\alpha\zeta\partial_\beta\zeta\rho^2}{1+\varkappa^2\rho^4\sin^2\zeta} + \frac{\partial_\alpha\psi_1\partial_\beta\psi_1\rho^2\cos^2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta} + \partial_\alpha\psi_2\partial_\beta\psi_2\rho^2\sin^2\zeta\Big),$$

$$\mathscr{L}_\mathfrak{s}^G = -\frac{g}{2}(1+\varkappa^2)^{\frac{1}{2}}\gamma^{\alpha\beta}\Big(\frac{\partial_\alpha\phi\partial_\beta\phi\,(1-r^2)}{1+\varkappa^2r^2} + \frac{\partial_\alpha r\partial_\beta r}{(1-r^2)(1+\varkappa^2r^2)} + \frac{\partial_\alpha\xi\partial_\beta\xi r^2}{1+\varkappa^2r^4\sin^2\xi} + \frac{\partial_\alpha\phi_1\partial_\beta\phi_1 r^2\cos^2\xi}{1+\varkappa^2r^4\sin^2\xi} + \partial_\alpha\phi_2\partial_\beta\phi_2 r^2\sin^2\xi\Big),$$

Explicit bosonic Lagrangian: metric parts

Arutyunov, Borsato, SF '13

$$\mathscr{L}=-\frac{g}{2}(1+\varkappa^2)^{\frac{1}{2}}\left(\gamma^{\alpha\beta}-\epsilon^{\alpha\beta}\right)\mathrm{str}\left[A_\alpha^{(2)}\frac{1}{1-\varkappa R_{\mathfrak{g}}\circ P_2}(A_\beta)\right],\qquad \varkappa=\frac{2\eta}{1-\eta^2}$$

- Choose a coset representative $\mathfrak{g}$
- Invert the operator $1-\varkappa R_{\mathfrak{g}}\circ P_2$
- Bosonic Lagrangian is given by the sum of the AdS and sphere parts

$$\mathscr{L}=\mathscr{L}_{\mathfrak{a}}+\mathscr{L}_{\mathfrak{s}}=\mathscr{L}_{\mathfrak{a}}^{G}+\mathscr{L}_{\mathfrak{a}}^{WZ}+\mathscr{L}_{\mathfrak{s}}^{G}+\mathscr{L}_{\mathfrak{s}}^{WZ}$$

- The metric pieces

$$\mathscr{L}_{\mathfrak{a}}^{G}=-\frac{g}{2}(1+\varkappa^2)^{\frac{1}{2}}\gamma^{\alpha\beta}\Big(-\frac{\partial_\alpha t\partial_\beta t\,(1+\rho^2)}{1-\varkappa^2\rho^2}+\frac{\partial_\alpha\rho\partial_\beta\rho}{(1+\rho^2)(1-\varkappa^2\rho^2)}$$
$$+\frac{\partial_\alpha\zeta\partial_\beta\zeta\rho^2}{1+\varkappa^2\rho^4\sin^2\zeta}+\frac{\partial_\alpha\psi_1\partial_\beta\psi_1\rho^2\cos^2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta}+\partial_\alpha\psi_2\partial_\beta\psi_2\rho^2\sin^2\zeta\Big),$$

$$\mathscr{L}_{\mathfrak{s}}^{G}=-\frac{g}{2}(1+\varkappa^2)^{\frac{1}{2}}\gamma^{\alpha\beta}\Big(\frac{\partial_\alpha\phi\partial_\beta\phi\,(1-r^2)}{1+\varkappa^2r^2}+\frac{\partial_\alpha r\partial_\beta r}{(1-r^2)(1+\varkappa^2r^2)}$$
$$+\frac{\partial_\alpha\xi\partial_\beta\xi r^2}{1+\varkappa^2r^4\sin^2\xi}+\frac{\partial_\alpha\phi_1\partial_\beta\phi_1r^2\cos^2\xi}{1+\varkappa^2r^4\sin^2\xi}+\partial_\alpha\phi_2\partial_\beta\phi_2r^2\sin^2\xi\Big),$$

Explicit bosonic Lagrangian: WZ parts

- The Wess-Zumino parts

$$\mathscr{L}_{\mathfrak{a}}^{WZ} = \frac{g}{2}\varkappa(1+\varkappa^2)^{\frac{1}{2}}\,\epsilon^{\alpha\beta}\,\frac{\rho^4\sin 2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta}\partial_\alpha\psi_1\partial_\beta\zeta\,,$$

$$\mathscr{L}_{\mathfrak{s}}^{WZ} = -\frac{g}{2}\varkappa(1+\varkappa^2)^{\frac{1}{2}}\,\epsilon^{\alpha\beta}\,\frac{r^4\sin 2\xi}{1+\varkappa^2 r^4\sin^2\xi}\partial_\alpha\phi_1\partial_\beta\xi\,.$$

- The deformed action is invariant under the shifts of t, ψ_k, ϕ, ϕ_k
- The ranges of ρ and r: $0\le\rho\le\frac{1}{\varkappa}$ and $0\le r\le 1$
- The deformed AdS is singular at $\rho = 1/\varkappa$
- In the undeformed case: $-Z_0^2+Z_1^2+Z_2^2+Z_3^2+Z_4^2-Z_5^2=-1$

$$Z_1+iZ_2=\rho\cos\zeta\, e^{i\psi_1}\,,\quad Z_3+iZ_4=\rho\sin\zeta\, e^{i\psi_2}\,,\quad Z_0+iZ_5=\sqrt{1+\rho^2}\, e^{it}$$

- while $\phi\,,\ \phi_1\,,\ \phi_2\,,\ \xi\,,\ r$ are related to Y_A, $Y_A^2=1$ as

$$Y_1+iY_2=r\cos\xi\, e^{i\phi_1}\,,\quad Y_3+iY_4=r\sin\xi\, e^{i\phi_2}\,,\quad Y_5+iY_6=\sqrt{1-r^2}\, e^{i\phi}$$

Explicit bosonic Lagrangian: WZ parts

- The Wess-Zumino parts

$$\mathscr{L}_{\mathfrak{a}}^{WZ} = \frac{g}{2}\varkappa(1+\varkappa^2)^{\frac{1}{2}}\,\epsilon^{\alpha\beta}\,\frac{\rho^4\sin 2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta}\partial_\alpha\psi_1\partial_\beta\zeta\,,$$

$$\mathscr{L}_{\mathfrak{s}}^{WZ} = -\frac{g}{2}\varkappa(1+\varkappa^2)^{\frac{1}{2}}\,\epsilon^{\alpha\beta}\,\frac{r^4\sin 2\xi}{1+\varkappa^2 r^4\sin^2\xi}\partial_\alpha\phi_1\partial_\beta\xi\,.$$

- The deformed action is invariant under the shifts of t, ψ_k, ϕ, ϕ_k
- The ranges of ρ and r: $0 \le \rho \le \frac{1}{\varkappa}$ and $0 \le r \le 1$
- The deformed AdS is singular at $\rho = 1/\varkappa$
- In the undeformed case: $-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 - Z_5^2 = -1$

$$Z_1 + iZ_2 = \rho\cos\zeta\, e^{i\psi_1}\,, \quad Z_3 + iZ_4 = \rho\sin\zeta\, e^{i\psi_2}\,, \quad Z_0 + iZ_5 = \sqrt{1+\rho^2}\, e^{it}$$

- while ϕ , ϕ_1 , ϕ_2 , ξ , r are related to Y_A, $Y_A^2 = 1$ as

$$Y_1 + iY_2 = r\cos\xi\, e^{i\phi_1}\,, \quad Y_3 + iY_4 = r\sin\xi\, e^{i\phi_2}\,, \quad Y_5 + iY_6 = \sqrt{1-r^2}\, e^{i\phi}$$

Explicit bosonic Lagrangian: WZ parts

- The Wess-Zumino parts

$$\mathscr{L}_{\mathfrak{a}}^{WZ} = \frac{g}{2}\varkappa(1+\varkappa^2)^{\frac{1}{2}}\,\epsilon^{\alpha\beta}\,\frac{\rho^4\sin 2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta}\partial_\alpha\psi_1\partial_\beta\zeta\,,$$

$$\mathscr{L}_{\mathfrak{s}}^{WZ} = -\frac{g}{2}\varkappa(1+\varkappa^2)^{\frac{1}{2}}\,\epsilon^{\alpha\beta}\,\frac{r^4\sin 2\xi}{1+\varkappa^2 r^4\sin^2\xi}\partial_\alpha\phi_1\partial_\beta\xi\,.$$

- The deformed action is invariant under the shifts of t, ψ_k, ϕ, ϕ_k
- The ranges of ρ and r: $0\le\rho\le\frac{1}{\varkappa}$ and $0\le r\le 1$
- The deformed AdS is singular at $\rho = 1/\varkappa$
- In the undeformed case: $-Z_0^2+Z_1^2+Z_2^2+Z_3^2+Z_4^2-Z_5^2=-1$

$$Z_1+iZ_2=\rho\cos\zeta\, e^{i\psi_1}\,,\quad Z_3+iZ_4=\rho\sin\zeta\, e^{i\psi_2}\,,\quad Z_0+iZ_5=\sqrt{1+\rho^2}\,e^{it}$$

