Sigma-model perturbation theory and AdS/CFT spectrum

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``Gauge Fields from Strings''

HUMBOLDT-UNIVERSITÄT ZU BERLIN

Based on work with L. Bianchi, B.Hoare

And with M.S Bianchi, L. Bianchi, A. Bres and E. Vescovi

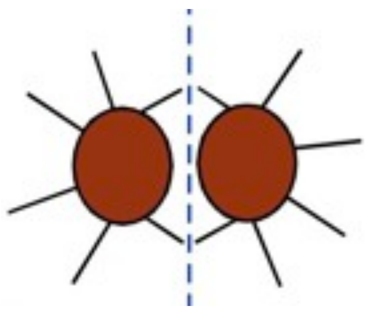
Supersymmetric Field Theories

Nordita Stockholm, August 15 2014

Outline

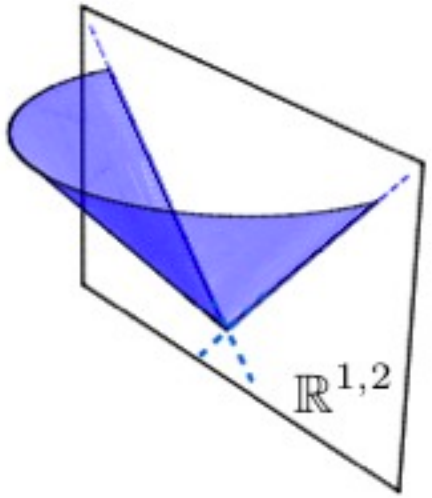
Sigma-model perturbation theory I

Unitarity methods for scattering in 2d



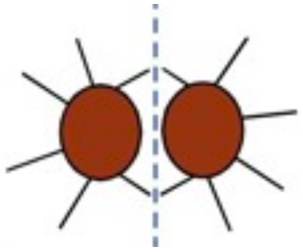
Sigma-model perturbation theory II

ABJM cusp anomaly at two loops and the interpolating function h(λ)



Sigma-model perturbation theory I

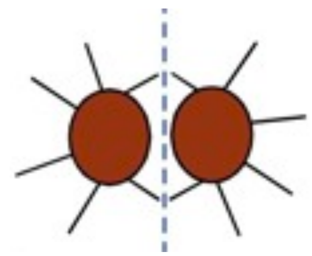
Unitarity methods for scattering in 2d

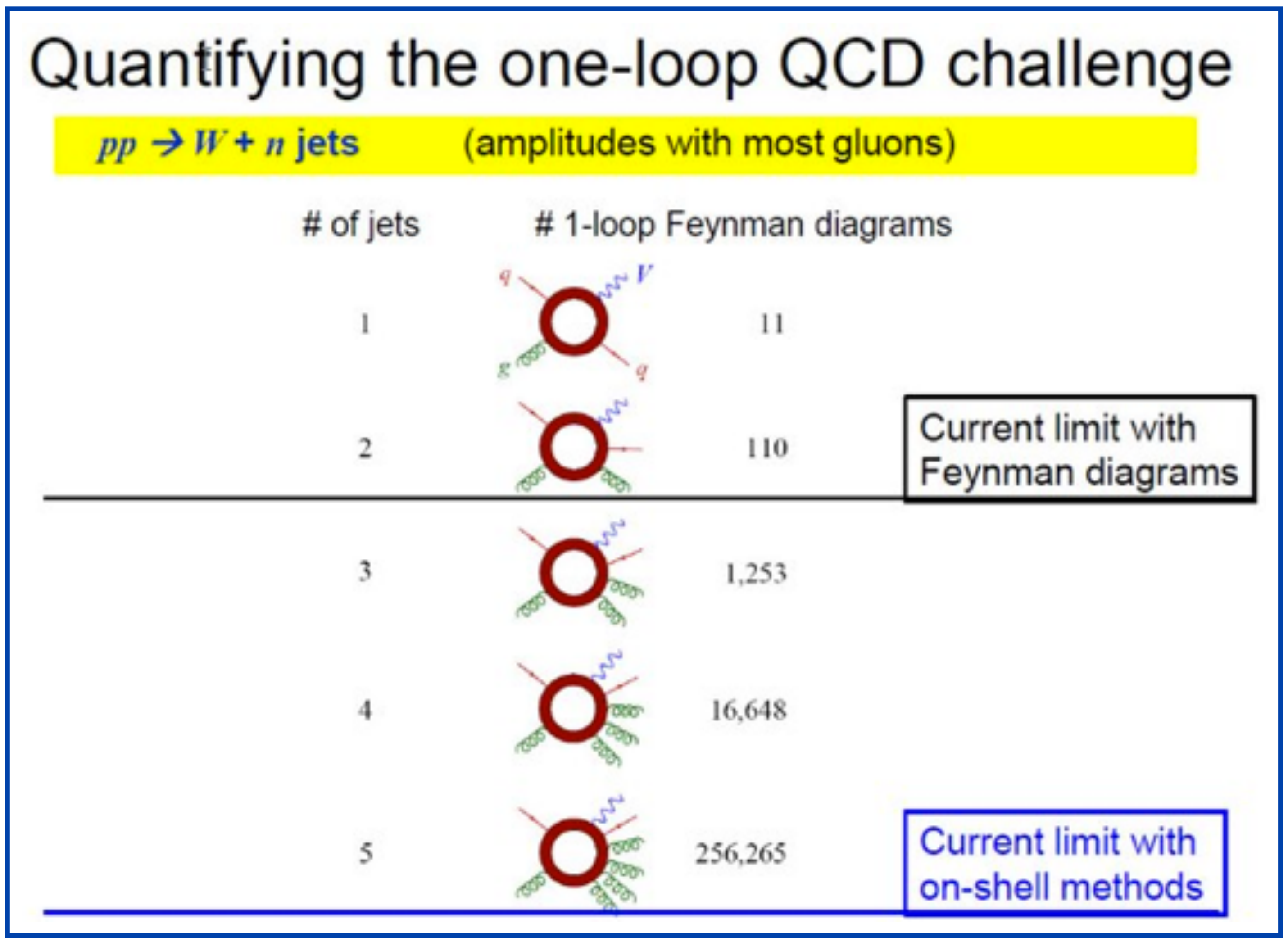


L. Bianchi, V.Forini, B.Hoare, arXiv: 1304.1798
O. T. Engelund, R. W. McKeown, R. Roiban, arXiv: 1304.4281
L. Bianchi, B. Hoare, arXiv: 1405.7947

Calculating scattering amplitudes efficiently

Remarkable efficiency of unitarity-based methods [Bern, Dixon, Dunbar, Kosower, 1994] for calculation of amplitudes in various qft's and various dimensions (non-abelian gauge theories, Chern-Simons theories, supergravity).





[from a L. Dixon talk]

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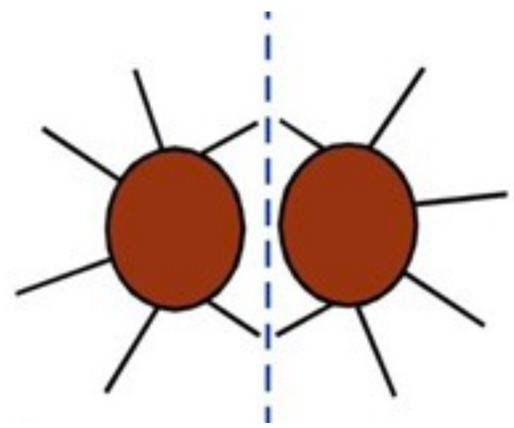
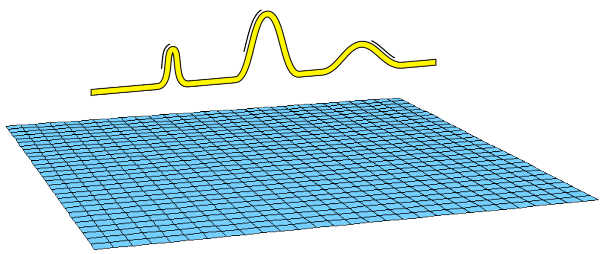


Goal: apply to evaluation of amplitudes of two-dimensional cases of interest.

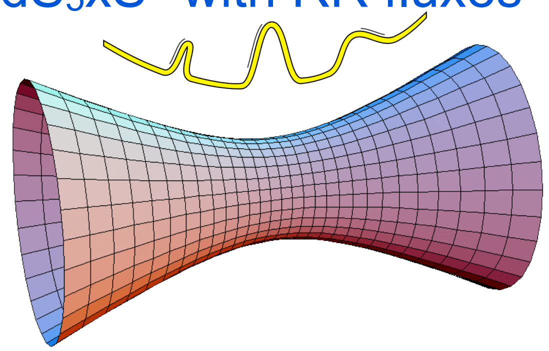
String worldsheet scattering

- Worldsheet amplitudes ($N \to \infty$, free strings), scattering of the (2d) lagrangean excitations.
  Non-trivial interactions due to highly non trivial background.

flat space



$AdS_5 x S^5$ with RR fluxes



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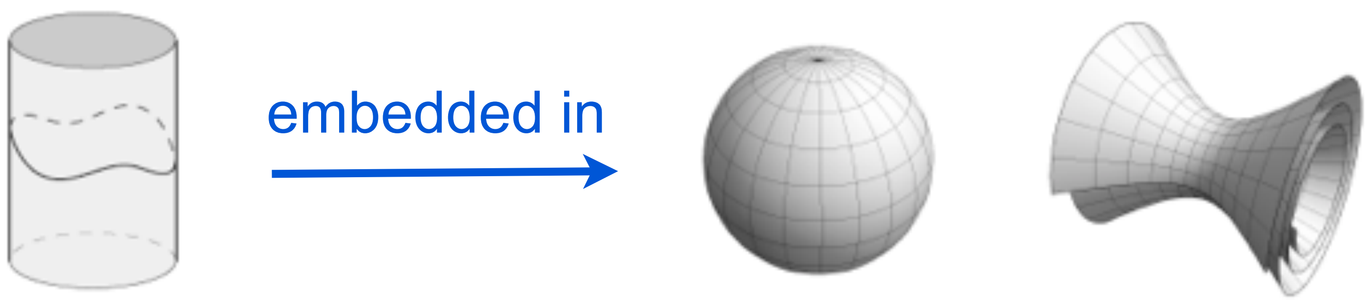

$AdS_5 \times S^5$

- Because of RR-background need a GS formulation

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \sqrt{-h}\, h^{ab} G_{MN}(X) \partial_a X^M \partial_b X^N + \text{fermions}$$

loop counting parameter $\hat{g} = \frac{2\pi}{\sqrt{\lambda}}$

- Work on a gauge-fixed sigma model (uniform light-cone gauge)

[Arutyunov, Frolov, Plefka, Zamaklar 2006]

$$H_{ws} = \int d\sigma\, \mathcal{H}_{ws} = -\int d\sigma\, p_- \equiv E - J$$

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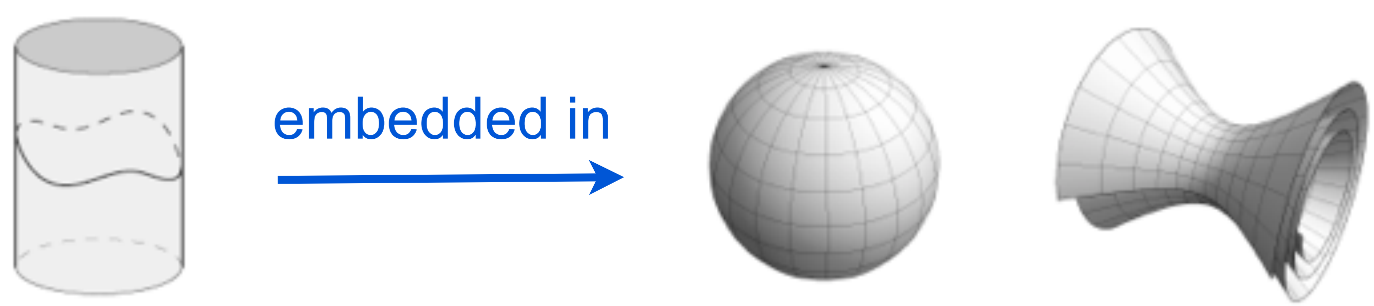

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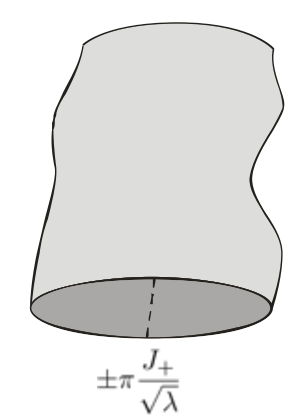
$$H_{ws} = \int d\sigma\, \mathcal{H}_{ws} = -\int d\sigma\, p_- \equiv E - J$$

- Decompactification limit $\dfrac{J_+}{\sqrt{\lambda}} \to \infty$ and large tension expansion $\hat{g} \to \infty$



$\longrightarrow$ sensible definition of a **perturbative worldsheet S-matrix**

AdS/CFT (internal) S-matrix I

[Klose McLoughlin Roiban Zarembo 2007]

- This S-matrix is the perturbative expansion of the exact AdS_5/CFT_4 S-matrix *aka* "spin chain S-matrix" : the rhs of asymptotic Bethe eqs

[Staudacher 2004]
[Beisert Staudacher 2005]
[Beisert 2005]

$$
\begin{aligned}
1 &= \prod_{j}^{K_4} \frac{x_{4j}^+}{x_{4j}^-} \\
1 &= \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{u_{2k}-u_{2j}-i}{u_{2k}-u_{2j}+i} \prod_{j=1}^{K_3} \frac{u_{2k}-u_{3j}+\frac{i}{2}}{u_{2k}-u_{3j}-\frac{i}{2}} \\
1 &= \prod_{j=1}^{K_2} \frac{u_{3k}-u_{2j}+\frac{i}{2}}{u_{3k}-u_{2j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3k}-x_{4j}^+}{x_{3k}-x_{4j}^-} \\
\left(\frac{x_{4k}^+}{x_{4k}^-}\right)^L &= \prod_{j=1}^{K_4} \left(\frac{u_{4k}-u_{4j}+i}{u_{4k}-u_{4j}-i}\, e^{2i\theta(x_{4k},x_{4j})}\right) \prod_{j=1}^{K_3} \frac{x_{4k}^- - x_{3j}}{x_{4k}^+ - x_{3j}} \prod_{j=1}^{K_5} \frac{x_{4k}^- - x_{5j}}{x_{4k}^+ - x_{5j}} \\
1 &= \prod_{j=1}^{K_6} \frac{u_{5k}-u_{6j}+\frac{i}{2}}{u_{5k}-u_{6j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5k}-x_{4j}^+}{x_{5k}-x_{4j}^-} \\
1 &= \prod_{\substack{j=1\\ j\neq k}}^{K_6} \frac{u_{6k}-u_{6j}-i}{u_{6k}-u_{6j}+i} \prod_{j=1}^{K_5} \frac{u_{6k}-u_{5j}+\frac{i}{2}}{u_{6k}-u_{5j}-\frac{i}{2}}
\end{aligned}
$$

$$x(u) = \tfrac{1}{2}u + \tfrac{1}{2}u\sqrt{1-2g^2/u^2}$$

$$x^\pm(u) = x(u \pm \tfrac{i}{2})$$

$$\delta D = \delta E = g^2 \sum_{j=1}^{K_4} \Big(\frac{i}{x_{4j}^+} - \frac{i}{x_{4j}^-}\Big)$$

$$g^2 = \frac{\lambda}{8\pi^2}$$

Describe the **exact asymptotic** spectrum of anomalous dimensions of local composite operators and energies of their dual string configurations.

AdS/CFT (internal) S-matrix II

Assuming integrability (consistency with Yang-Baxter equation) and using global symmetries one can:

- derive exact dispersion relation $\epsilon = \sqrt{1 + h(\lambda)^2 \sin^2 \frac{p}{2}}$ [Beisert 2006]

- derive two-particle S-matrix entering the Bethe equations [Staudacher 2004] [Beisert Staudacher 2005] [Beisert 2005]

$$S_{12} = S^0 \, \mathbf{S_{12}}$$

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$$S_{12} = S^0 \, \mathbf{S_{12}}$$

up to one (/more) scalar factor(/s), fixed with additional constraints like “crossing symmetry” and semiclassical string data. [Janik 2005]

- The scalar phase is the hardest thing to compute, crucial for the spectrum. Particularly in some models relevant in AdS_3/CFT_2 where solutions to crossing-like equations are difficult to determine.

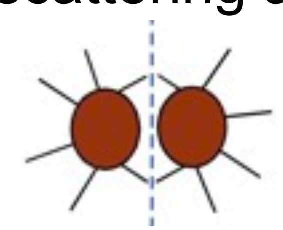
Ben Hoare talk later

Motivation

- Provide 2d scattering perturbation theory with efficient tools.
- Extract information about the overall factors of scattering matrix.
- Provide tests of quantum integrability for certain string backgrounds.
- Methodological: techniques never really applied in two dimensions.

Initiate the use of unitarity-based methods for perturbative S-matrix in massive two-dimensional field theories.

Construct one-loop $2 \to 2$ scattering amplitude with <u>standard</u> unitarity



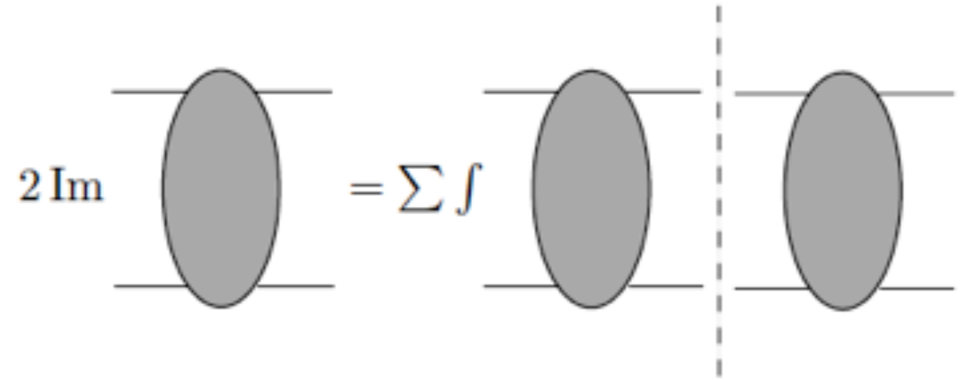
directly from the corresponding on-shell tree-level amplitudes.

Unitarity cuts method

- Consequence of unitarity of the S-matrix (optical theorem).

$$S = 1 + i\,T \xrightarrow[S^\dagger = S^{-1}]{\text{unitarity}} 2\,\mathrm{Im}(T) = T\,T^\dagger$$

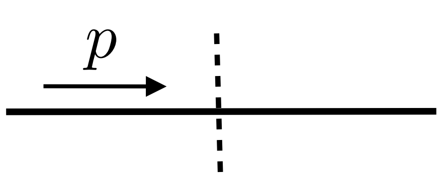
Inserting a complete set of states



- Relates a certain loop amplitude to a lower order one.

- Imaginary part of the amplitude contains the branch-cut information.

- Cutting (Cutkosky) rules ex.



$$\longrightarrow \quad 2\pi i\,\delta(p^2 - m^2)$$

- **Unitarity cuts method**: revert the order, **find n-loop amplitude fusing lower order ones**

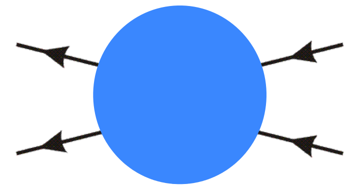
- Only the singular part can be reconstructed (logs or polilogs.)

- Cut-constructibility of a theory always to be verified.

  (Special known case in 4d: massless susy gauge theories are 1-loop cut-constructibles).

Two-dimensional scattering

Two-body scattering process of a theory invariant under space and time translations



described via the four-point amplitude

$$\langle \Phi^P(p_3)\Phi^Q(p_4)\,|\mathbb{S}|\,\Phi_M(p_1)\Phi_N(p_2)\rangle = (2\pi)^2\delta^{(d)}(p_1+p_2-p_3-p_4)\,\mathcal{A}^{PQ}_{MN}(p_1,p_2,p_3,p_4)$$

For d=2 and in the single mass case, scattering 2 → 2 is simple.

Particles either preserve or exchange their momenta

$$\delta^{(2)}(p_1+p_2-p_3-p_4) = J(p_1,p_2)\left(\delta(\mathrm{p}_1-\mathrm{p}_3)\delta(\mathrm{p}_2-\mathrm{p}_4)+\delta(\mathrm{p}_1-\mathrm{p}_4)\delta(\mathrm{p}_2-\mathrm{p}_3)\right)$$

The Jacobian $J(p_1,p_2) = 1/(\partial\epsilon_{\mathrm{p}_1}/\partial\mathrm{p}_1 - \partial\epsilon_{\mathrm{p}_2}/\partial\mathrm{p}_2)$ depends on dispersion relation.

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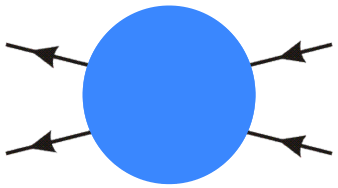

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S-matrix element defined by

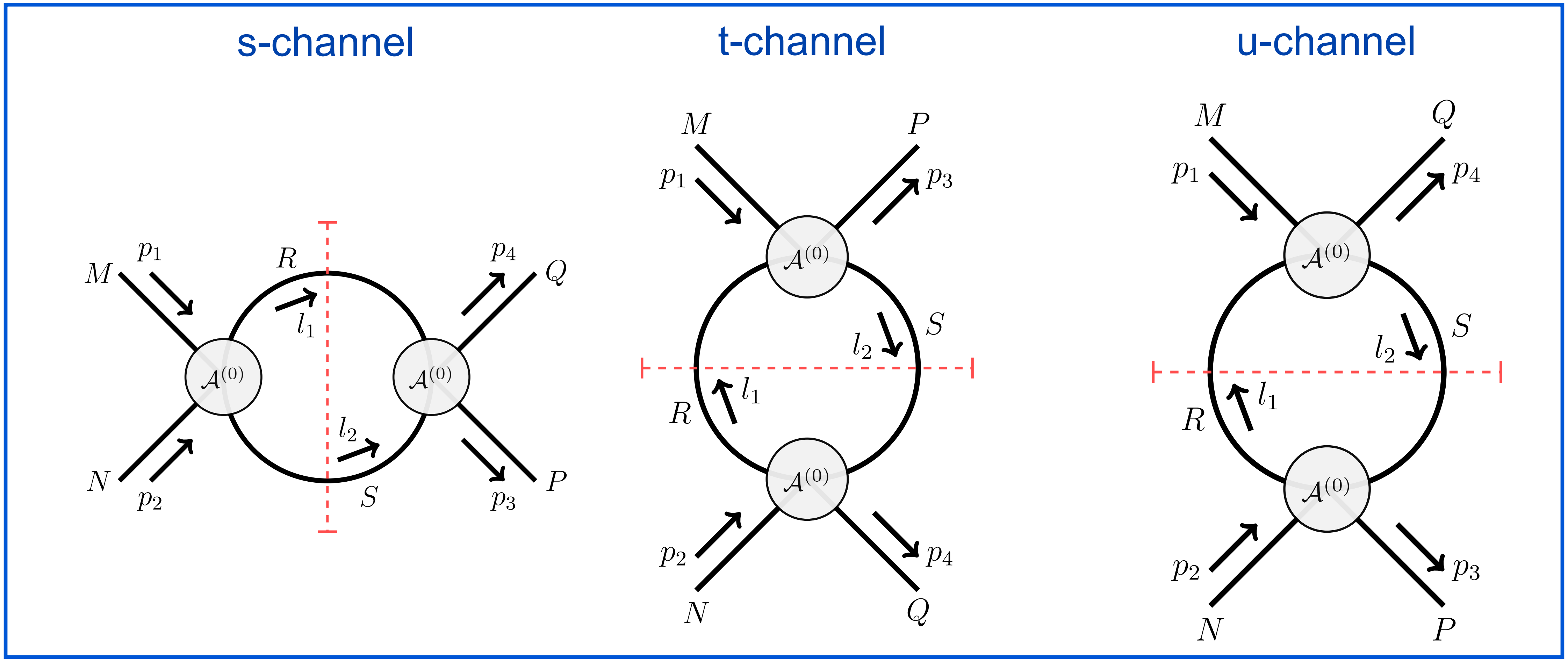
$$S^{PQ}_{MN}(p_1,p_2) \equiv \frac{J(p_1,p_2)}{4\epsilon_1\epsilon_2}\mathcal{A}^{PQ}_{MN}(p_1,p_2,p_1,p_2)$$

Dispersion relation for asymptotic states (equal masses =1): $\epsilon_i^2 = 1+\mathrm{p}_\mathrm{i}^2$

Fix ordering of incoming states $\mathrm{p}_1 > \mathrm{p}_2$.