- while ϕ , ϕ_1 , ϕ_2 , ξ , r are related to Y_A, $Y_A^2=1$ as

$$Y_1+iY_2=r\cos\xi\, e^{i\phi_1}\,,\quad Y_3+iY_4=r\sin\xi\, e^{i\phi_2}\,,\quad Y_5+iY_6=\sqrt{1-r^2}\,e^{i\phi}$$

Small $\varkappa$ limit

SF, unpublished

The S_q^5 part reduces to S^5 and $\mathscr{L}_{\mathfrak{s}}^{WZ}$ vanishes. Let $g(1+\varkappa^2)^{\frac{1}{2}} = 1$

$$ds_{\mathfrak{a}}^2 = -\frac{dt^2\,(1+\rho^2)}{1-\varkappa^2\rho^2} + \frac{d\rho^2}{(1+\rho^2)(1-\varkappa^2\rho^2)} + \frac{d\zeta^2\,\rho^2}{1+\varkappa^2\rho^4\sin^2\zeta} + \frac{d\psi_1^2\,\rho^2\cos^2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta} + d\psi_2^2\,\rho^2\sin^2\zeta\,,$$

$$B_{\mathfrak{a}} = \frac{\varkappa}{2}\frac{\rho^4\sin 2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta}d\psi_1\wedge d\zeta$$

1. $\varkappa \to 0$, all the coordinates are finite. One gets AdS_5 with $R = -20$
2. $\varkappa \to 0$, $\rho \to 1/\rho/\varkappa$, $t \to \varkappa t$, $\psi_2 \to \varkappa\psi_2$; $\rho > 1$
   $B_{\mathfrak{a}}$ becomes a total derivative and the metric

$$ds_{\mathfrak{a}}^2 = -\frac{dt^2}{\rho^2-1} + \frac{d\rho^2}{\rho^2-1} + \rho^2\frac{d\zeta^2}{\sin^2\zeta} + \rho^2\cot^2\zeta\,d\psi_1^2 + \frac{\sin^2\zeta}{\rho^2}d\psi_2^2$$

The curvature $R = -4 - 2\frac{\rho^2+1}{\rho^2(\rho^2-1)} - 2\frac{\cos 2\zeta}{\rho^2} < 0$

Small $\varkappa$ limit

SF, unpublished

The S^5_q part reduces to S^5 and $\mathscr{L}^{WZ}_{\mathfrak{s}}$ vanishes. Let $g(1+\varkappa^2)^{\frac{1}{2}} = 1$

$$ds^2_{\mathfrak{a}} = -\frac{dt^2\,(1+\rho^2)}{1-\varkappa^2\rho^2} + \frac{d\rho^2}{(1+\rho^2)\,(1-\varkappa^2\rho^2)} + \frac{d\zeta^2\,\rho^2}{1+\varkappa^2\rho^4\sin^2\zeta} + \frac{d\psi_1^2\,\rho^2\cos^2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta} + d\psi_2^2\,\rho^2\sin^2\zeta\,,$$

$$B_{\mathfrak{a}} = \frac{\varkappa}{2}\frac{\rho^4\sin 2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta}d\psi_1\wedge d\zeta$$

1. $\varkappa\to 0$, all the coordinates are finite. One gets AdS_5 with $R=-20$
2. $\varkappa\to 0,\ \ \rho\to 1/\rho/\varkappa,\ \ t\to\varkappa t,\ \ \psi_2\to\varkappa\psi_2;\ \ \ \rho>1$
   $B_{\mathfrak{a}}$ becomes a total derivative and the metric

$$ds^2_{\mathfrak{a}} = -\frac{dt^2}{\rho^2-1} + \frac{d\rho^2}{\rho^2-1} + \rho^2\frac{d\zeta^2}{\sin^2\zeta} + \rho^2\cot^2\zeta\,d\psi_1^2 + \frac{\sin^2\zeta}{\rho^2}d\psi_2^2$$

The curvature $R = -4 - 2\frac{\rho^2+1}{\rho^2(\rho^2-1)} - 2\frac{\cos 2\zeta}{\rho^2} < 0$

Small $\varkappa$ limit

SF, unpublished

The S^5_q part reduces to S^5 and $\mathscr{L}^{WZ}_{\mathfrak{s}}$ vanishes. Let $g(1+\varkappa^2)^{\frac{1}{2}} = 1$

$$ds^2_{\mathfrak{a}} = -\frac{dt^2\,(1+\rho^2)}{1-\varkappa^2\rho^2} + \frac{d\rho^2}{(1+\rho^2)\,(1-\varkappa^2\rho^2)} + \frac{d\zeta^2\,\rho^2}{1+\varkappa^2\rho^4\sin^2\zeta} + \frac{d\psi_1^2\,\rho^2\cos^2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta} + d\psi_2^2\,\rho^2\sin^2\zeta\,,$$

$$B_{\mathfrak{a}} = \frac{\varkappa}{2}\frac{\rho^4\sin 2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta}d\psi_1\wedge d\zeta$$

3. $\varkappa\to 0,\ \ \rho\to\rho/\sqrt{\varkappa},\ \ t\to\sqrt{\varkappa}t,\ \ \zeta\to\zeta_0+\sqrt{\varkappa}\zeta,$
$\psi_1\to\sqrt{\varkappa}\psi_1/\cos\zeta_0,\ \ \psi_2\to\sqrt{\varkappa}\psi_2/\sin\zeta_0.$
$B_{\mathfrak{a}}$ and the metric

$$ds^2_{\mathfrak{a}} = \rho^2(-dt^2+d\psi_2^2) + \frac{\rho^2(d\zeta^2+d\psi_1^2)}{1+\rho^4\sin^2\zeta_0} + \frac{d\rho^2}{\rho^2}\,,$$

$$B_{\mathfrak{a}} = \frac{\rho^4\sin\zeta_0}{1+\rho^4\sin^2\zeta_0}d\psi_1\wedge d\zeta$$

This is the Maldacena-Russo background!

Maldacena, Russo '99

Small $\varkappa$ limit

SF, unpublished

The S^5_q part reduces to S^5 and $\mathscr{L}^{WZ}_{\mathfrak{s}}$ vanishes. Let $g(1+\varkappa^2)^{\frac{1}{2}} = 1$

$$ds^2_{\mathfrak{a}} = -\frac{dt^2\,(1+\rho^2)}{1-\varkappa^2\rho^2} + \frac{d\rho^2}{(1+\rho^2)\,(1-\varkappa^2\rho^2)} + \frac{d\zeta^2\,\rho^2}{1+\varkappa^2\rho^4\sin^2\zeta} + \frac{d\psi_1^2\,\rho^2\cos^2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta} + d\psi_2^2\,\rho^2\sin^2\zeta\,,$$

$$B_{\mathfrak{a}} = \frac{\varkappa}{2}\frac{\rho^4\sin 2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta}d\psi_1\wedge d\zeta$$

3. $\varkappa\to 0,\ \rho\to\rho/\sqrt{\varkappa},\ t\to\sqrt{\varkappa}t,\ \zeta\to\zeta_0+\sqrt{\varkappa}\zeta,$
$\psi_1\to\sqrt{\varkappa}\psi_1/\cos\zeta_0,\ \psi_2\to\sqrt{\varkappa}\psi_2/\sin\zeta_0.$
$B_{\mathfrak{a}}$ and the metric

$$ds^2_{\mathfrak{a}} = \rho^2(-dt^2+d\psi_2^2) + \frac{\rho^2(d\zeta^2+d\psi_1^2)}{1+\rho^4\sin^2\zeta_0} + \frac{d\rho^2}{\rho^2}\,,$$

$$B_{\mathfrak{a}} = \frac{\rho^4\sin\zeta_0}{1+\rho^4\sin^2\zeta_0}d\psi_1\wedge d\zeta$$

This is the Maldacena-Russo background!