Scattering in d=2: unitarity cuts (1)

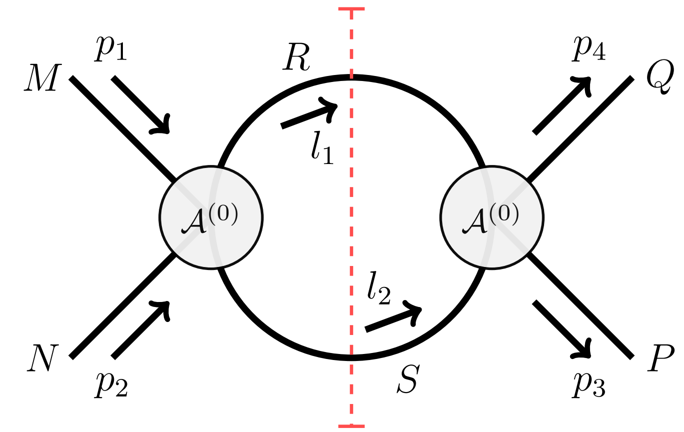
One-loop result from unitarity techniques: contributions from three cut-diagrams



Example: s-cut contribution. Glue tree-amplitudes.

$$\mathcal{A}^{(1)}{}^{PQ}_{MN}(p_1,p_2,p_3,p_4)|_{s-cut} = \frac{1}{2}\int \frac{d^2 l_1}{(2\pi)^2}\int \frac{d^2 l_2}{(2\pi)^2}\; i\pi\delta^+({l_1}^2-1)\; i\pi\delta^+(l_2^2-1)$$
$$\times\, \mathcal{A}^{(0)}{}^{RS}_{MN}(p_1,p_2,l_1,l_2)\mathcal{A}^{(0)}{}^{PQ}_{SR}(l_2,l_1,p_3,p_4)$$

Scattering in d=2: unitarity cuts (2)



- Use 2-momentum conservation at the first vertex

$$\widetilde{\mathcal{A}}^{(1)}{}^{PQ}_{MN}(p_1,p_2,p_3,p_4)|_{s-cut} = \frac{1}{2}\int \frac{d^2 l_1}{(2\pi)^2}\, i\pi\delta^+({l_1}^2-1)\, i\pi\delta^+((l_1-p_1-p_2)^2-1)$$
$$\times \widetilde{\mathcal{A}}^{(0)}{}^{RS}_{MN}(p_1,p_2,l_1,-l_1+p_1+p_2)\, \widetilde{\mathcal{A}}^{(0)}{}^{PQ}_{SR}(-l_1+p_1+p_2,l_1,p_3,p_4)$$

- Use the zeroes of δ - functions in the $\widetilde{\mathcal{A}}^{(0)}$: loop momenta are completely frozen.

  Can pull tree-level amplitudes out of the integral (like $f(x)\,\delta(x) = f(0)\,\delta(x)$)

- Restore loop momentum off-shell $i\pi\delta^+(l_1^2-1) \longrightarrow \dfrac{1}{l_1^2-1}$

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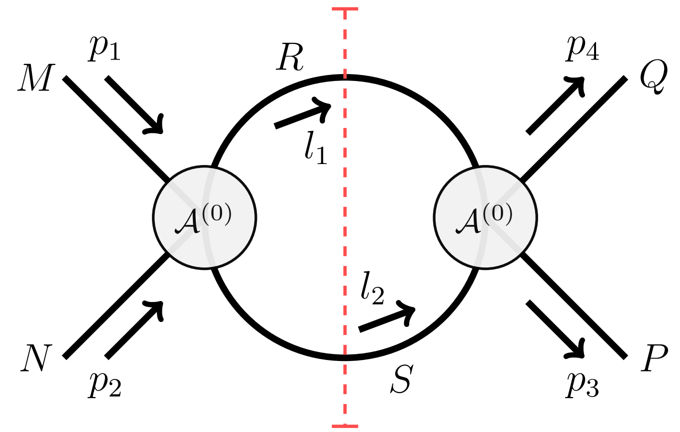

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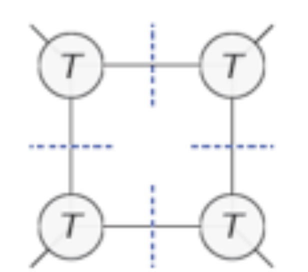
Two-particle cuts in d=2 at one loop are maximal cuts.

Expect same as quadrupole cuts in d=4: $A_4^{1-loop} = \sum (A_4^{tree})^4\, I_{box}$



4-points amplitude at one-loop

Tree-level amplitudes can be pulled out of the integral, evaluated at those zeroes.

A simple sum over discrete solutions of the on-shell conditions

$$
\begin{aligned}
\widetilde{\mathcal{A}}^{(1)\,PQ}_{\;MN}(p_1,p_2,p_3,p_4) = \;& \frac{I(p_1+p_2)}{2}\Big[\widetilde{\mathcal{A}}^{(0)\,RS}_{\;MN}(p_1,p_2,p_1,p_2)\widetilde{\mathcal{A}}^{(0)\,PQ}_{\;SR}(p_2,p_1,p_3,p_4) \\
& + \widetilde{\mathcal{A}}^{(0)\,RS}_{\;MN}(p_1,p_2,p_2,p_1)\widetilde{\mathcal{A}}^{(0)\,PQ}_{\;SR}(p_1,p_2,p_3,p_4)\Big] \\
+ \;& I(p_1-p_3)\widetilde{\mathcal{A}}^{(0)\,SP}_{\;MR}(p_1,p_3,p_1,p_3)\widetilde{\mathcal{A}}^{(0)\,RQ}_{\;SN}(p_1,p_2,p_3,p_4) \\
+ \;& I(p_1-p_4)\widetilde{\mathcal{A}}^{(0)\,SQ}_{\;MR}(p_1,p_4,p_1,p_4)\widetilde{\mathcal{A}}^{(0)\,RP}_{\;SN}(p_1,p_2,p_4,p_3)
\end{aligned}
$$

weighted by scalar “bubble” integrals

$$I(p) = \int \frac{d^2 q}{(2\pi)^2}\frac{1}{(q^2-1+i\epsilon)((q-p)^2-1+i\epsilon)}$$

Inherently **finite** formula.

One of initial motivation of our work: ordinary Feynman diagrammatics was problematic (divergencies did not cancel). Recently clarified in [Roiban, Sundin, Tseytlin, Wulff 14]

Final formula for the S-matrix

Final formula for the S-matrix (choose $p_3 = p_1, \quad p_4 = p_2$)

$$
\begin{aligned}
S^{(1)}{}^{PQ}_{MN}(p_1, p_2) = \frac{1}{4(\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2)} \Big[& \tilde{S}^{(0)}{}^{RS}_{MN}(p_1, p_2)\tilde{S}^{(0)}{}^{PQ}_{RS}(p_1, p_2) I(p_1 + p_2) \\
& + \tilde{S}^{(0)}{}^{SP}_{MR}(p_1, p_1)\tilde{S}^{(0)}{}^{RQ}_{SN}(p_1, p_2) I(0) \\
& + \tilde{S}^{(0)}{}^{SQ}_{MR}(p_1, p_2)\tilde{S}^{(0)}{}^{PR}_{SN}(p_1, p_2) I(p_1 - p_2) \Big] \,,
\end{aligned}
$$

where $$\tilde{S}^{(0)}(p_1, p_2) = 4(\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2) S^{(0)}(p_1, p_2)$$

Sum of products of two tree-level amplitudes weighted by scalar bubble integrals

$$
\begin{aligned}
I_s &\equiv I(p_1 + p_2) = \frac{1}{\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2} - \frac{\operatorname{arsinh}(\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2)}{4\pi i\,(\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2)} \\
I_t &\equiv I(0) = \frac{1}{4\pi i} \\
I_u &\equiv I(p_1 - p_2) = \frac{\operatorname{arsinh}(\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2)}{4\pi i\,(\epsilon_2\,\mathrm{p}_1 - \epsilon_1\,\mathrm{p}_2)}
\end{aligned}
$$

Possible absence of rational terms: formula cannot be completely general !
Needs to be tested on various examples.

Final formula for the S-matrix

$[M]=0$ bosons
$[M]=1$ fermions

Final formula for the S-matrix (choose $p_3=p_1, \quad p_4=p_2$)

$$
\begin{aligned}
S^{(1)}{}^{PQ}_{MN}(p_1,p_2) =& \frac{1}{4(\epsilon_2\,\mathrm{p}_1-\epsilon_1\,\mathrm{p}_2)}\Big[\tilde{S}^{(0)}{}^{RS}_{MN}(p_1,p_2)\tilde{S}^{(0)}{}^{PQ}_{RS}(p_1,p_2)I(p_1+p_2)\\
&+(-1)^{[P][S]+[R][S]}\,\tilde{S}^{(0)}{}^{SP}_{MR}(p_1,p_1)\tilde{S}^{(0)}{}^{RQ}_{SN}(p_1,p_2)I(0)\\
&+(-1)^{[P][R]+[Q][S]+[R][S]+[P][Q]}\,\tilde{S}^{(0)}{}^{SQ}_{MR}(p_1,p_2)\tilde{S}^{(0)}{}^{PR}_{SN}(p_1,p_2)I(p_1-p_2)\Big]
\end{aligned}
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\end{aligned}
$$

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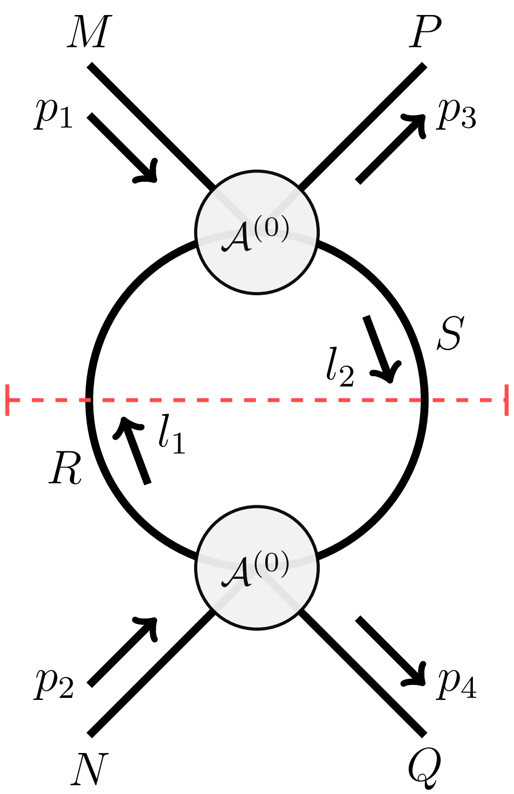
Remarks

- The t-channel cut is special.

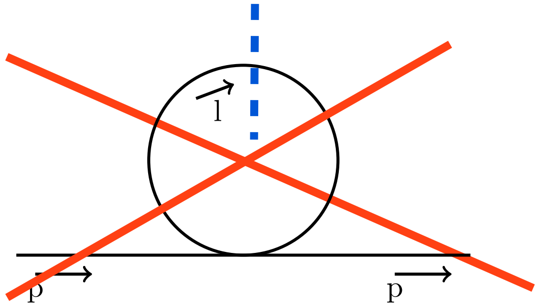
  - Using first $\delta(\mathrm{p}_1 - \mathrm{p}_3)\delta(\mathrm{p}_2 - \mathrm{p}_4)$
    makes it ill-defined and requires a prescription:
    use delta-function **only at the end** of the calculation

  - Asymmetrical wrt choice of the vertex
    used to solve momenta:
    leads to a consistency condition



$$\tilde{S}^{(0)}{}^{SP}_{MR}(p_1,p_1)\,\tilde{S}^{(0)}{}^{RQ}_{SN}(p_1,p_2) \;=\; \tilde{S}^{(0)}{}^{PS}_{MR}(p_1,p_2)\,\tilde{S}^{(0)}{}^{QR}_{SN}(p_2,p_2)$$

- We are NOT including contributions from tadpoles (no physical cuts)



- A inherently finite result says **nothing** about UV-finiteness or renormalizability.
  Might be missing rational terms following from regularization procedure.

Cut-constructibility to be always checked

Relativistic models

- Bosonic models:
  - generalized sine-Gordon: gauged WZW model for a coset G/H = SO(n + 1)/SO(n) plus an integrable potential (n=1: sine-Gordon, n=2: complex sine-Gordon)

The method works <u>up to a finite shift in the coupling</u>.

- Supersymmetric generalizations (``Pohlmeyer reductions'' of string theories):
  - $\mathcal{N} = 1, 2$ supersymmetric sine-Gordon

The method reproduces the <u>full</u> result.

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In two cases (complex sine-Gordon and Pohlmeyer-reduced AdS_3xS^3 theory) cut-constructibility is highly non trivial!

Theory only integrable at classical level. Quantum counterterms restoring various properties of integrability (e.g. Yang-Baxter equation).

It is this “quantum integrable” result that the unitarity method gives.

AdS/CFT S-matrix: exact and perturbative structure

In the asymptotic case, matrix structure of the exact S-matrix is uniquely fixed by a (centrally extended) PSU(2|2)^2 symmetry algebra.

From symmetries and integrability follows a **group factorization**

$$\mathbb{S} = e^{i\,\theta}\ \hat{S}^{PSU(2|2)} \otimes \hat{S}^{PSU(2|2)}$$

$$\hat{S}_{AB}^{CD} = \begin{cases} A\delta_a^c\delta_b^d + B\delta_a^d\delta_b^c \\ D\delta_\alpha^\gamma\delta_\beta^\delta + E\delta_\alpha^\delta\delta_\beta^\gamma \\ C\epsilon_{ab}\epsilon^{\gamma\delta} & F\epsilon_{\alpha\beta}\epsilon^{cd} \\ G\delta_a^c\delta_\beta^\delta & H\delta_a^d\delta_\beta^\gamma \\ L\delta_\alpha^\gamma\delta_b^d & K\delta_\alpha^\delta\delta_b^c \end{cases}$$

Each factor has manifest $SU(2)\times SU(2)$ invariance

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Each factor has manifest $SU(2)\times SU(2)$ invariance

Logarithms

Matrix structure,
rational dependence
on momenta

A perturbative check should recover the tensor structure, group factorization and exponentiation of the logarithms.

Perturbative structure of worldsheet S-matrix

Expansion of symmetry-determined and phase parts ($\theta^{(0)}$ absorbed in $T^{(0)}$)

$$\hat{S} = \mathbf{1} + i \sum_{n=1} g^{-n} \hat{T}^{(n-1)} \qquad\qquad \theta = \sum_{n=1}^{\infty} g^{-n} \theta^{(n-1)}$$

requires one-loop logarithms to contribute only to the diagonal terms

$$S = \mathbf{1} + \frac{i}{g} \hat{T}^{(0)} + \frac{i}{g^2} \left(\hat{T}^{(1)} + \theta^{(1)} \, \mathbf{1} \right)$$

Goal: compute one loop worldsheet S-matrix
"bootstrapping" it from tree level.

String worldsheet S-matrix

Superstring action

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \sqrt{-h}\, h^{ab} G_{MN}(X) \partial_a X^M \partial_b X^N + \text{fermions}$$

- Green-Schwarz formulation for fermions $\qquad \varrho_a = \partial_a x^\mu E_\mu{}^A \Gamma_A$

quadratic part $\qquad L_F = i(\sqrt{-g} g^{ab} \delta^{IJ} - \epsilon^{ab} s^{IJ}) \bar{\theta}^I \varrho_a D_b \theta^J$

$$D_a \theta^I = \left(\partial_a + \frac{1}{4} \partial_a x^\mu \omega_\mu{}^{AB} \Gamma_{AB} \right) \theta^I + \frac{1}{2} \varrho_a \Gamma_{01234} \epsilon^{IJ} \theta^J$$

- Use an interpolating lightcone -gauge

[Arutyunov Frolov Zamaklar 06]

$$X^+ = (1+a)\, t + a\, \varphi \equiv \tau + a\, \sigma$$

AdS_5 $\qquad S^5$

$a = 1/2 \qquad$ light-cone gauge

$a = 0 \qquad$ temporal gauge

Gauge fixing

[Klose McLoughlin Roiban Zarembo 06]

Bosonic lagrangean to quartic order in the fields $X = (Y, Z)$

$$
\begin{aligned}
L = {} & \frac{1}{2}\,(\partial_{\mathrm{a}} X)^2 - \frac{1}{2}\,X^2 + \frac{1}{4}\,Z^2\,(\partial_{\mathrm{a}} Z)^2 - \frac{1}{4}\,Y^2\,(\partial_{\mathrm{a}} Y)^2 + \frac{1}{4}\,(Y^2 - Z^2)\left(\dot{X}^2 + \acute{X}^2\right) \\
& - \frac{1-2a}{8}\,(X^2)^2 + \frac{1-2a}{4}\,(\partial_{\mathrm{a}} X \cdot \partial_{\mathrm{b}} X)^2 - \frac{1-2a}{8}\,\left[(\partial_{\mathrm{a}} X)^2\right]^2 .
\end{aligned}
$$

Lorentz invariance (quadratic part) broken by interactions.

Massive states with relativistic dispersion relation $\epsilon = \sqrt{1 + p^2}$

$$
\epsilon = \sqrt{1 + \frac{\lambda}{\pi^2}\,\sin^2\frac{p}{2}}
$$

loop corrections

Bosonic part invariant under $SO(4) \times SO(4)$.

Worldsheet fields

- Worldsheet fields (embedding coordinates in AdS_5xS^5)

$$T,\ \Phi,\ Y^m,\ Z^m,\ \text{fermions}$$

can be represented as bispinors $SO(4) \simeq (SU(2) \times SU(2))/\mathbb{Z}_2$

$$Y_{a\dot{a}} = (\sigma_m)_{a\dot{a}}\, Y^m, \qquad Z_{\alpha\dot{\alpha}} = (\sigma_\mu)_{\alpha\dot{\alpha}}\, Z^\mu \qquad a, \dot{a}, \alpha, \dot{\alpha} = 1, 2$$

Bosons and fermions form bi-fundamental representation of $\mathrm{PSU}(2|2)_L \times \mathrm{PSU}(2|2)_R$

- Formal definition of a bi-fundamental supermultiplet $\Phi_{A\dot{A}}$, $A = (a|\alpha)$ $\dot{A} = (\dot{a}|\dot{\alpha})$ providing a basis for the definition of the S-matrix.

- Two-particle S-matrix is 256 x 256

$$\mathbb{S}\, |\Phi_{A\dot{A}}(p)\Phi_{B\dot{B}}(p')\rangle = |\Phi_{C\dot{C}}(p)\Phi_{D\dot{D}}(p')\rangle\, \mathbb{S}_{A\dot{A}B\dot{B}}^{C\dot{C}D\dot{D}}(p, p')$$

Worldsheet S-matrix: explicit perturbative evaluation

- Expansion of worldsheet S-matrix in coupling: defines the T-matrix

$$\mathbb{S} = \mathbb{1} + \frac{1}{\hat{g}}\,\mathbb{T}^{(0)} + \frac{1}{\hat{g}^2}\,\mathbb{T}^{(1)} + \ldots = \mathbb{1} + \mathbb{T} \qquad\qquad \hat{g} = \frac{\sqrt{\lambda}}{2\pi}$$

- **Tree level result**: first non trivial order in the perturbative expansion.
  Obtained applying LSZ reduction to quartic vertices of the lagrangean.

[Klose McLoughlin Roiban Zarembo 06]

✓ It exhibits group factorization $\mathbb{T} = \mathbb{1} \otimes \mathrm{T} + \mathrm{T} \otimes \mathbb{1}$

$$\begin{aligned}
\mathrm{T}_{ab}^{cd} &= \mathrm{A}\,\delta_a^c\delta_b^d + \mathrm{B}\,\delta_a^d\delta_b^c\,, & \mathrm{T}_{ab}^{\gamma\delta} &= \mathrm{C}\,\epsilon_{ab}\epsilon^{\gamma\delta}\,,\\
\mathrm{T}_{\alpha\beta}^{\gamma\delta} &= \mathrm{D}\,\delta_\alpha^\gamma\delta_\beta^\delta + \mathrm{E}\,\delta_\alpha^\delta\delta_\beta^\gamma\,, & \mathrm{T}_{\alpha\beta}^{cd} &= \mathrm{F}\,\epsilon_{\alpha\beta}\epsilon^{cd}\,,\\
\mathrm{T}_{a\beta}^{c\delta} &= \mathrm{G}\,\delta_a^c\delta_\beta^\delta\,, & \mathrm{T}_{\alpha b}^{\gamma d} &= \mathrm{L}\,\delta_\alpha^\gamma\delta_b^d\,,\\
\mathrm{T}_{a\beta}^{\gamma d} &= \mathrm{H}\,\delta_a^d\delta_\beta^\gamma\,, & \mathrm{T}_{\alpha b}^{c\delta} &= \mathrm{K}\,\delta_\alpha^\delta\delta_b^c\,.
\end{aligned}$$

where e.g. $\quad A(p_1,p_2) = \frac{1}{4}\left[(1-2a)(\epsilon_2 \mathrm{p}_1 - \epsilon_1 \mathrm{p}_2) + \frac{(p_1-p_2)^2}{\epsilon_2 \mathrm{p}_1 - \epsilon_1 \mathrm{p}_2}\right]$

✓ Coincide with the related expansion of the exact spin chain S-matrix.

Worldsheet S-matrix at one loop via unitarity cuts: result

$$
\begin{aligned}
S_{AB}^{CD}(\mathrm{p}_1,\mathrm{p}_2) &= \exp\left(i\varphi_a(\mathrm{p}_1,\mathrm{p}_2)\right)\ \tilde{S}_{AB}^{CD} \\
&= \exp\Big(\underline{-\frac{i}{2\hat{g}}(e_2\mathrm{p}_1-e_1\mathrm{p}_2)(a-\tfrac{1}{2})} + \frac{i}{\hat{g}^2}\tilde{\varphi}(\mathrm{p}_1,\mathrm{p}_2)\Big)\ \tilde{S}_{AB}^{CD} + \mathcal{O}\Big(\frac{1}{\hat{g}^3}\Big)
\end{aligned}
$$

where

Ex. $$A^{(1)} = 1 + \frac{i}{4\,\hat{g}}\frac{(\mathrm{p}_1-\mathrm{p}_2)^2}{\epsilon_2\mathrm{p}_1-\epsilon_1\mathrm{p}_2} + \frac{1}{4\,\hat{g}^2}\Big(\mathrm{p}_1\mathrm{p}_2 - \frac{(\mathrm{p}_1+\mathrm{p}_2)^4}{8(\epsilon_2\mathrm{p}_1-\epsilon_1\mathrm{p}_2)^2}\Big)$$

and

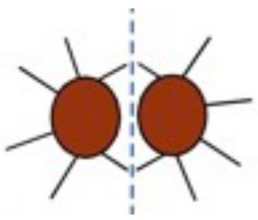
$$\tilde{\varphi}(\mathrm{p}_1,\mathrm{p}_2) = \frac{1}{2\pi}\frac{\mathrm{p}_1^2\mathrm{p}_2^2\big((\epsilon_2\mathrm{p}_1-\epsilon_1\mathrm{p}_2)-(\epsilon_1\epsilon_2-\mathrm{p}_1\mathrm{p}_2)\,\mathrm{arsinh}[\epsilon_2\mathrm{p}_1-\epsilon_1\mathrm{p}_2]\big)}{(\epsilon_2\mathrm{p}_1-\epsilon_1\mathrm{p}_2)^2}$$

- ✓ All **logarithmic** dependence encoded in the scalar factor (as required from integrability!)
- ✓ All rational dependence coincides with related expansion of EXACT worldsheet S-matrix
- ✓ All gauge dependence encoded in the scalar factor (as required from physical arguments!)

Remarks and a wish list

- Enough evidence that for large class of 2-d models (relativistic and not) four-points one-loop amplitudes are cut-constructible

  > Standard unitarity (2-particle cuts) reproduces **all rational terms**, **up to shifts in the coupling**.



- Efficient way for

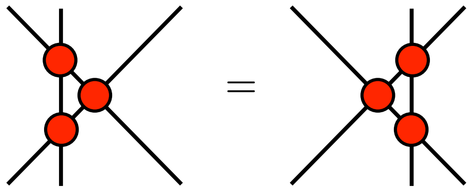
  > Checks of quantum integrability aspects (e.g. group factorization).

  > Proposing/checking matrix structure and overall phases for models relevant for the AdS/CFT correspondence. **L. Bianchi, B. Hoare, arXiv: 1405.7947**

- Cut-constructibility "criterion"

  > Integrability is crucial asset

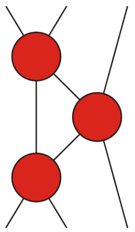
  > Structure of the one-loop S-matrix derived by unitarity cuts *automatically* satisfies the Yang-Baxter equation



$$\mathbb{S}_{12}\mathbb{S}_{13}\mathbb{S}_{23} = \mathbb{S}_{23}\mathbb{S}_{13}\mathbb{S}_{12}$$

Wish list

- ★ Two loops rational terms (all logarithms reproduced in [Engelund McEwan Roiban 2013]

- ★ Higher points: factorization should emerge $S_{3\to3} = \left(S_{2\to2}\right)^3$ 

- ★ Extend to other interesting integrable string backgrounds (require a tree-level S-matrix! and massless modes treatment) [Basso 2010]

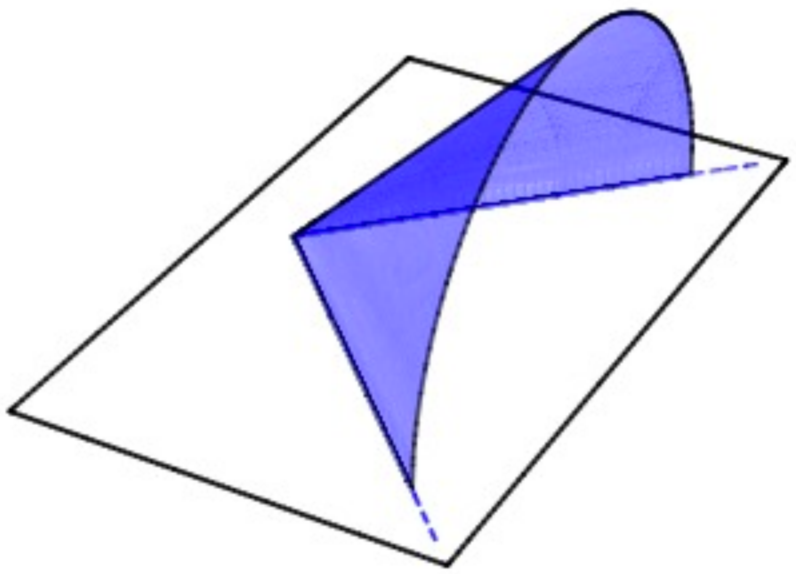
- ★ Extend to off-shell objects, including form factors and correlation functions.