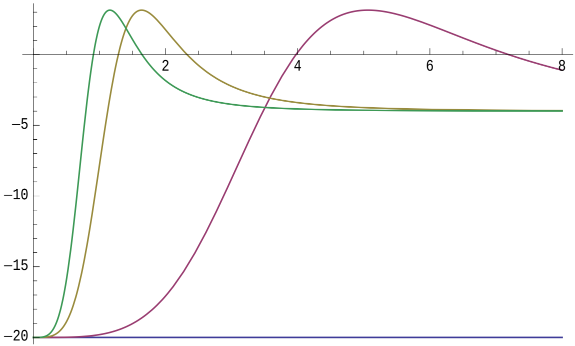
Maldacena, Russo '99

Small $\varkappa$ limit

SF, unpublished

The curvature

$$R = -4 + \frac{40}{1 + \rho^4 \sin^2 \zeta_0} - \frac{56}{(1 + \rho^4 \sin^2 \zeta_0)^2}$$



Large $\varkappa$ limit

Hoare, Roiban, Tseytlin '14

$$ds^2_{\mathfrak{a}} = g(1+\varkappa^2)^{\frac{1}{2}}\Big(-\frac{dt^2\,(1+\rho^2)}{1-\varkappa^2\rho^2} + \frac{d\rho^2}{(1+\rho^2)\,(1-\varkappa^2\rho^2)} + \frac{d\zeta^2\,\rho^2}{1+\varkappa^2\rho^4\sin^2\zeta} + \frac{d\psi_1^2\,\rho^2\cos^2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta} + d\psi_2^2\,\rho^2\sin^2\zeta\Big)$$

$$ds^2_{\mathfrak{s}} = g(1+\varkappa^2)^{\frac{1}{2}}\Big(\frac{d\phi^2\,(1-r^2)}{1+\varkappa^2 r^2} + \frac{dr^2}{(1-r^2)\,(1+\varkappa^2 r^2)} + \frac{d\xi^2\,r^2}{1+\varkappa^2 r^4\sin^2\xi} + \frac{d\phi_1^2\,r^2\cos^2\xi}{1+\varkappa^2 r^4\sin^2\xi} + d\phi_2^2\,r^2\sin^2\xi\Big)$$

- $\varkappa\to\infty$, $g\to\varkappa g$, $\rho\to 1/\rho$, $\psi_2\to\psi_2/\varkappa$, $r\to 1/r$, $\phi_2=\phi_2/\varkappa$
- The WZ term becomes a total derivative and the metric

$$\frac{1}{g}ds^2_{\mathfrak{a}} = dt^2\left(1+\rho^2\right) - \frac{d\rho^2}{1+\rho^2} + \frac{d\zeta^2\,\rho^2}{\sin^2\zeta} + d\psi_1^2\,\rho^2\cot^2\zeta + \frac{d\psi_2^2}{\rho^2}\sin^2\zeta$$

$$\frac{1}{g}ds^2_{\mathfrak{s}} = d\phi^2\left(r^2-1\right) + \frac{dr^2}{r^2-1} + \frac{d\xi^2\,r^2}{\sin^2\xi} + d\phi_1^2\,r^2\cot^2\xi + \frac{d\phi_2^2}{r^2}\sin^2\xi$$

- T-duality along the ψ_2 and ϕ_2 directions gives $\mathrm{dS}_5\times\mathrm{H}^5$

Large $\varkappa$ limit

Hoare, Roiban, Tseytlin '14

$$ds_{\mathfrak{a}}^2 = g(1+\varkappa^2)^{\frac{1}{2}}\Big(-\frac{dt^2\,(1+\rho^2)}{1-\varkappa^2\rho^2}+\frac{d\rho^2}{(1+\rho^2)\,(1-\varkappa^2\rho^2)} +\frac{d\zeta^2\,\rho^2}{1+\varkappa^2\rho^4\sin^2\zeta}+\frac{d\psi_1^2\,\rho^2\cos^2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta}+d\psi_2^2\,\rho^2\sin^2\zeta\Big)$$

$$ds_{\mathfrak{s}}^2 = g(1+\varkappa^2)^{\frac{1}{2}}\Big(\frac{d\phi^2\,(1-r^2)}{1+\varkappa^2r^2}+\frac{dr^2}{(1-r^2)\,(1+\varkappa^2r^2)} +\frac{d\xi^2\,r^2}{1+\varkappa^2r^4\sin^2\xi}+\frac{d\phi_1^2\,r^2\cos^2\xi}{1+\varkappa^2r^4\sin^2\xi}+d\phi_2^2\,r^2\sin^2\xi\Big)$$

- $\varkappa\to\infty$, $g\to\varkappa g$, $\rho\to 1/\rho$, $\psi_2\to\psi_2/\varkappa$, $r\to 1/r$, $\phi_2=\phi_2/\varkappa$
- The WZ term becomes a total derivative and the metric

$$\frac{1}{g}ds_{\mathfrak{a}}^2 = dt^2\left(1+\rho^2\right)-\frac{d\rho^2}{1+\rho^2}+\frac{d\zeta^2\,\rho^2}{\sin^2\zeta}+d\psi_1^2\,\rho^2\cot^2\zeta+\frac{d\psi_2^2}{\rho^2}\sin^2\zeta$$

$$\frac{1}{g}ds_{\mathfrak{s}}^2 = d\phi^2\left(r^2-1\right)+\frac{dr^2}{r^2-1}+\frac{d\xi^2\,r^2}{\sin^2\xi}+d\phi_1^2\,r^2\cot^2\xi+\frac{d\phi_2^2}{r^2}\sin^2\xi$$

- T-duality along the ψ_2 and ϕ_2 directions gives $dS_5\times H^5$

Large $\varkappa$ limit: T-duality along the time direction

SF, unpublished

$$ds_{\mathfrak{a}}^2 = g(1+\varkappa^2)^{\frac{1}{2}}\Big(-\frac{dt^2\,(1+\rho^2)}{1-\varkappa^2\rho^2} + \frac{d\rho^2}{(1+\rho^2)\,(1-\varkappa^2\rho^2)} + \frac{d\zeta^2\,\rho^2}{1+\varkappa^2\rho^4\sin^2\zeta} + \frac{d\psi_1^2\,\rho^2\cos^2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta} + d\psi_2^2\,\rho^2\sin^2\zeta\Big)$$

$$ds_{\mathfrak{s}}^2 = g(1+\varkappa^2)^{\frac{1}{2}}\Big(\frac{d\phi^2\,(1-r^2)}{1+\varkappa^2 r^2} + \frac{dr^2}{(1-r^2)\,(1+\varkappa^2 r^2)} + \frac{d\xi^2\,r^2}{1+\varkappa^2 r^4\sin^2\xi} + \frac{d\phi_1^2\,r^2\cos^2\xi}{1+\varkappa^2 r^4\sin^2\xi} + d\phi_2^2\,r^2\sin^2\xi\Big)$$

- $\varkappa\to\infty$, $g\to\varkappa g$, $\rho\to\rho/\varkappa$, $t\to t/\varkappa$, $r\to r/\varkappa$, $\phi\to\phi/\varkappa$
- The WZ term becomes 0, and the metric

$$\frac{1}{g}ds_{\mathfrak{a}}^2 = -\frac{dt^2}{1-\rho^2} + \frac{d\rho^2}{1-\rho^2} + \rho^2\big(d\zeta^2 + d\psi_1^2\cos^2\zeta + d\psi_2^2\sin^2\zeta\big)$$

$$\frac{1}{g}ds_{\mathfrak{s}}^2 = \frac{d\phi^2}{1+r^2} + \frac{dr^2}{1+r^2} + r^2\big(d\xi^2 + d\phi_1^2\cos^2\xi + d\phi_2^2\sin^2\xi\big)$$

- T-duality along the t and ϕ directions gives $\mathrm{dS}_5\times\mathrm{H}^5$

Large $\varkappa$ limit: T-duality along the time direction

SF, unpublished

$$ds_{\mathfrak{a}}^2 = g(1+\varkappa^2)^{\frac{1}{2}}\Big(-\frac{dt^2\,(1+\rho^2)}{1-\varkappa^2\rho^2} + \frac{d\rho^2}{(1+\rho^2)\,(1-\varkappa^2\rho^2)} + \frac{d\zeta^2\,\rho^2}{1+\varkappa^2\rho^4\sin^2\zeta} + \frac{d\psi_1^2\,\rho^2\cos^2\zeta}{1+\varkappa^2\rho^4\sin^2\zeta} + d\psi_2^2\,\rho^2\sin^2\zeta\Big)$$

$$ds_{\mathfrak{s}}^2 = g(1+\varkappa^2)^{\frac{1}{2}}\Big(\frac{d\phi^2\,(1-r^2)}{1+\varkappa^2 r^2} + \frac{dr^2}{(1-r^2)\,(1+\varkappa^2 r^2)} + \frac{d\xi^2\,r^2}{1+\varkappa^2 r^4\sin^2\xi} + \frac{d\phi_1^2\,r^2\cos^2\xi}{1+\varkappa^2 r^4\sin^2\xi} + d\phi_2^2\,r^2\sin^2\xi\Big)$$

- $\varkappa \to \infty$, $g \to \varkappa g$, $\rho \to \rho/\varkappa$, $t \to t/\varkappa$, $r \to r/\varkappa$, $\phi \to \phi/\varkappa$
- The WZ term becomes 0, and the metric

$$\frac{1}{g}ds_{\mathfrak{a}}^2 = -\frac{dt^2}{1-\rho^2} + \frac{d\rho^2}{1-\rho^2} + \rho^2\left(d\zeta^2 + d\psi_1^2\cos^2\zeta + d\psi_2^2\sin^2\zeta\right)$$

$$\frac{1}{g}ds_{\mathfrak{s}}^2 = \frac{d\phi^2}{1+r^2} + \frac{dr^2}{1+r^2} + r^2\left(d\xi^2 + d\phi_1^2\cos^2\xi + d\phi_2^2\sin^2\xi\right)$$