[Klose McLoughlin 2012/2013]
[Engelund McEwan Roiban 2013]

String sigma-model perturbation theory II

ABJM cusp anomaly at two loops and the interpolating function h(λ)

L. Bianchi, M.S. Bianchi, A. Bres, VF, E. Vescovi, arxiv:1407.4788



AdS_4/CFT_3 and integrability

- Planar AdS_4/CFT_3 system

  $\mathcal{N}=6$ super Chern-Simons theory in 3d **and** Type IIA strings in $AdS_4 \times \mathbb{CP}^3$

  believed to be integrable: Bethe equations, quantum spectral curve approach.

- Similarities with AdS_5/CFT_4, but two important differences:

1. The string background is **not** maximally supersymmetric

Construction of the superstring action is *complicated:* coset sigma-model does not cover full superspace, issues with *κ-symmetry gauge-fixing.*

2. **Dispersion relation** entering all-integrability based calculations

$$\epsilon = \sqrt{1 + 4\, h^2(\lambda) \sin^2 \frac{p}{2}}$$

is in terms of an **unknown,** here **non-trivial**, interpolating function of the coupling.

Integrable couplings

- In $\mathcal{N}=4$ SYM the function is "trivial": $$h(\lambda_{YM}) = \frac{\sqrt{\lambda_{YM}}}{4\pi}$$

  Seen perturbatively at weak and strong coupling.

  [Gross Mikhailov Roiban 2002] [Santambrogio Zanon 2002] [Sieg 2010]
  [Hofman Maldacena 2006] [Klose, McLoughlin, Minahan, Zarembo 2007]

  Checked exactly via comparison between integrability and localisation results for the ``Brehmstrahulung function'' of N=4 SYM.

  [Correa, Henn, Sever, Maldacena 2012]

- In ABJM non-trivial dependence on the t'Hooft coupling

$$\begin{aligned} h^2(\lambda) &= \lambda^2 - \frac{2\,\pi^3}{3}\,\lambda^4 + \mathcal{O}\left(\lambda^6\right) && \lambda \ll 1 \\ h(\lambda) &= \sqrt{\frac{\lambda}{2}} - \frac{\log 2}{2\pi} + \mathcal{O}(\sqrt{\lambda})^{-1} && \lambda \gg 1 \end{aligned}$$

[Gaiotto Giombi Yin 08] [Grignani Harmark Orselli] [Nihsioka Takayanagi 08]
[Minahan, Ohlsson Sax, Sieg 09] [Leoni, Mauri, Minahan, Ohlsson Sax, Santambrogio, Sieg, Tartaglino Mazzucchellu 10]

A conjecture for the ABJM integrable coupling

Finite coupling dependence unknown from first principles.
Not yet integrability-derived ABJM ``Brehmstrahlung function''

[Lewkowycz Maldacena 2013] [Bianchi, Griguolo, Leoni, Penati, Seminara 2014]

- A conjecture exist

$$\lambda = \frac{\sinh 2\pi h(\lambda)}{2\pi}\, {}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,\frac{3}{2};-\sinh^2 2\pi h(\lambda)\right)$$

[Gromov Sizov 2014]

extrapolated by "similarities" between two all-order calculations:

> one based on integrability: "slope-function" as exact solution of the ABJM spectral curve

[Cavaglia', Fioravanti, Gromov Tateo 2014]

> one based on localization: expectation1/6 BPS Wilson loop

[Marino, Putrov, 10] [Drukker, Marino, Putrov, 10]

$$h(\lambda) = \lambda - \frac{\pi^2}{3}\lambda^3 + \frac{5\pi^4}{12}\lambda^5 - \frac{893\pi^6}{1260}\lambda^7 + \mathcal{O}(\lambda^9) \qquad \lambda \ll 1$$

$$h(\lambda) = \sqrt{\frac{1}{2}\left(\lambda - \frac{1}{24}\right)} - \frac{\log 2}{2\pi} + \mathcal{O}\left(e^{-2\pi\sqrt{2\lambda}}\right) \qquad \lambda \gg 1$$

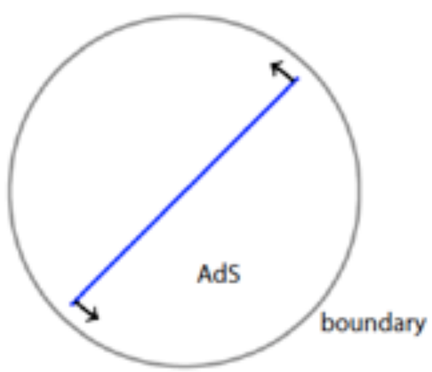
Cusp anomaly in AdS$_5$/CFT$_4$

- Weak coupling:

  > anomalous dimension of twist operators in large spin limit $\Delta_{\text{twist}} \sim f(\lambda)\, \ln S\,, \qquad S \gg 1$

  > governs renormalization of light-like cusped Wilson loops $\langle W_{\text{cusp}} \rangle \sim e^{-f(\lambda)\phi \ln \frac{\Lambda}{\epsilon}}$

- Strong coupling: corresponding string configurations are related



$\rightarrow$

$$E_{\text{classical}} \sim f(\lambda)\, \ln S\,, \qquad S \gg 1$$

$$\langle W_{\text{cusp}} \rangle = Z_{\text{string}} = \int [dX\, d\theta] e^{-S[X,\theta]}$$

[Gubser, Klebanov, Polyakov,02] [Kruczenski,02] [Kruczenski, Tirziu, Roiban, Tseytlin 07]

- Integrability gives an **all-order** equation for cusp anomaly $f(\lambda)$, BES equation matching all known perturbative results. [Beisert Eden Staduacher 2006]

ABJM cusp anomaly

- Despite nontrivial differences of the cusp physics in ABJM

  [MS Bianchi, Griguolo, Penati, Seminara 2013,14] [Marmiroli 2013] [Lewkowitz Maldacena 2013]

  BES equation is only slightly modified, therefore the prediction

  [Gromov Vieira 2008]

$$f_{\text{ABJM}}(\lambda) = \frac{1}{2} f_{\mathcal{N}=4}(\lambda_{\text{YM}})\Big|_{\frac{\sqrt{\lambda_{\text{YM}}}}{4\pi} \to h(\lambda)}$$

  from which, knowing already the N=4 SYM case,

  [Basso Korchemsky Kotanski 2007]
  [Roiban Tseytlin 2007]

$$f_{\text{ABJM}}(\lambda) = 2h(\lambda) - \frac{3\log 2}{2\pi} - \frac{K}{8\pi^2}\frac{1}{h(\lambda)} + \cdots \qquad \lambda \gg 1$$

- A **direct** string sigma-model evaluation of the LHS

$$f_{\text{ABJM}}(\lambda) = \underbrace{\sqrt{2\lambda}}_{\text{[ABJM]}} - \underbrace{\frac{5\log 2}{2\pi}}_{\text{[several papers]}} + \mathcal{O}(\sqrt{\lambda})^{-1}$$

  will give also an extimation of the rhs.

$$h(\lambda) = \sqrt{\frac{\lambda}{2}} - \frac{\log 2}{2\pi} + \mathcal{O}(\sqrt{\lambda})^{-1}$$

Superstrings on AdS_4xCP^3

- Solution of Type IIA sugra preserving 24 out of 32 supersymmetries. [Nilsson Pope, 84]

- Supercoset approach à la [Metsaev Tseytlin 98] . Sigma-model action based on

$$\frac{OSp(6|4)}{U(3)\times SO(1,3)}$$

[Arutyunov Frolov 08]
[Stefanski 08]

has 24 fermionic dof: interpreted as *partial* kappa-symmetry gauge-fixing of full action. Remaining kappa-symmetry generically removes 8 fermions.

For strings only moving in AdS_4 kappa-symmetry is enhanced:
only 12 (out of 16) physical fermions! Coset model misses 4 physical fermions.

- For these configurations (and higher order perturbation theory) start from *complete* AdS_4xCP^3 action and make alternative kappa-symmetry gauge fixing.

[Gomis Sorokin Wulff 08] [Grassi Sorokin Wulff 09] [Uvarov 09]

AdS_4xCP^3 action in AdS light-cone gauge

- Obtained in [Uvarov, 09,10] from double dimensional reduction from D=11 membrane action based on supercoset OSp(8/4)/(SO(1,3)x S7)

  [de Wit, Peeters, Plefka, Sevrin 98]

- κ-symmetry gauge-fixing: light-cone gauge, with light-cone in the boundary of AdS_4

In AdS_5xS^5 two ways to fix light-cone gauge corresponding to two inequivalent geodesics:

1. one wrapping a big circle in S^5

   > widely used in AdS/CFT (BMN, pp-waves, world-sheet S-matrix)

   > quite complicated (non-polynomial) form.

2. one running entirely inside AdS_5

   > dramatically simplifies fermionic part of the action

[Metsaev Tseytlin 00]
[Metsaev Thorn Tseytlin 00]

AdS_4xCP^3 light-cone gauge action

- Convenient to use Poincaré patch

$$ds^2_{AdS_4} = \frac{dw^2 + dx^+ dx^- + dx^1 dx^1}{w^2} \qquad x^\pm = x^2 \pm x^0$$

Take the natural light-cone coordinates on $\mathbb{R}^{1,2}$

- To fix bosonic diffeos choose a "modified" conformal gauge $\gamma^{ij} = \mathrm{diag}(-w^2, w^{-2})$

combined with standard light-cone gauge $x^+ = p^+ \tau \, , \qquad p^+ = \mathrm{const}$

- To fix κ-symmetry, choose $\Gamma^+ \Theta = (\Gamma^0 + \Gamma^2)\Theta = 0$

> Divide the 32-dimensional spinors associated to odd generators of OSp(8/4) in:

θ fermions (superPoincare)
η fermions (superconformal)

> Kill that half of the fermions related to generators with negative charge wrt J^{+-} from Lorentz group acting on Minkowski boundary

AdS4xCP3 light-cone gauge action

- AdS lc gauge-fixed action: **at most quartic** in the remaining16 fermions

$$(\theta_a, \bar\theta^a) \quad (\theta_4, \bar\theta^4) \qquad\qquad (\eta_a, \bar\eta^a) \quad (\eta_4, \bar\eta^4) \qquad\qquad a = 1, 2, 3$$

3+3 1+1 3+3 1+1

$$S = -\frac{T}{2}\int d\tau\, d\sigma\, L \qquad\qquad w \equiv e^{2\varphi}$$

$$L = \gamma^{ij}\Big\{\frac{e^{-4\varphi}}{4}\left(\partial_i x^+ \partial_j x^- + \partial_i x^1 \partial_j x^1\right) + \partial_i\varphi\partial_j\varphi + g_{MN}\partial_i z^M \partial_j z^N$$

$$+ e^{-4\varphi}\left[\partial_i x^+ \varpi_j + \partial_i x^+ \partial_j z^M \underline{h_M} + e^{-4\varphi} B \partial_i x^+ \partial_j x^+\right]\Big\}$$

$$- 2\,\varepsilon^{ij} e^{-4\varphi}\left(\underline{\omega_i}\partial_j x^+ + e^{-2\varphi}\underline{C}\partial_i x^1 \partial_j x^+ + \partial_i x^+ \partial_j z^M \underline{\ell_M}\right)$$

$$\varpi_i = i\left(\partial_i\theta_a\bar\theta^a - \theta_a\partial_i\bar\theta^a + \partial_i\theta_4\bar\theta^4 - \theta_4\partial_i\bar\theta^4 + \partial_i\eta_a\bar\eta^a - \eta_a\partial_i\bar\eta^a + \partial_i\eta_4\bar\eta^4 - \eta_4\partial_i\bar\eta^4\right)$$

$$B = 8\left[(\hat\eta_a\hat{\bar\eta}^a)^2 + \varepsilon_{abc}\hat{\bar\eta}^a\hat{\bar\eta}^b\hat{\bar\eta}^c\bar\eta^4 + \varepsilon^{abc}\hat\eta_a\hat\eta_b\hat\eta_c\eta_4 + 2\eta_4\bar\eta^4\left(\hat\eta_a\hat{\bar\eta}^a - \theta_4\bar\theta^4\right)\right]$$

- Only SU(3) symmetry is manifest (will be inherited by fluctuations on the cusp)

- No studies at quantum level so far. One of our aims is to check its quantum consistency (finiteness) and shows that is in fact simple to handle.

Anomalous radius shift

- Original ABJM **dictionary proposal** (R is the CP[3] radius) [ABJM 2008]

$$T = \boxed{\frac{R^2}{2\pi\alpha'} = 2\sqrt{2\lambda}}$$

$$\lambda = \frac{N}{k}$$

is modified to (in planar limit)

$$T = \boxed{\frac{R^2}{2\pi\alpha'} = 2\sqrt{2\left(\lambda - \frac{1}{24}\right)}}$$

[Bergman Hirano 2009]

(due to an orbifold singularity of the original, M-theory, background)

$$2\sqrt{2\lambda} - \frac{1}{12\sqrt{2}\,\lambda}$$

plays a role at 2-loop order in perturbation theory

- String perturbative expansion - in inverse string (effective) tension - is not affected. Shift: assumed, new input which plays a role when expressing the result in terms of λ.

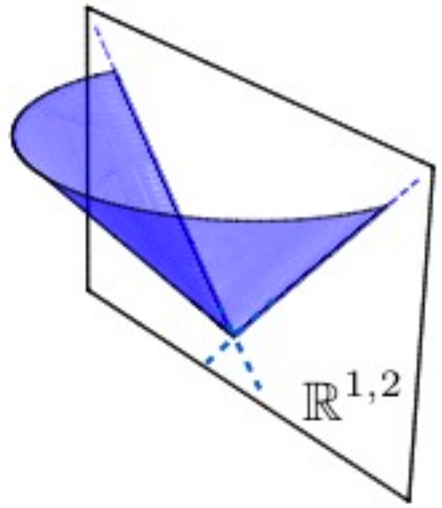
Perturbative evaluation of path integral around the cusp

- Classical solution

$$w \equiv e^{2\varphi} = \sqrt{\frac{\tau}{\sigma}} \qquad x^+ = \tau \qquad x^- = -\frac{1}{2\sigma}$$

describe a surface bounded by a null cusp, as at the AdS_4 boundary $0 = z^2 = -2x^+x^-$.

- To extract cusp anomaly, compute partition function around it.



$$\langle W_{cusp} \rangle = Z_{string} \equiv \int \mathcal{D}[x, w, z, \eta, \theta] \, e^{-S_E}$$

Expand around the solution $X = X_{\rm cl} + \tilde{X}$

and evaluate the path integral perturbatively.

$$Z_{string} \equiv e^{-\frac{1}{2} f(\lambda) V} \qquad V : \text{(infinite) 2d volume, } \sim \log S$$

As solution is “homogeneous”, i.e. fluctuation lagrangean has constant coefficients, one can factor out V.

$$f(g) = g \left[1 + \frac{a_1}{g} + \frac{a_2}{g^2} + \ldots \right] , \qquad g = \frac{T}{2} \; .$$

One loop

Very smooth calculation.

8 bosonic modes

1 real scalar $\tilde{x}^1$ with mass $\frac{1}{\sqrt{2}}$,

1 real scalar $\tilde{\varphi}$ with mass 1,

3 complex massless z^a, $a = 1, 2, 3$.

8 fermionic modes

2 massless modes,

6 massive excitations with mass $\frac{1}{2}$.

Their determinant is easily evaluated

$$
\begin{aligned}
-\ln Z_1 &= \frac{1}{2}\int \frac{d^2p}{(2\pi)^2}\left\{\ln(p^2+1)+\ln\left(p^2+\frac{1}{2}\right)+6\ln(p^2)-2\ln(p^2)-6\ln\left(p^2+\frac{1}{4}\right)\right\} \\
&= -\frac{5\ln 2}{16\pi}\underbrace{\int dtds}_{V}
\end{aligned}
$$

One-loop finiteness, expected result: $a_1 = -\dfrac{5\log 2}{2\pi}$

[McLoughlin, Roiban, Tseytlin 08] [Alday Arutyunov Bykov 08]

Two loops

Expand the action up to quartic order in fluctuations
and compute all **connected vacuum** Feynman diagrams.

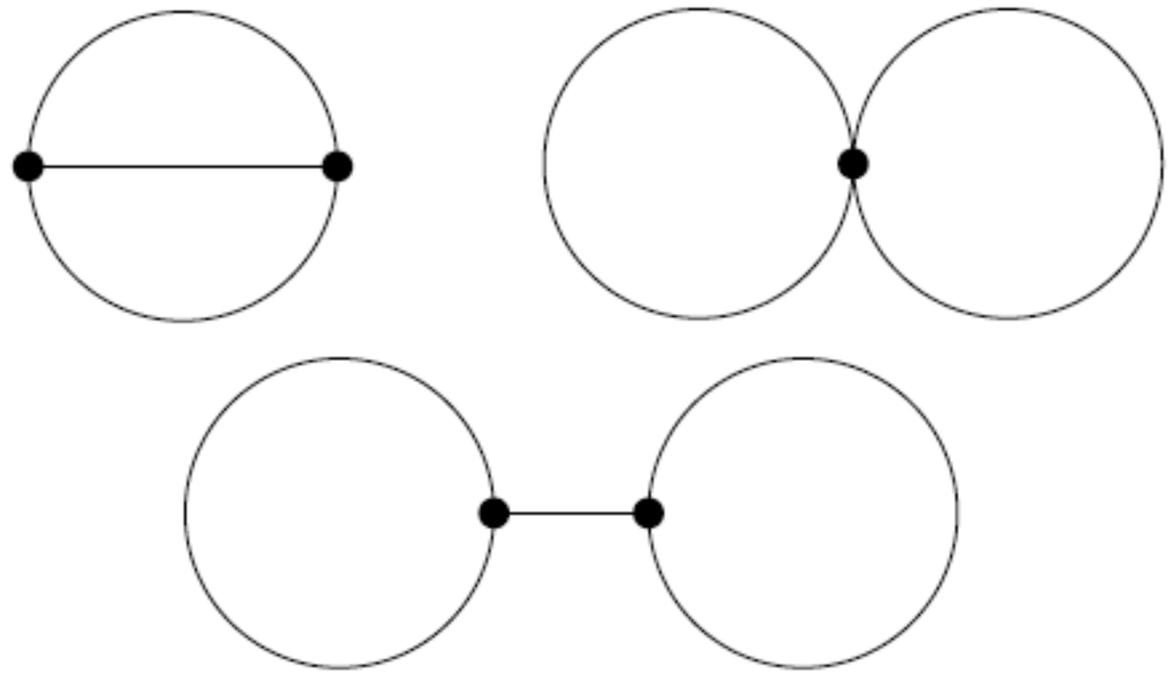
$$
\begin{aligned}
\mathcal{L}_{(3)} = \; & -8\varphi(\partial_s x^1)^2 - 2\varphi(x^1)^2 + 8\varphi x^1(\partial_s x) + 4\varphi^2(\partial_t\varphi - \partial_s\varphi) + 4\varphi[(\partial_t\varphi)^2 - (\partial_s\varphi)^2] \\
& +4\varphi(\partial_t z^a\partial_t\bar z_a - \partial_s z^a\partial_s\bar z_a) + 2\varepsilon_{abc}\partial_t z^a\bar\eta^b\bar\eta^c - 2\varepsilon^{abc}\partial_t\bar z_a\eta_b\eta_c + 4\partial_t\bar z_a\bar\eta^a\bar\eta^4 - 4\partial_t z^a\eta_a\eta_4 \\
& i\left\{\left[2i\varepsilon_{acb}z^c\bar\eta^b\partial_s\bar\theta^a - i\varepsilon_{acb}z^c\bar\eta^b\bar\theta^a - 8\varphi\eta_a\partial_s\bar\theta^a + 4\varphi\eta_a\bar\theta^a - 2i\varepsilon^{adc}\eta_a\left(\partial_s\bar z_d\theta_c + \bar z_d\partial_s\theta_c - \tfrac{1}{2}\bar z_d\theta_c\right)\right] + \text{c.c.}\right\} \\
& -4i\varphi(\partial_s\theta_4\bar\eta^4 - \partial_s\eta_4\bar\theta^4 + \eta_4\partial_s\bar\theta^4 - \theta_4\partial_s\bar\eta^4) + 8i\eta_a\bar\eta^a\partial_s x^1 - 4i\eta_a\bar\eta^a x^1 + 4i\theta_4\bar\theta^4\partial_s x^1 - 2i\theta_4\bar\theta^4 x^1 \\
& +4i\eta_4\bar\eta^4\partial_s x^1 - 2i\eta_4\bar\eta^4 x^1 + 4\partial_s\bar z_a\bar\eta^a\bar\theta^4 + 4\partial_s z^a\eta_a\theta_4
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{(4)} = \; & 32\varphi^2(\partial_s x^1)^2 + 8\varphi^2(x^1)^2 - 32\varphi^2 x^1(\partial_s x^1) + \tfrac{4}{3}\varphi^4 + \tfrac{16}{3}\varphi^3(\partial_t\varphi) + 8\varphi^2(\partial_t\varphi)^2 \\
& +\tfrac{16}{3}\varphi^3(\partial_s\varphi) + 8\varphi^2(\partial_s\varphi)^2 + 8\varphi^2(\partial_t z^a\partial_t\bar z_a + \partial_s z^a\partial_s\bar z_a) + \tfrac{1}{3}\left[\bar z_a\partial_t z^a\bar z_b\partial_t z^b + z^a\partial_t\bar z_a z^b\partial_t\bar z_b\right. \\
& \left.- z^b\bar z_b\partial_t z^a\partial_t\bar z_a - \bar z_a z^b\partial_t z^a\partial_t\bar z_b + \bar z_a\partial_s z^a\bar z_b\partial_s z^b + z^a\partial_s\bar z_a z^b\partial_s\bar z_b - z^b\bar z_b\partial_s z^a\partial_s\bar z_a - \bar z_a z^b\partial_s z^a\partial_s\bar z_b\right] \\
& -4i\partial_t\bar z_a(z^a\eta_b\bar\eta^b + \bar\eta^a z^b\eta_b) - 4i\varepsilon^{acb}\partial_t\bar z_a\bar z_c\eta_b\bar\eta^4 - 2i\varepsilon_{acb}\partial_t z^a z^c\bar\eta^b\eta_4 + 4i(\theta_4\bar\theta^4 + \eta_4\bar\eta^4)(\partial_t z^b\bar z_b - \partial_t\bar z_b z^b) \\
& +8\left[(\eta_a\bar\eta^a)^2 + \varepsilon_{abc}\bar\eta^a\bar\eta^b\bar\eta^c\bar\eta^4 + \varepsilon^{abc}\eta_a\eta_b\eta_c\eta_4 + 2\eta_4\bar\eta^4(\eta_a\bar\eta^a - \theta_4\bar\theta^4)\right] + i\left\{+2z^a\bar z_a\bar\eta_b\partial_s\theta_b - z^a\bar z_a\bar\eta^b\theta_b\right. \\
& -2\bar\eta^a\bar z_a z^b\partial_s\theta_b + \bar\eta^a\bar z_a z^b\theta_b - 8i\varepsilon_{acb}\varphi z^c\bar\eta^b\partial_s\bar\theta^a + 4i\varepsilon_{acb}\varphi z^c\bar\eta^b\bar\theta^a + 16\varphi^2\eta_a\partial_s\bar\theta^a - 8\varphi^2\eta_a\bar\theta^a \\
& -2\eta_a\partial_s\bar\theta^a|z|^2 + \eta_a\bar\theta^a|z|^2 + 2\eta_a\partial_s\bar\theta^c\bar z_c z^a - \eta_a\bar\theta^c\bar z_c z^a + 8i\varphi\eta_a\varepsilon^{acb}\bar z_c\partial_s\theta_b - 4i\varphi\eta_a\varepsilon^{acb}\bar z_c\theta_b + \text{c.c.} \\
& +8\varphi^2(\partial_s\theta_4\bar\eta^4 - \partial_s\eta_4\bar\theta^4 + \eta_4\partial_s\bar\theta^4 - \theta_4\partial_s\bar\eta^4) + 8i\varepsilon_{acb}z^c\bar\eta^b\bar\eta^a\partial_s x^1 - 4i\varepsilon_{acb}z^c\bar\eta^b\bar\eta^a x^1 \\
& -8i\varepsilon^{adc}\eta_a\bar z_d\eta_c\partial_s x^1 + 4i\varepsilon^{adc}\eta_a\bar z_d\eta_c x^1 - 48\varphi\eta_a\bar\eta^a\partial_s x^1 + 24\varphi\eta_a\bar\eta^a x^1 - 24\varphi\theta_4\bar\theta^4\partial_s x^1 \\
& .+ 12\varphi\theta_4\bar\theta^4 x^1 - 24\varphi\eta_4\bar\eta^4\partial_s x^1 + 12\varphi\eta_4\bar\eta^4 x^1 - 4\varepsilon^{acb}\partial_s\bar z_a\bar z_c\eta_b\bar\theta^4 - 4\varepsilon_{acb}\partial_s z^a z^c\bar\eta^b\theta^4 \\
& \left.+16i\varphi\partial_s\bar z_a\bar\eta^a\bar\theta^4 + 16i\varphi\partial_s z^a\eta_a\theta_4 + 4\left[\theta_4\bar\eta^4\partial_s z^b\bar z_b - \theta_4\bar\eta^4\partial_s\bar z_b z^b - \eta_4\bar\theta^4\partial_s z^b\bar z_b + \eta_4\bar\theta^4\partial_s\bar z_b z^b\right]\right\}
\end{aligned}
$$

($\mathbb{CP}^3$ parametrized by relative of Fubini-Study metric in terms of complex variables z^a and $\bar z_a$, transforming in the $\mathbf{3}$ and $\bar{\mathbf{3}}$ of $SU(3)$).