- T-duality along the t and ϕ directions gives $dS_5 \times H^5$

Mirror of $AdS_5 \times S^5 \equiv (AdS_5 \times S^5)_{\varkappa=\infty}$

Arutyunov, van Tongeren '14

- Metric and curvature ($g = 1$)

$$ds^2_{\mathfrak{a}} = -\frac{dt^2}{1-\rho^2} + \frac{d\rho^2}{1-\rho^2} + \rho^2 \left(d\zeta^2 + d\psi_1^2 \cos^2\zeta + d\psi_2^2 \sin^2\zeta\right), \quad R_{\mathfrak{a}} = 4\frac{1-2\rho^2}{1-\rho^2}$$

$$ds^2_{\mathfrak{s}} = \frac{d\phi^2}{1+r^2} + \frac{dr^2}{1+r^2} + r^2 \left(d\xi^2 + d\phi_1^2 \cos^2\xi + d\phi_2^2 \sin^2\xi\right), \quad R_{\mathfrak{s}} = -4\frac{1+2r^2}{1+r^2}$$

- Dilaton $\Phi = \Phi_0 - \frac{1}{2}\log(1-\rho^2)(1+r^2)$
- The five-form $F = 4e^{-\Phi}(\omega_\phi - \omega_t)$
- No Killing spinors $\Longrightarrow$ susy is not realised by superisometries
- In l.c. gauge $t = \tau$, $p_\phi = 1$, it is the mirror of the $AdS_5 \times S^5$ strings
- Particular case of the general mirror duality Arutynov, de Leeuw, van Tongeren '14

Mirror of $\mathrm{AdS}_5\times\mathrm{S}^5 \equiv (\mathrm{AdS}_5\times\mathrm{S}^5)_{\varkappa=\infty}$

Arutyunov, van Tongeren '14

- Metric and curvature ($g=1$)

$$ds^2_{\mathfrak{a}} = -\frac{dt^2}{1-\rho^2} + \frac{d\rho^2}{1-\rho^2} + \rho^2\left(d\zeta^2 + d\psi_1^2\cos^2\zeta + d\psi_2^2\sin^2\zeta\right), \quad R_{\mathfrak{a}} = 4\frac{1-2\rho^2}{1-\rho^2}$$

$$ds^2_{\mathfrak{s}} = \frac{d\phi^2}{1+r^2} + \frac{dr^2}{1+r^2} + r^2\left(d\xi^2 + d\phi_1^2\cos^2\xi + d\phi_2^2\sin^2\xi\right), \quad R_{\mathfrak{s}} = -4\frac{1+2r^2}{1+r^2}$$

- Dilaton $\quad \Phi = \Phi_0 - \frac{1}{2}\log(1-\rho^2)(1+r^2)$
- The five-form $\quad F = 4e^{-\Phi}(\omega_\phi - \omega_t)$
- No Killing spinors $\Longrightarrow$ susy is not realised by superisometries
- In l.c. gauge $t=\tau$, $p_\phi = 1$, it is the mirror of the $\mathrm{AdS}_5\times\mathrm{S}^5$ strings
- Particular case of the general mirror duality Arutynov, de Leeuw, van Tongeren '14

(S, J) folded string

SF, Roiban in progress

Folded string spinning in $(AdS_5)_q$ and rotating about a circle in $(S^5)_q$.

- Ansatz is the same as for the undeformed case SF, Tseytlin '02

$$t=\kappa\tau\,,\quad \psi_1=\omega\tau\,,\quad \rho=\rho(\sigma)\,,\quad \phi=\nu\tau\,,\quad \zeta=\psi_2=r=\xi=\phi_1=\phi_2=0$$

- The WZ term does not contribute. Virasoro constraint gives

$$(\rho')^2=(1+\rho^2)\big(\omega^2\varkappa^2\rho^4-(\omega^2-\kappa^2-\varkappa^2\nu^2)\rho^2+\kappa^2-\nu^2\big)$$

- We want $\rho'=0$ at $\rho=\rho_0$, and rhs >0 for $\rho<\rho_0$. Thus

$$\kappa>\nu\,,\quad \omega^2>\kappa^2+\varkappa^2\nu^2 \quad\Longrightarrow\quad \nu^2<\frac{\omega^2}{1+\varkappa^2}$$

- ρ_0 satisfies

$$\omega^2\varkappa^2\rho_0^4-(\omega^2-\kappa^2-\varkappa^2\nu^2)\rho_0^2+\kappa^2-\nu^2=0\,,$$

$$(\rho_0^{\mp})^2=\frac{\omega^2-\kappa^2-\varkappa^2\nu^2\mp\sqrt{(\omega^2-\kappa^2-\varkappa^2\nu^2)^2-4\omega^2\varkappa^2\,(\kappa^2-\nu^2)}}{2\varkappa^2\omega^2}$$

- Since $(\rho')^2>0$ at $\rho=1/\varkappa$ both roots must be less than $1/\varkappa^2$

$$\kappa\le\omega\sqrt{1+\varkappa^2}-\varkappa\sqrt{\omega^2-\nu^2}$$

In what follows we denote $\rho_0\equiv\rho_0^-$.

(S, J) folded string

SF, Roiban in progress

Folded string spinning in $(AdS_5)_q$ and rotating about a circle in $(S^5)_q$.

- Ansatz is the same as for the undeformed case SF, Tseytlin '02

$$t = \kappa\tau\,, \quad \psi_1 = \omega\tau\,, \quad \rho = \rho(\sigma)\,, \quad \phi = \nu\tau\,, \quad \zeta = \psi_2 = r = \xi = \phi_1 = \phi_2 = 0$$

- The WZ term does not contribute. Virasoro constraint gives

$$(\rho')^2 = (1+\rho^2)\big(\omega^2\varkappa^2\rho^4 - (\omega^2 - \kappa^2 - \varkappa^2\nu^2)\rho^2 + \kappa^2 - \nu^2\big)$$

- We want $\rho' = 0$ at $\rho = \rho_0$, and rhs > 0 for $\rho < \rho_0$. Thus

$$\kappa > \nu\,, \quad \omega^2 > \kappa^2 + \varkappa^2\nu^2 \quad \Longrightarrow \quad \nu^2 < \frac{\omega^2}{1+\varkappa^2}$$

- ρ_0 satisfies

$$\omega^2\varkappa^2\rho_0^4 - (\omega^2 - \kappa^2 - \varkappa^2\nu^2)\rho_0^2 + \kappa^2 - \nu^2 = 0\,,$$

$$(\rho_0^{\mp})^2 = \frac{\omega^2 - \kappa^2 - \varkappa^2\nu^2 \mp \sqrt{(\omega^2 - \kappa^2 - \varkappa^2\nu^2)^2 - 4\omega^2\varkappa^2\,(\kappa^2 - \nu^2)}}{2\varkappa^2\omega^2}$$

- Since $(\rho')^2 > 0$ at $\rho = 1/\varkappa$ both roots must be less than $1/\varkappa^2$

$$\kappa \le \omega\sqrt{1+\varkappa^2} - \varkappa\sqrt{\omega^2 - \nu^2}$$

In what follows we denote $\rho_0 \equiv \rho_0^-$.

(S, J) folded string

- The periodicity condition on $\rho(\sigma)$ implies

$$2\pi = \int_0^{2\pi} d\sigma = 4\int_0^{\rho_0} \frac{d\rho}{\sqrt{(1+\rho^2)\big(\omega^2\varkappa^2\rho^4 - (\omega^2-\kappa^2-\varkappa^2\nu^2)\rho^2 + \kappa^2 - \nu^2\big)}}$$

- Angular momentum $\mathcal{J} = J/g(1+\varkappa^2)^{\frac{1}{2}}$

$$\mathcal{J} = \nu\int_0^{2\pi} d\sigma = 2\pi\nu$$

- Spin $\mathcal{S} = S/g(1+\varkappa^2)^{\frac{1}{2}}$

$$\mathcal{S} = \omega\int_0^{2\pi} d\sigma\, \rho^2 = 4\omega\int_0^{\rho_0} \frac{d\rho\, \rho^2}{\sqrt{(1+\rho^2)\big(\omega^2\varkappa^2\rho^4 - (\omega^2-\kappa^2-\varkappa^2\nu^2)\rho^2 + \kappa^2 - \nu^2\big)}}$$

- Energy $\mathcal{E} = E/g(1+\varkappa^2)^{\frac{1}{2}}$

$$\mathcal{E} = \kappa\int_0^{2\pi} d\sigma \frac{1+\rho^2}{1-\varkappa^2\rho^2} = 4\kappa\int_0^{\rho_0} \frac{d\rho\sqrt{1+\rho^2}}{(1-\varkappa^2\rho^2)\sqrt{\omega^2\varkappa^2\rho^4 - (\omega^2-\kappa^2-\varkappa^2\nu^2)\rho^2 + \kappa^2 - \nu^2}}$$

Large S expansion

- corresponds to $\omega \to \infty$, and $\rho_0 \to \rho_0^+$. Then

$$\kappa \to \omega\sqrt{1+\varkappa^2} - \varkappa\sqrt{\omega^2-\nu^2}\,, \quad \rho_0^2 \to \sqrt{1+\frac{1}{\varkappa^2}}\sqrt{1-\frac{\nu^2}{\omega^2}} - 1 < \frac{1}{2\varkappa^2}$$

- No matter how fast the string rotates, it *never* hits the singularity!
- If $\frac{\nu^2}{\omega^2} \to \frac{1}{1+\varkappa^2}$ then $\rho_0 \to 0$
- Let $\mathcal{J} = 0$

$$\mathcal{E} = \frac{w_0}{\varkappa}\mathcal{L} + \frac{2}{\varkappa}\log\frac{w_0-1}{w_0+1} - \frac{32}{k_0\varkappa^2}e^{-\mathcal{L}} + 32\left(\frac{2\mathcal{L}}{k_0\varkappa^2} - \frac{8+11\varkappa^2}{k_0\varkappa^4} - \frac{16}{w_0 k_0^2\varkappa^3}\right)e^{-2\mathcal{L}} + \cdots$$

$$\mathcal{L} = \frac{\varkappa}{k_0}\mathcal{S} + \frac{2}{k_0}\log\left(\frac{1+k_0}{1-k_0}\right), \quad w_0 = \sqrt{1+\frac{\varkappa}{\sqrt{1+\varkappa^2}}}\,, \quad k_0 = \sqrt{1-\frac{\varkappa}{\sqrt{1+\varkappa^2}}}$$

- Compare with

Gubser, Klebanov, Polyakov '02

$$\mathcal{E} = \mathcal{S} + 2\log\mathcal{S} + 8\log 2 - 2 + \sum_{n=1}^{\infty} P_n(\log\mathcal{S})e^{-n\log\mathcal{S}}$$

Large S expansion at $\varkappa = \infty$

- Take the limit $\varkappa \to \infty$ and rescale $\mathcal{E} \to \mathcal{E}/\varkappa$ and $\mathcal{S} \to \mathcal{S}/\varkappa^2$

$$\mathcal{E} = 2\mathcal{S} + 4\sqrt{2} + 2\log\left(3 - 2\sqrt{2}\right) - 32\sqrt{2}e^{-\mathcal{L}} + 32\left(4\mathcal{S} - 19\sqrt{2}\right)e^{-2\mathcal{L}} + \cdots,$$

$$\mathcal{L} = \sqrt{2}\mathcal{S} + 4$$

- Is there scattering theory at $\mathcal{S} = \infty$? Can the terms $\sim e^{-n\mathcal{L}}$ be obtained from TBA?
- If $\frac{\mathcal{J}}{\mathcal{S}}$ is constant

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1\, e^{-\alpha\mathcal{S}} + \cdots, \quad \alpha = \sqrt{\mathfrak{L}} + \frac{2\mathfrak{L}}{(\mathfrak{L}-1)\mathcal{S}} + \frac{2(\mathfrak{L}-2)\sqrt{\mathfrak{L}}(1+\mathfrak{L})}{(\mathfrak{L}-1)^3\mathcal{S}^2} + \cdots,$$

$$\mathcal{E}_0 = \mathfrak{L}\mathcal{S} + \frac{4\sqrt{\mathfrak{L}}}{\mathfrak{L}-1} - 4\mathrm{csch}^{-1}\left(\sqrt{\mathfrak{L}-1}\right) + \frac{8\,(\mathfrak{L}-2)}{(\mathfrak{L}-1)^3\,\mathcal{S}} + \cdots,$$

$$\mathcal{E}_1 = -\frac{32\sqrt{\mathfrak{L}}}{(\mathfrak{L}-1)^2} - \frac{64(\mathfrak{L}-2)(1+3\mathfrak{L})}{(\mathfrak{L}-1)^4\mathcal{S}} + \cdots, \quad \mathfrak{L} = 1 + \sqrt{1 + \frac{\mathcal{J}^2}{\mathcal{S}^2}}$$

Large S expansion at $\varkappa = \infty$

- Take the limit $\varkappa \to \infty$ and rescale $\mathcal{E} \to \mathcal{E}/\varkappa$ and $\mathcal{S} \to \mathcal{S}/\varkappa^2$

$$\mathcal{E} = 2\mathcal{S} + 4\sqrt{2} + 2\log\left(3 - 2\sqrt{2}\right) - 32\sqrt{2}e^{-\mathcal{L}} + 32\left(4\mathcal{S} - 19\sqrt{2}\right)e^{-2\mathcal{L}} + \cdots,$$

$$\mathcal{L} = \sqrt{2}\mathcal{S} + 4$$

- Is there scattering theory at $\mathcal{S} = \infty$? Can the terms $\sim e^{-n\mathcal{L}}$ be obtained from TBA?
- If $\frac{\mathcal{J}}{\mathcal{S}}$ is constant

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1\, e^{-\alpha\mathcal{S}} + \cdots, \quad \alpha = \sqrt{\mathfrak{L}} + \frac{2\mathfrak{L}}{(\mathfrak{L}-1)\mathcal{S}} + \frac{2(\mathfrak{L}-2)\sqrt{\mathfrak{L}}(1+\mathfrak{L})}{(\mathfrak{L}-1)^3\mathcal{S}^2} + \cdots,$$

$$\mathcal{E}_0 = \mathfrak{L}\mathcal{S} + \frac{4\sqrt{\mathfrak{L}}}{\mathfrak{L}-1} - 4\text{csch}^{-1}\left(\sqrt{\mathfrak{L}-1}\right) + \frac{8\,(\mathfrak{L}-2)}{(\mathfrak{L}-1)^3\,\mathcal{S}} + \cdots,$$

$$\mathcal{E}_1 = -\frac{32\sqrt{\mathfrak{L}}}{(\mathfrak{L}-1)^2} - \frac{64(\mathfrak{L}-2)(1+3\mathfrak{L})}{(\mathfrak{L}-1)^4\mathcal{S}} + \cdots, \quad \mathfrak{L} = 1 + \sqrt{1 + \frac{\mathcal{J}^2}{\mathcal{S}^2}}$$

Dilaton

Arutyunov, Borsato, SF in progress

Introduce the coordinates

$$x^0 = t\,,\quad x^1 = \psi_2\,,\quad x^2 = \psi_1\,,\quad x^3 = \sin^2\zeta\,,\quad x^4 = \rho^2\,,$$
$$x^5 = \phi_3\,,\quad x^6 = \phi_2\,,\quad x^7 = \phi_1\,,\quad x^8 = \sin^2\xi\,,\quad x^9 = r^2\,.$$

- The gravity fields can depend only on x^3, x^4, x^8, x^9.
- The only nonvanishing components of $H_{\mu\nu\rho}$ are H_{234} and H_{789}
- Dilaton equation

$$4\nabla^2\Phi - 4(\nabla_\mu\Phi)^2 + R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} = 0$$

- $\Phi = \Phi_0 + \Phi_a(x^3, x^4) + \Phi_s(x^8, x^9) + \Phi_{as}(x^3, x^4, x^8, x^9)$

$$4\nabla^2\Phi_a - 4(\nabla_\mu\Phi_a)^2 + R_a - \frac{1}{2}H_{234}H^{234} = \frac{c(\varkappa)}{L^2}\,,\quad L^2 = g(1+\varkappa^2)^{\frac{1}{2}}$$

Dilaton

- Perturbative solution ($x \equiv x^3\,,\ y \equiv x^4$)

$$\Phi_a = -\log\left(f_a\sqrt{(1+\varkappa^2 x y^2)(1-\varkappa^2 y)}\right)$$

$$f_a = 1 + y\left(\frac{2}{3}+\frac{x}{3}\right)\varkappa^2 + y\left(-\frac{5}{108}-\frac{x}{27}+\frac{y}{2}+\frac{xy}{4}+\frac{x^2y}{6}\right)\varkappa^4$$
$$+y\left(\frac{113}{4860}+\frac{47x}{2430}-\frac{y}{18}-\frac{7xy}{180}-\frac{4x^2y}{135}+\frac{2y^2}{5}+\frac{xy^2}{5}+\frac{2x^2y^2}{15}+\frac{x^3y^2}{10}\right)\varkappa^6+\cdots$$

$$c(\kappa) = -20-\frac{20\varkappa^2}{3}+\frac{56\varkappa^4}{27}-\frac{256\varkappa^6}{243}+\frac{14761\varkappa^8}{21870}-\frac{47633\varkappa^{10}}{98415}+\frac{61367791\varkappa^{12}}{165337200}+\cdots$$

$$\Phi_s(x,y) = \Phi_a(x,-y)$$

- Equation for $\Phi_{as}(x^3,x^4,x^8,x^9)$

$$\nabla^2\Phi_{as} - (\nabla_\mu\Phi_{as})^2 - 2\nabla^\mu\Phi_a\nabla_\mu\Phi_{as} - 2\nabla^\mu\Phi_s\nabla_\mu\Phi_{as} = 0$$

- Symmetry: $\Phi_{as}(x^8,-x^9,x^3,-x^4) = \Phi_{as}(x^3,x^4,x^8,x^9)$
- In the $\mathrm{AdS}_2\times \mathrm{S}^2$ case $\Phi_{as}\neq 0$ Lunin, Roiban, Tseytlin '14