Two loops

- At this order, possible topologies of connected vacuum diagrams are sunset, double bubble, double tadpole.



where vertices carry up to two derivatives.

- Finiteness is not obvious, each diagram is separately divergent.
  Some simplicity occurring by bosonic propagators being diagonal.

- Massless fermions (main difference wrt to AdS_5xS^5 case) behave as effectively decoupled

Two loops

- Standard reduction allows to rewrite every integral as linear combination of the two scalar integrals

$$\mathrm{I}[m_1^2, m_2^2, m_3^2] = \int \frac{d^2p\, d^2q\, d^2r}{(2\pi)^4} \frac{\delta^{(2)}(p+q+r)}{(p^2+m_1^2)\,(q^2+m_2^2)\,(r^2+m_3^2)}$$

$$\mathrm{I}[m^2] = \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2+m^2} \qquad \text{UV divergent}$$

- Summing up diagrams, all divergences cancel, finite contributions always reduce to the three-propagator integral

$$\boxed{I(2m^2, m^2, m^2) = \frac{K}{8\pi^2 m^2}} \qquad K = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

which is responsible for the appearance of the Catalan constant K.

- Two-loop result:

$$-\ln Z_2 = \frac{V_2}{T}\left[\frac{1}{2} I\left(1, \tfrac{1}{2}, \tfrac{1}{2}\right) - \frac{3}{8} I\left(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}\right)\right] = -\frac{1}{4}\frac{V_2}{T} I\left(1, \tfrac{1}{2}, \tfrac{1}{2}\right) = -\frac{K}{16\,\pi^2}\frac{V_2}{T}$$

Final result

- The two loop ABJM cusp anomaly at strong coupling

$$f_{\mathrm{ABJM}}(\lambda) = \sqrt{2\lambda} - \frac{5\log 2}{2\pi} - \left(\frac{K}{4\pi^2} + \frac{1}{24}\right)\frac{1}{\sqrt{2\lambda}} + \mathcal{O}\left(\lambda^{-1}\right)$$

Final result

- The two loop ABJM cusp anomaly at strong coupling $\left(\tilde{\lambda} \equiv \lambda - \frac{1}{24} \right)$

$$f_{\mathrm{ABJM}}\left(\tilde{\lambda}\right) = \sqrt{2\tilde{\lambda}} - \frac{5\log 2}{2\pi} - \frac{K}{4\pi^2\sqrt{2\tilde{\lambda}}} + \mathcal{O}(\sqrt{\tilde{\lambda}})^{-2}$$

The N=4 SYM result has *different factors* (effect of ratio of AdS_4 and CP^3 radii, # of transverse bosons, # of massive fermions) *in front of same structures*.

- Using integrability prediction

$$f_{\mathrm{ABJM}}(\lambda) = 2h(\lambda) - \frac{3\log 2}{2\pi} - \frac{K}{8\pi^2}\frac{1}{h(\lambda)} + \cdots$$

we get for the interpolating function at strong coupling

$$h(\lambda) = \sqrt{\frac{\lambda}{2}} - \frac{\log 2}{2\pi} - \frac{1}{48\sqrt{2\lambda}} + \mathcal{O}(\sqrt{\lambda})^{-2}$$

coinciding with strong coupling expansion of [Gromov Sizov 2014] conjecture

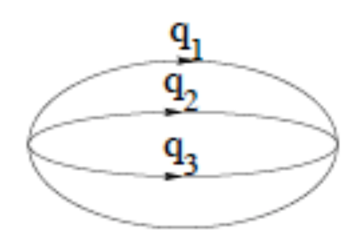
$$h(\lambda) = \sqrt{\frac{1}{2}\left(\lambda - \frac{1}{24}\right)} - \frac{\log 2}{2\pi} + \mathcal{O}\left(e^{-2\pi\sqrt{2\lambda}}\right)$$

Concluding remarks & outlook

- ✓ Two-loop calculation of ABJM cusp anomaly at strong coupling.
- ✓ First non-trivial perturbative check of $h(\lambda)$ at strong coupling.
- ✓ Quantum consistency (UV-finiteness) of this AdS_4xCP^3 action.
- ✓ Indirect evidence of quantum integrability of Type IIA string in AdS_4xCP^3

- ★ Calculate $f(\lambda)$ in backgrounds relevant for the AdS_3/CFT_2 correspondence.

- ★ Three loop calculation: should involve products of $K \ln 2$ and ζ_3
  Transcendentality properties studied, but yet unknown integrals.
  Interesting for seeing divergence cancellation mechanism.



- ★ Finite coupling "stringy" test of $h(\lambda)$ could be via lattice, à la [McEwan, Roiban, 13]: partition function of the *discretized* AdS light-cone gauge action in the background of the null cusp solution.

tack själv!

EXTRAS

Brehmstrahlung function of N=4 SYM

$$\Gamma_{cusp}(\phi) = -B(\lambda, N)\phi^2 \qquad \langle W \rangle \sim e^{-\Gamma_{\text{cusp}}(\phi,\lambda)\log\frac{L_{\text{IR}}}{\epsilon_{\text{UV}}}}$$

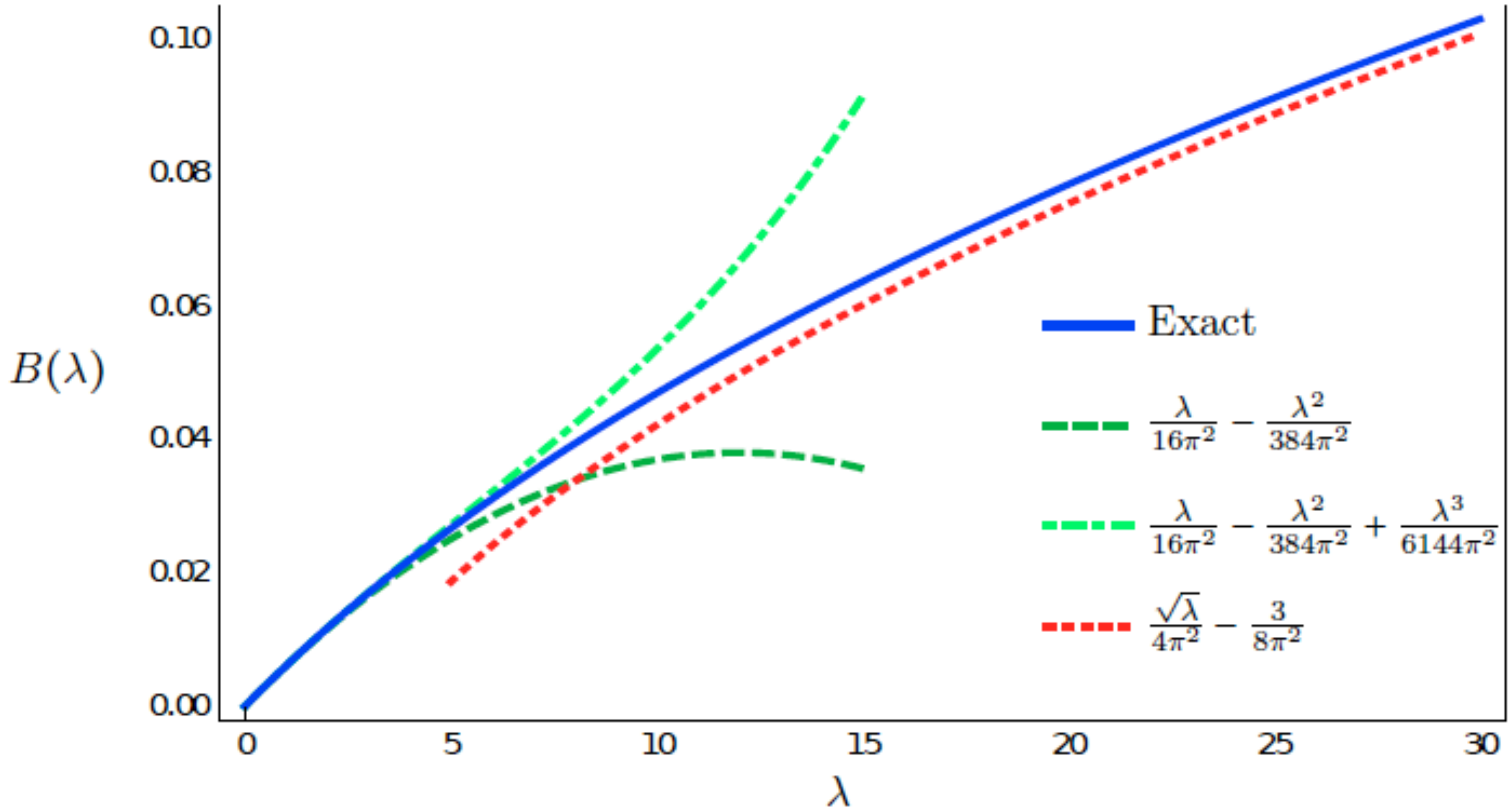


Figure 2: Plot of the Bremsstrahlung function B in the planar limit (solid blue curve). At weak coupling, the lower and upper dashed green curves denote the two- and three-loop approximation, respectively. It is interesting to note that the radius of convergence of the weak coupling expansion is given by the first zero of I_1 in (4), which is at $\lambda \sim -14.7$. As one can see in the plot, the perturbative formulas become unreliable in that region. At the same time, we see that the first two orders of the strong coupling result (red dotted curve) give a qualitatively good approximation starting from that region.

Brehmstrahlung function of N=4 SYM

$$
\begin{aligned}
B &= \frac{1}{2\pi^2}\lambda\partial_\lambda \log\langle W_{\circledcirc}\rangle \\
\langle W_{\circledcirc}\rangle &= \frac{1}{N} L^1_{N-1}\left(-\frac{\lambda}{4N}\right)e^{\frac{\lambda}{8N}}\ , \qquad\qquad \lambda = g^2_{YM}N \\
B &= \frac{1}{4\pi^4}\frac{\sqrt{\lambda}I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} + o(1/N^2)
\end{aligned}
$$

where L is the modified Laguerre polynomial and W_0 is the 1/2 BPS circular Wilson loop.

The last line gives the planar expression.

[Erickson Semenoff Zarembo 00]
[Drukker Gross 00]
[Pestun 07]

Semiclassical quantization for “non homogenous solutions”

- Standard quantization of a soliton
  - > Background field method

$$\mathcal{L} \xrightarrow{\ \phi = \phi_{cl} + \frac{\tilde{\phi}}{\sqrt[4]{\lambda}}\ } \tilde{\mathcal{L}}_{\text{fluct}}$$

  - > Effective action

$$\Gamma = -\ln Z = -\ln \frac{\text{det fermions}}{\sqrt{\text{det bosons}}}$$

  - > 1-loop energy

$$E_1 = \frac{\Gamma_1}{\kappa \mathcal{T}}\,, \qquad \mathcal{T} \equiv \int d\tau \to \infty$$

- Stationary solution → 1-dimensional determinants

$$\det\Big[-\partial_\tau^2 - \partial_\sigma^2 + M^2(\sigma)\Big] = \mathcal{T}\int \frac{d\omega}{2\pi}\Big[-\partial_\sigma^2 + \omega^2 + M^2(\sigma)\Big]$$

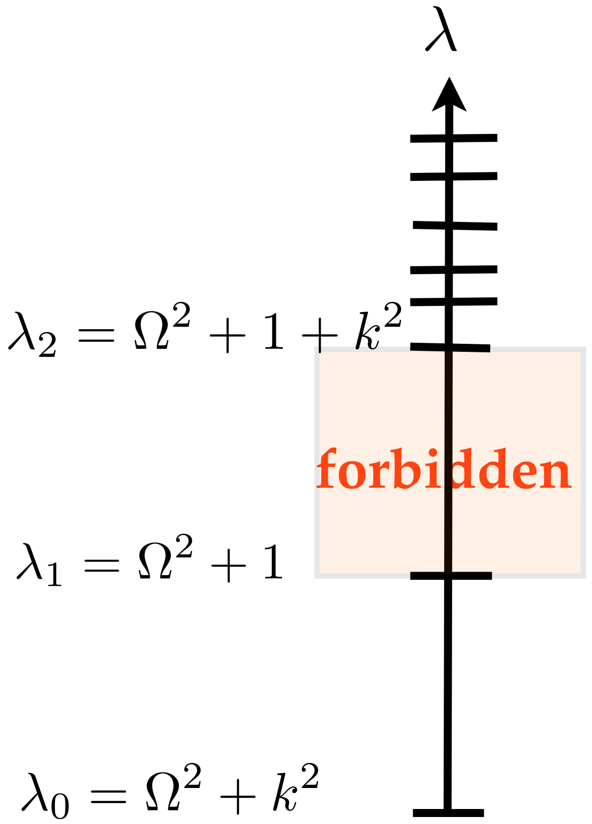
$$\rho'^2 = \kappa^2\,\text{cn}^2\Big[\frac{\kappa\,\sigma}{\epsilon}, -\epsilon^2\Big]$$

Semiclassical quantization EXACTLY

Fluctuation operators obey eigenvalue equations that can be cast in a known form

$$\boxed{\Big\{ -\partial_x^2 + 2k^2\,\mathrm{sn}^2[x+\mathbb{K},k^2] + \Omega^2 \Big\}\,\beta_i(x) = \lambda\,\beta_i(x)}$$

Lamé equation
with periodic b.c.