Dilaton

- Perturbative solution ($x \equiv x^3\,,\ y \equiv x^4$)

$$\Phi_a = -\log\left(f_a\sqrt{(1+\varkappa^2 x y^2)(1-\varkappa^2 y)}\right)$$

$$f_a = 1 + y\left(\frac{2}{3}+\frac{x}{3}\right)\varkappa^2 + y\left(-\frac{5}{108}-\frac{x}{27}+\frac{y}{2}+\frac{xy}{4}+\frac{x^2 y}{6}\right)\varkappa^4$$
$$+y\left(\frac{113}{4860}+\frac{47x}{2430}-\frac{y}{18}-\frac{7xy}{180}-\frac{4x^2y}{135}+\frac{2y^2}{5}+\frac{xy^2}{5}+\frac{2x^2y^2}{15}+\frac{x^3y^2}{10}\right)\varkappa^6+\cdots$$

$$c(\kappa) = -20-\frac{20\varkappa^2}{3}+\frac{56\varkappa^4}{27}-\frac{256\varkappa^6}{243}+\frac{14761\varkappa^8}{21870}-\frac{47633\varkappa^{10}}{98415}+\frac{61367791\varkappa^{12}}{165337200}+\cdots$$

$$\Phi_s(x,y) = \Phi_a(x,-y)$$

- Equation for $\Phi_{as}(x^3, x^4, x^8, x^9)$

$$\nabla^2\Phi_{as} - (\nabla_\mu\Phi_{as})^2 - 2\nabla^\mu\Phi_a\nabla_\mu\Phi_{as} - 2\nabla^\mu\Phi_s\nabla_\mu\Phi_{as} = 0$$

- Symmetry: $\Phi_{as}(x^8, -x^9, x^3, -x^4) = \Phi_{as}(x^3, x^4, x^8, x^9)$
- In the $AdS_2\times S^2$ case $\Phi_{as} \neq 0$ — Lunin, Roiban, Tseytlin '14

Dilaton

- Perturbative solution ($x \equiv x^3\,,\ y \equiv x^4$)

$$\Phi_a = -\log\left(f_a\sqrt{(1+\varkappa^2 xy^2)(1-\varkappa^2 y)}\right)$$

$$f_a = 1 + y\left(\frac{2}{3}+\frac{x}{3}\right)\varkappa^2 + y\left(-\frac{5}{108}-\frac{x}{27}+\frac{y}{2}+\frac{xy}{4}+\frac{x^2y}{6}\right)\varkappa^4$$
$$+y\left(\frac{113}{4860}+\frac{47x}{2430}-\frac{y}{18}-\frac{7xy}{180}-\frac{4x^2y}{135}+\frac{2y^2}{5}+\frac{xy^2}{5}+\frac{2x^2y^2}{15}+\frac{x^3y^2}{10}\right)\varkappa^6+\cdots$$

$$c(\kappa) = -20-\frac{20\varkappa^2}{3}+\frac{56\varkappa^4}{27}-\frac{256\varkappa^6}{243}+\frac{14761\varkappa^8}{21870}-\frac{47633\varkappa^{10}}{98415}+\frac{61367791\varkappa^{12}}{165337200}+\cdots$$

$$\Phi_s(x,y) = \Phi_a(x,-y)$$

- Equation for $\Phi_{as}(x^3,x^4,x^8,x^9)$

$$\nabla^2\Phi_{as} - (\nabla_\mu\Phi_{as})^2 - 2\nabla^\mu\Phi_a\nabla_\mu\Phi_{as} - 2\nabla^\mu\Phi_s\nabla_\mu\Phi_{as} = 0$$

- Symmetry: $\Phi_{as}(x^8,-x^9,x^3,-x^4) = \Phi_{as}(x^3,x^4,x^8,x^9)$
- In the $AdS_2\times S^2$ case $\Phi_{as} \neq 0$ — Lunin, Roiban, Tseytlin '14

On RR-fields

- $F_{01234}\,,\ \ F_{56789}$ do not vanish
- $\partial_\rho(\sqrt{-g}F^{\mu\nu\rho}) + \frac{1}{6}\sqrt{-g}F^{\mu\nu\delta\eta\tau}H_{\delta\eta\tau} = 0 \implies$

  $F_{013}\,, F_{014}\,, F_{018}\,, F_{019}\,, F_{563}\,, F_{564}\,, F_{568}\,, F_{569} \neq 0 \implies C_{01}\,,\ C_{56} \neq 0$

- $\partial_\rho(\sqrt{-g}F^{\mu_1\cdots\mu_4\rho}) = -\frac{1}{36}\epsilon^{\mu_1\cdots\mu_4\mu_5\cdots\mu_{10}}H_{\mu_5\mu_6\mu_7}F_{\mu_8\mu_9\mu_{10}} \implies$

  $F_{01234}\,, F_{01238}\,, F_{01239}\,, F_{01248}\,, F_{01249}\,, F_{01378}\,, F_{01379}\,, F_{01478}\,, F_{01479}\,, F_{01789} \neq 0$

- Einstein equations require $C_{\mu\nu} \neq 0\,,\ \mu = 0,1,2\,,\ \nu = 5,6,7$
- This generates more nonvanishing five-form components
- Does the axion vanish ?
- $e^\Phi F$ might have simpler structure

The light-cone gauge Hamiltonian for the world-sheet action

- Introduce the light-cone coordinates

$$x_- = \phi - t, \qquad x_+ = (1-a)\,t + a\,\phi$$

- Impose the uniform light-cone gauge

$$x_+ = \tau\,, \quad p_+ = 1\,, \quad 2r = \int_{-r}^{r} \mathrm{d}\sigma\, p_+ = (1-a)J + aE\,, \quad H = -\int_{-r}^{r} \mathrm{d}\sigma\, p_- = E - J$$

Solving the Virasoro constraints for p_-, one gets the l.c. Hamiltonian

$$\mathcal{H} = -p_-(p_\mu, x^\mu, x'^\mu).$$

- Rescaling the fields with powers of g and expanding in g, one finds an expansion of the l.c. action in powers of fields

$$S = \int \mathrm{d}\tau \mathrm{d}\sigma \left(p_\mu \dot{x}^\mu - \mathcal{H}_2 - \frac{1}{g}\mathcal{H}_4 - \cdots \right),$$

Quartic Hamiltonian

The quadratic Lagrangian is $\mathscr{L}_2 = P_{a\dot{a}}\dot{Y}^{a\dot{a}} + P_{\alpha\dot{\alpha}}\dot{Z}^{\alpha\dot{\alpha}} - \mathcal{H}_2$ where

$$\mathcal{H}_2 = \frac{1}{4}(P_{a\dot{a}}P^{a\dot{a}} + P_{\alpha\dot{\alpha}}P^{\alpha\dot{\alpha}}) + \frac{1+\varkappa^2}{2}(Y_{a\dot{a}}Y^{a\dot{a}} + Z_{\alpha\dot{\alpha}}Z^{\alpha\dot{\alpha}} + Y'_{a\dot{a}}Y'^{a\dot{a}} + Z'_{\alpha\dot{\alpha}}Z'^{\alpha\dot{\alpha}})$$

The quartic Hamiltonian becomes $\mathcal{H}_4 = \mathcal{H}_4^G + \mathcal{H}_4^{WZ}$

$$\begin{aligned}
\mathcal{H}_4^G &= 2(1+\varkappa^2)\left(Z_{\alpha\dot{\alpha}}Z^{\alpha\dot{\alpha}}Z'_{\beta\dot{\beta}}Z'^{\beta\dot{\beta}} - Y_{a\dot{a}}Y^{a\dot{a}}Y'_{b\dot{b}}Y'^{b\dot{b}}\right) \\
&+2(1+\varkappa^2)^2\left(Z_{\alpha\dot{\alpha}}Z^{\alpha\dot{\alpha}}Y'_{b\dot{b}}Y'^{b\dot{b}} - Y_{a\dot{a}}Y^{a\dot{a}}Z'_{\beta\dot{\beta}}Z'^{\beta\dot{\beta}}\right) \\
&+\frac{\varkappa^2}{2}\left(Z_{\alpha\dot{\alpha}}Z^{\alpha\dot{\alpha}}P_{\beta\dot{\beta}}P^{\beta\dot{\beta}} - Y_{a\dot{a}}Y^{a\dot{a}}P_{b\dot{b}}P^{b\dot{b}}\right) \\
&+\frac{(2a-1)}{8}\Bigg[\frac{1}{4}(P_{a\dot{a}}P^{a\dot{a}} + P_{\alpha\dot{\alpha}}P^{\alpha\dot{\alpha}})^2 - 4(1+\varkappa^2)^2(Y_{a\dot{a}}Y^{a\dot{a}} + Z_{\alpha\dot{\alpha}}Z^{\alpha\dot{\alpha}})^2 \\
&+2(1+\varkappa^2)(P_{a\dot{a}}P^{a\dot{a}} + P_{\alpha\dot{\alpha}}P^{\alpha\dot{\alpha}})(Y'_{a\dot{a}}Y'^{a\dot{a}} + Z'_{\alpha\dot{\alpha}}Z'^{\alpha\dot{\alpha}}) + 4(1+\varkappa^2)^2(Y'_{a\dot{a}}Y'^{a\dot{a}} + Z'_{\alpha\dot{\alpha}}Z'^{\alpha\dot{\alpha}})^2 \\
&-4(1+\varkappa^2)(P_{a\dot{a}}Y'^{a\dot{a}} + P_{\alpha\dot{\alpha}}Z'^{\alpha\dot{\alpha}})^2\Bigg],
\end{aligned}$$

$$\mathcal{H}_4^{WZ} = 8i\varkappa(1+\varkappa^2)^{\frac{1}{2}}\left[Z^{3\dot{4}}Z^{4\dot{3}}(P_{3\dot{3}}Z'^{3\dot{3}} - P_{4\dot{4}}Z'^{4\dot{4}}) + Y^{1\dot{2}}Y^{2\dot{1}}(P_{1\dot{1}}Y'^{1\dot{1}} - P_{2\dot{2}}Y'^{2\dot{2}})\right].$$

The gauge dependent terms multiplying $(2a-1)$ are invariant under SO(8).

Tree level bosonic S-matrix

- The S-matrix $\mathbb{S}$ is related to the $\mathbb{T}$-matrix as $\mathbb{S}=\mathbb{1}+\frac{i}{g}\mathbb{T}$.
- States $a^\dagger_{M\dot{M}}(p)a^\dagger_{N\dot{N}}(p')\,|0\rangle$, $M=(a,\alpha)$ and $\dot{M}=(\dot{a},\dot{\alpha})$, $p>p'$:

$$|Y_{a\dot{c}}Y'_{b\dot{d}}\rangle\ ,\ |Y_{a\dot{c}}Z'_{\beta\dot{\delta}}\rangle\ ,\ |Z_{\alpha\dot{\gamma}}Y'_{b\dot{d}}\rangle\ ,\ |Z_{\alpha\dot{\gamma}}Y'_{\beta\dot{\delta}}\rangle$$

- In the undeformed case the invariance of $\mathbb{S}$ with respect to $\mathfrak{psu}(2|2)\oplus\mathfrak{psu}(2|2)$ leads to the factorisation

$$\mathbb{T}^{P\dot{P},Q\dot{Q}}_{M\dot{M},N\dot{N}}=(-1)^{\epsilon_{\dot{M}}(\epsilon_N+\epsilon_Q)}T^{PQ}_{MN}\delta^{\dot{P}}_{\dot{M}}\delta^{\dot{Q}}_{\dot{N}}+(-1)^{\epsilon_Q(\epsilon_{\dot{M}}+\epsilon_{\dot{P}})}\delta^P_M\delta^Q_N T^{\dot{P}\dot{Q}}_{\dot{M}\dot{N}}\ .$$

  1. Dotted and undotted indices are referred to two copies of $\mathfrak{psu}(2|2)$
  2. ϵ_M and $\epsilon_{\dot{M}}$ are 0 for bosonic (Latin) indices and 1 for fermionic (Greek) ones

- We find that the deformed $\mathbb{T}$-matrix also admits the same factorisation

$$T^{cd}_{ab}=A\,\delta^c_a\delta^d_b+B\,\delta^d_a\delta^c_b+W\,\epsilon_{ab}\delta^d_a\delta^c_b\,,$$
$$T^{\gamma\delta}_{\alpha\beta}=D\,\delta^\gamma_\alpha\delta^\delta_\beta+E\,\delta^\delta_\alpha\delta^\gamma_\beta+W\,\epsilon_{\alpha\beta}\,\delta^\delta_\alpha\delta^\gamma_\beta\,,$$
$$T^{c\delta}_{a\beta}=G\,\delta^c_a\delta^\delta_\beta\,,\qquad T^{\gamma d}_{\alpha b}=L\,\delta^\gamma_\alpha\delta^d_b\,,$$

Tree level bosonic S-matrix

- The S-matrix $\mathbb{S}$ is related to the $\mathbb{T}$-matrix as $\mathbb{S} = \mathbb{1} + \frac{i}{g}\mathbb{T}$.
- States $a^\dagger_{M\dot{M}}(p) a^\dagger_{N\dot{N}}(p') |0\rangle$, $M = (a, \alpha)$ and $\dot{M} = (\dot{a}, \dot{\alpha})$, $p > p'$:

$$|Y_{a\dot{c}} Y'_{b\dot{d}}\rangle \ , \ |Y_{a\dot{c}} Z'_{\beta\dot{\delta}}\rangle \ , \ |Z_{\alpha\dot{\gamma}} Y'_{b\dot{d}}\rangle \ , \ |Z_{\alpha\dot{\gamma}} Y'_{\beta\dot{\delta}}\rangle$$

- In the undeformed case the invariance of $\mathbb{S}$ with respect to $\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2)$ leads to the factorisation

$$\mathbb{T}^{P\dot{P},Q\dot{Q}}_{M\dot{M},N\dot{N}} = (-1)^{\epsilon_{\dot{M}}(\epsilon_N+\epsilon_Q)} T^{PQ}_{MN} \delta^{\dot{P}}_{\dot{M}} \delta^{\dot{Q}}_{\dot{N}} + (-1)^{\epsilon_Q(\epsilon_{\dot{M}}+\epsilon_{\dot{P}})} \delta^P_M \delta^Q_N T^{\dot{P}\dot{Q}}_{\dot{M}\dot{N}} \, .$$

  1. Dotted and undotted indices are referred to two copies of $\mathfrak{psu}(2|2)$
  2. ϵ_M and $\epsilon_{\dot{M}}$ are 0 for bosonic (Latin) indices and 1 for fermionic (Greek) ones

- We find that the deformed $\mathbb{T}$-matrix also admits the same factorisation

$$T^{cd}_{ab} = A\,\delta^c_a \delta^d_b + B\,\delta^d_a \delta^c_b + W\,\epsilon_{ab}\delta^d_a \delta^c_b \, ,$$

$$T^{\gamma\delta}_{\alpha\beta} = D\,\delta^\gamma_\alpha \delta^\delta_\beta + E\,\delta^\delta_\alpha \delta^\gamma_\beta + W\,\epsilon_{\alpha\beta}\,\delta^\delta_\alpha \delta^\gamma_\beta \, ,$$

$$T^{c\delta}_{a\beta} = G\,\delta^c_a \delta^\delta_\beta \, , \qquad T^{\gamma d}_{\alpha b} = L\,\delta^\gamma_\alpha \delta^d_b \, ,$$

Tree level bosonic S-matrix

- The S-matrix $\mathbb{S}$ is related to the $\mathbb{T}$-matrix as $\mathbb{S} = \mathbb{1} + \frac{i}{g}\mathbb{T}$.
- States $a^\dagger_{M\dot{M}}(p)a^\dagger_{N\dot{N}}(p')\,|0\rangle$, $M = (a, \alpha)$ and $\dot{M} = (\dot{a}, \dot{\alpha})$, $p > p'$:

$$|Y_{a\dot{c}}Y'_{b\dot{d}}\rangle\ ,\ |Y_{a\dot{c}}Z'_{\beta\dot{\delta}}\rangle\ ,\ |Z_{\alpha\dot{\gamma}}Y'_{b\dot{d}}\rangle\ ,\ |Z_{\alpha\dot{\gamma}}Y'_{\beta\dot{\delta}}\rangle$$

- In the undeformed case the invariance of $\mathbb{S}$ with respect to $\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2)$ leads to the factorisation

$$\mathbb{T}^{P\dot{P},Q\dot{Q}}_{M\dot{M},N\dot{N}} = (-1)^{\epsilon_{\dot{M}}(\epsilon_N+\epsilon_Q)}\mathcal{T}^{PQ}_{MN}\delta^{\dot{P}}_{\dot{M}}\delta^{\dot{Q}}_{\dot{N}} + (-1)^{\epsilon_Q(\epsilon_{\dot{M}}+\epsilon_{\dot{P}})}\delta^P_M\delta^Q_N\mathcal{T}^{\dot{P}\dot{Q}}_{\dot{M}\dot{N}}\ .$$

  1. Dotted and undotted indices are referred to two copies of $\mathfrak{psu}(2|2)$
  2. ϵ_M and $\epsilon_{\dot{M}}$ are 0 for bosonic (Latin) indices and 1 for fermionic (Greek) ones

- We find that the deformed $\mathbb{T}$-matrix also admits the same factorisation

$$\mathcal{T}^{cd}_{ab} = A\,\delta^c_a\delta^d_b + B\,\delta^d_a\delta^c_b + W\,\epsilon_{ab}\delta^d_a\delta^c_b\,,$$

$$\mathcal{T}^{\gamma\delta}_{\alpha\beta} = D\,\delta^\gamma_\alpha\delta^\delta_\beta + E\,\delta^\delta_\alpha\delta^\gamma_\beta + W\,\epsilon_{\alpha\beta}\,\delta^\delta_\alpha\delta^\gamma_\beta\,,$$

$$\mathcal{T}^{c\delta}_{a\beta} = G\,\delta^c_a\delta^\delta_\beta\,, \qquad \mathcal{T}^{\gamma d}_{\alpha b} = L\,\delta^\gamma_\alpha\delta^d_b\,,$$

Tree level bosonic S-matrix

Introduce $\nu = \frac{\varkappa}{(1+\varkappa^2)^{\frac{1}{2}}} = \frac{2\eta}{1+\eta^2}\,, \quad \omega = (1+\varkappa^2)^{\frac{1}{2}}\sqrt{1+p^2} = \sqrt{\frac{1+p^2}{1-\nu^2}}\,.$

The coefficients are

$$A(p,p') = \frac{1-2a}{4}(p\omega' - p'\omega) + \frac{1}{4}\frac{(p-p')^2+\nu^2(\omega-\omega')^2}{p\omega'-p'\omega}\,,$$

$$B(p,p') = -E(p,p') = \frac{pp'+\nu^2\omega\omega'}{p\omega'-p'\omega}\,,$$

$$D(p,p') = \frac{1-2a}{4}(p\omega' - p'\omega) - \frac{1}{4}\frac{(p-p')^2+\nu^2(\omega-\omega')^2}{p\omega'-p'\omega}\,,$$

$$G(p,p') = -L(p',p) = \frac{1-2a}{4}(p\omega' - p'\omega) - \frac{1}{4}\frac{\omega^2-\omega'^2}{p\omega'-p'\omega}\,,$$

$$W(p,p') = i\nu$$

1. W corresponds to the contribution of the Wess-Zumino term
2. The matrix $\mathcal{T}$ is recovered from its matrix elements as follows
$$\mathcal{T} = \mathcal{T}_{MN}^{PQ}\, E_P^M \otimes E_Q^N = \mathcal{T}_{ab}^{cd}\, E_c^a \otimes E_d^b + \mathcal{T}_{\alpha\beta}^{\gamma\delta}\, E_\gamma^\alpha \otimes E_\delta^\beta + \mathcal{T}_{a\beta}^{c\delta}\, E_c^a \otimes E_\delta^\beta + \mathcal{T}_{\alpha b}^{\gamma d}\, E_\gamma^\alpha \otimes E_d^b$$
3. $\mathcal{T}$ satisfies the classical Yang-Baxter equation for any ν
$$[\mathcal{T}_{12}(p_1,p_2), \mathcal{T}_{13}(p_1,p_3) + \mathcal{T}_{23}(p_2,p_3)] + [\mathcal{T}_{13}(p_1,p_3), \mathcal{T}_{23}(p_2,p_3)] = 0$$

Tree level bosonic S-matrix

Introduce $\nu = \frac{\varkappa}{(1+\varkappa^2)^{\frac{1}{2}}} = \frac{2\eta}{1+\eta^2}\,, \quad \omega = (1+\varkappa^2)^{\frac{1}{2}}\sqrt{1+p^2} = \sqrt{\frac{1+p^2}{1-\nu^2}}\,.$

The coefficients are

$$A(p,p') = \frac{1-2a}{4}(p\omega' - p'\omega) + \frac{1}{4}\frac{(p-p')^2 + \nu^2(\omega-\omega')^2}{p\omega' - p'\omega}\,,$$

$$B(p,p') = -E(p,p') = \frac{pp' + \nu^2\omega\omega'}{p\omega' - p'\omega}\,,$$

$$D(p,p') = \frac{1-2a}{4}(p\omega' - p'\omega) - \frac{1}{4}\frac{(p-p')^2 + \nu^2(\omega-\omega')^2}{p\omega' - p'\omega}\,,$$

$$G(p,p') = -L(p',p) = \frac{1-2a}{4}(p\omega' - p'\omega) - \frac{1}{4}\frac{\omega^2 - \omega'^2}{p\omega' - p'\omega}\,,$$

$$W(p,p') = i\nu$$

1. W corresponds to the contribution of the Wess-Zumino term
2. The matrix $\mathcal{T}$ is recovered from its matrix elements as follows
$$\mathcal{T} = \mathcal{T}_{MN}^{PQ}\, E_P^M \otimes E_Q^N = \mathcal{T}_{ab}^{cd}\, E_c^a \otimes E_d^b + \mathcal{T}_{\alpha\beta}^{\gamma\delta}\, E_\gamma^\alpha \otimes E_\delta^\beta + \mathcal{T}_{a\beta}^{c\delta}\, E_c^a \otimes E_\delta^\beta + \mathcal{T}_{\alpha b}^{\gamma d}\, E_\gamma^\alpha \otimes E_d^b$$
3. $\mathcal{T}$ satisfies the classical Yang-Baxter equation for any ν
$$[\mathcal{T}_{12}(p_1,p_2), \mathcal{T}_{13}(p_1,p_3) + \mathcal{T}_{23}(p_2,p_3)] + [\mathcal{T}_{13}(p_1,p_3), \mathcal{T}_{23}(p_2,p_3)] = 0$$

Comparison with q-deformed S-matrix

- The q-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ S-matrix is — Hoare, Hollowood, Miramontes '11

$$\mathbf{S} = S_{\mathfrak{su}(2)} S \,\hat{\otimes}\, S\,, \quad S_{\mathfrak{su}(2)} = \frac{e^{ia(p_2\mathcal{E}_1 - p_1\mathcal{E}_2)}}{\sigma_{12}^2} \frac{x_1^+ + \xi}{x_1^- + \xi} \frac{x_2^- + \xi}{x_2^+ + \xi} \cdot \frac{x_1^- - x_2^+}{x_1^+ - x_2^-} \frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}}$$

1. S is the $\mathfrak{psu}(2|2)_q$-invariant S-matrix — Beisert, Koroteev '08
2. σ is the dressing factor
3. $\mathcal{E}$ is the q-deformed dispersion relation whose large g expansion starts with ω

- Crossing equation gives the dressing factor — Hoare, Hollowood, Miramontes '11

$$\sigma_{12} = e^{i\theta_{12}}\,, \quad \theta_{12} = \chi(x_1^+, x_2^+) + \chi(x_1^-, x_2^-) - \chi(x_1^+, x_2^-) - \chi(x_1^-, x_2^+),$$

$$\chi(x_1, x_2) = i \oint \frac{dz}{2\pi i} \frac{1}{z - x_1} \oint \frac{dz'}{2\pi i} \frac{1}{z' - x_2} \log \frac{\Gamma_{q^2}(1 + \frac{ig}{2}(u(z) - u(z')))}{\Gamma_{q^2}(1 - \frac{ig}{2}(u(z) - u(z')))}.$$

$\Gamma_q(x)$ is the q-deformed Gamma function

Comparison with q-deformed S-matrix

- The q-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ S-matrix is Hoare, Hollowood, Miramontes '11

$$\mathbf{S} = S_{\mathfrak{su}(2)} S \hat{\otimes} S \,, \quad S_{\mathfrak{su}(2)} = \frac{e^{ia(p_2\mathcal{E}_1 - p_1\mathcal{E}_2)}}{\sigma_{12}^2} \frac{x_1^+ + \xi}{x_1^- + \xi} \frac{x_2^- + \xi}{x_2^+ + \xi} \cdot \frac{x_1^- - x_2^+}{x_1^+ - x_2^-} \frac{1 - \frac{1}{x_1^- x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}}$$

1. S is the $\mathfrak{psu}(2|2)_q$-invariant S-matrix Beisert, Koroteev '08
2. σ is the dressing factor
3. $\mathcal{E}$ is the q-deformed dispersion relation whose large g expansion starts with ω

- Crossing equation gives the dressing factor Hoare, Hollowood, Miramontes '11

$$\sigma_{12} = e^{i\theta_{12}} \,, \quad \theta_{12} = \chi(x_1^+, x_2^+) + \chi(x_1^-, x_2^-) - \chi(x_1^+, x_2^-) - \chi(x_1^-, x_2^+),$$

$$\chi(x_1, x_2) = i \oint \frac{dz}{2\pi i} \frac{1}{z - x_1} \oint \frac{dz'}{2\pi i} \frac{1}{z' - x_2} \log \frac{\Gamma_{q^2}(1 + \frac{ig}{2}(u(z) - u(z')))}{\Gamma_{q^2}(1 - \frac{ig}{2}(u(z) - u(z')))}.$$

$\Gamma_q(x)$ is the q-deformed Gamma function

Comparison with q-deformed S-matrix

- Large g expansion of **S** requires $q = e^{-\upsilon/g}$ where υ is kept fixed
- Due to the factorisation one compares the $\mathcal{T}$-matrix with the **T**-matrix

$$S^{1/2}_{\mathfrak{su}(2)}\, \mathbb{1}_g\, S = \mathbb{1} + \frac{i}{g}\mathbf{T}$$

- Computing the **T**-matrix and comparing it with the $\mathcal{T}$-matrix, one finds that they are equal to each other iff

$$\upsilon = \nu \implies q = e^{-\nu/g}$$

Summary

- Found NSNS background fields in the string frame
- Successfully matched in the large tension limit the tree-level world-sheet bosonic S-matrix with the q-deformed S-matrix obtained from symmetries
- Analysed the GKP string and its generalisations, and found that the strings never touched the singularity. One-loop correction?
- Interesting open problems
  - Use the NSNS background fields and the type IIB supergravity equations of motion to find the full supergravity background
  - Expand the sigma model action up to quadratic order in fermions and extract RR fields
  - Symmetries of the deformed background
  - Dual gauge theory
  - Analyse various spinning string solutions and derive finite-gap integral equations
  - Rigid strings are described by a q-deformed Neumann model

    Arutynov, Medina-Rincon '14
  - Exact spectrum through the mirror TBA
  - Minimal surfaces solutions