Spectrum non-trivial, but solution is known exactly

Gelfand Yaglom theorem: to compute determinant, solve an associated initial value problem, then the determinant is simply given in terms of the solution

[Hermite 1872]

$$\beta_\pm(x) = \frac{H(x\pm\alpha)}{\Theta(x)}\,e^{\mp Z(\alpha)x}$$

$$\mathrm{sn}(\alpha,k^2) \quad = \quad \sqrt{1+\frac{1}{k^2}\Big(1+\frac{\pi^2\,\omega^2}{4\,\mathbb{K}^2(k^2)}\Big)}$$

$$\boxed{\det\mathcal{O}_\beta = 4\sinh^2 2\mathbb{K}\,Z(\alpha)}$$

Exact one-loop partition function

$$\Gamma_1 = -\frac{\mathcal{T}}{4\pi}\int_{\mathbb{R}} d\omega \ln \frac{\det^8 \mathcal{O}_\psi}{\det^2 \mathcal{O}_\beta \,\det \mathcal{O}_\phi \,\det^5(-\partial^2)}$$

$$\det\mathcal{O}_\beta = \sinh^2[2\,\mathbb{K}(k^2)\,Z(\alpha)] \qquad \mathrm{sn}(\alpha,k^2) = \sqrt{1+\frac{1}{k^2}\Big(1+\frac{\pi^2\,\omega^2}{4\,\mathbb{K}^2(k^2)}\Big)}$$

$$\det\mathcal{O}_\phi = \sinh^2\Big[\mathbb{K}\big(\frac{4k}{(1+k)^2}\big)\,Z(\tilde\alpha)\Big] \qquad \mathrm{sn}\Big(\tilde\alpha,\frac{4k}{(1+k)^2}\Big) = \sqrt{\frac{(1+k)^2}{8k}\Big[2+\frac{\pi^2\,\omega^2}{\mathbb{K}^2(\frac{4k}{(1+k)^2})}\Big]}$$

$$\det\mathcal{O}_\psi = \cosh^2\Big[\mathbb{K}\big(\frac{4k}{(1+k)^2}\big)\,Z(\hat\alpha)\Big] \qquad \mathrm{sn}\Big(\hat\alpha,\frac{4k}{(1+k)^2}\Big) = \sqrt{\frac{(1+k)^2}{4k}\Big[1+\frac{\pi^2\,\omega^2}{\mathbb{K}^2(\frac{4k}{(1+k)^2})}\Big]}$$

from which the one-loop energy $\qquad E_1 = \dfrac{\Gamma_1}{\kappa\mathcal{T}}\,, \qquad \mathcal{T}\equiv\displaystyle\int d\tau \to \infty$

Not easy to deal with, but the expanded integrand is rich of information

The phase

Bilinear of local charges

$$\theta = \sum_{n=0}^{\infty} \hat{g}^{1-n}\theta_{12}^{(n)}$$

$$\begin{aligned}\theta^{(n)} &= \chi^{(n)}(x_1^+, x_2^+) - \chi^{(n)}(x_1^+, x_2^-) - \chi^{(n)}(x_1^-, x_2^+) + \chi^{(n)}(x_1^-, x_2^-) \\ &- \chi^{(n)}(x_2^+, x_1^+) + \chi^{(n)}(x_2^+, x_1^-) + \chi^{(n)}(x_2^-, x_1^+) - \chi^{(n)}(x_2^-, x_1^-)\end{aligned}$$

with Zhukovsky variables encoding dispersion relation

$$x_p^{\pm} = \frac{\pi e^{\pm \frac{i}{2}p}}{\sqrt{\lambda}\sin\frac{p}{2}}\left(1 + \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}\right)$$

Beyond one loop each contribution is rational.

$$\begin{aligned}\chi^{(1)}(x_1, x_2) &= -\frac{1}{2\pi}\mathrm{Li}_2\frac{\sqrt{x_1} - 1/\sqrt{x_2}}{\sqrt{x_1} - \sqrt{x_2}} - \frac{1}{2\pi}\mathrm{Li}_2\frac{\sqrt{x_1} + 1/\sqrt{x_2}}{\sqrt{x_1} + \sqrt{x_2}} \\ &+ \frac{1}{2\pi}\mathrm{Li}_2\frac{\sqrt{x_1} + 1/\sqrt{x_2}}{\sqrt{x_1} - \sqrt{x_2}} + \frac{1}{2\pi}\mathrm{Li}_2\frac{\sqrt{x_1} - 1/\sqrt{x_2}}{\sqrt{x_1} + \sqrt{x_2}}\end{aligned}$$

Crossing equation

$$i\theta(x_j, x_k) + i\theta(1/x_j, x_k) = 2\log h(x_j, x_k)$$

$$h(x_j, x_k) = \frac{x_k^-}{x_k^+}\frac{(1 - \frac{1}{x_j^- x_k^-})(x_j^- - x_k^+)}{(1 - \frac{1}{x_j^+ x_k^-})(x_j^+ - x_k^+)}$$

Unitarity-cut result and the Yang-Baxter equation

[Bianchi Hoare, 2014]

Compact formula for the one-loop contribution (explicitating bubble integrals)

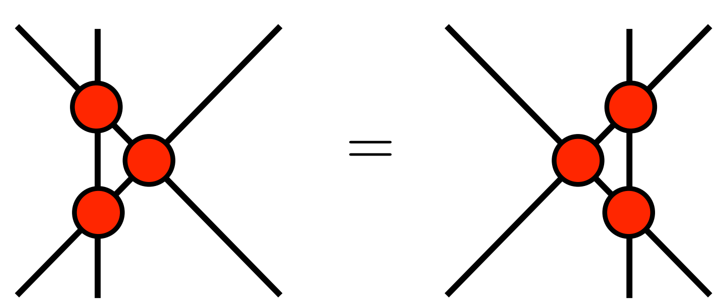
Generalization to different masses.

$$T^{(1)} = \underbrace{\frac{\theta}{2\pi}(T^{(0)} \textcircled{u} T^{(0)} - T^{(0)} \textcircled{s} T^{(0)})}_{\text{logarithmic terms}} + \underbrace{\frac{i}{2} T^{(0)} \textcircled{s} T^{(0)}}_{\text{rational imaginary}} + \underbrace{\frac{1}{16\pi}(\frac{1}{m^2}\widetilde{T}^{(0)} \underset{\leftarrow}{\textcircled{t}} T^{(0)} + \frac{1}{m'^2} T^{(0)} \underset{\rightarrow}{\textcircled{t}} \widetilde{T}^{(0)})}_{\text{rational real}}$$

$$\theta \equiv \text{arcsinh}\,(\frac{e'p - ep'}{mm'}) \qquad e = \sqrt{p^2 + m^2}\ , \qquad e' = \sqrt{p'^2 + m'^2}$$

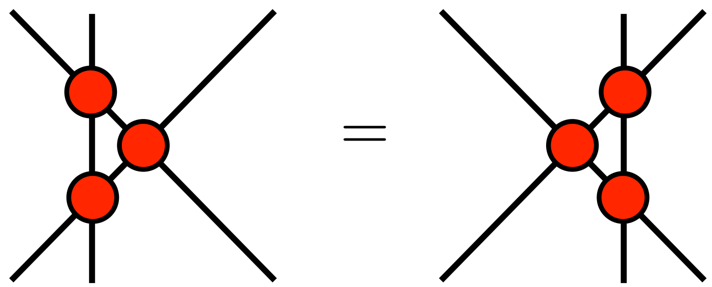
Useful to be used in the Yang-Baxter equation, a cubic matrix equation necessarily satisfied by integrable theories.

$$\mathbb{S}_{12}\mathbb{S}_{13}\mathbb{S}_{23} = \mathbb{S}_{23}\mathbb{S}_{13}\mathbb{S}_{12}\ ,$$



Unitarity-cut result and the Yang-Baxter equation

$$\mathbb{S}_{12}\mathbb{S}_{13}\mathbb{S}_{23} = \mathbb{S}_{23}\mathbb{S}_{13}\mathbb{S}_{12} \ ,$$

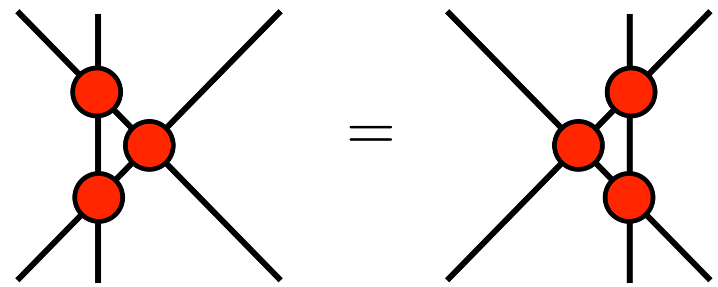


- $[\mathbb{T}^{(0)}_{12}, \mathbb{T}^{(0)}_{13}] + [\mathbb{T}^{(0)}_{12}, \mathbb{T}^{(0)}_{23}] + [\mathbb{T}^{(0)}_{13}, \mathbb{T}^{(0)}_{23}] = 0$ Classical Yang-Baxter

- $$[\mathbb{T}^{(0)}_{12}, \mathbb{T}^{(1)}_{13}] + [\mathbb{T}^{(0)}_{12}, \mathbb{T}^{(1)}_{23}] + [\mathbb{T}^{(0)}_{13}, \mathbb{T}^{(1)}_{23}] - [\mathbb{T}^{(0)}_{13}, \mathbb{T}^{(1)}_{12}] - [\mathbb{T}^{(0)}_{23}, \mathbb{T}^{(1)}_{12}] - [\mathbb{T}^{(0)}_{23}, \mathbb{T}^{(1)}_{13}] =$$
$$\mathbb{T}^{(0)}_{23}\mathbb{T}^{(0)}_{13}\mathbb{T}^{(0)}_{12} - \mathbb{T}^{(0)}_{12}\mathbb{T}^{(0)}_{13}\mathbb{T}^{(0)}_{23}$$

Unitarity-cut result and the Yang-Baxter equation

$$\mathbb{S}_{12}\mathbb{S}_{13}\mathbb{S}_{23} = \mathbb{S}_{23}\mathbb{S}_{13}\mathbb{S}_{12} \ ,$$

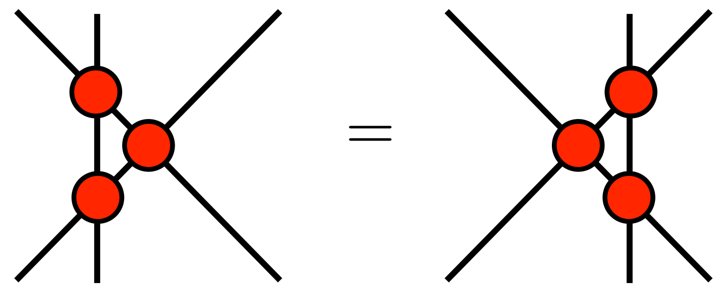


- $[\mathbb{T}^{(0)}_{12}, \mathbb{T}^{(0)}_{13}] + [\mathbb{T}^{(0)}_{12}, \mathbb{T}^{(0)}_{23}] + [\mathbb{T}^{(0)}_{13}, \mathbb{T}^{(0)}_{23}] = 0$ Classical Yang-Baxter
- $[\mathbb{T}^{(0)}_{12}, \mathbb{T}^{(1)}_{13}] + [\mathbb{T}^{(0)}_{12}, \mathbb{T}^{(1)}_{23}] + [\mathbb{T}^{(0)}_{13}, \mathbb{T}^{(1)}_{23}] - [\mathbb{T}^{(0)}_{13}, \mathbb{T}^{(1)}_{12}] - [\mathbb{T}^{(0)}_{23}, \mathbb{T}^{(1)}_{12}] - [\mathbb{T}^{(0)}_{23}, \mathbb{T}^{(1)}_{13}] = \mathbb{T}^{(0)}_{23}\mathbb{T}^{(0)}_{13}\mathbb{T}^{(0)}_{12} - \mathbb{T}^{(0)}_{12}\mathbb{T}^{(0)}_{13}\mathbb{T}^{(0)}_{23}$

$$T^{(1)} = \underbrace{\frac{\theta}{2\pi}(T^{(0)} \textcircled{u}\, T^{(0)} - T^{(0)} \textcircled{s}\, T^{(0)})}_{\text{logarithmic terms}} + \underbrace{\frac{i}{2} T^{(0)} \textcircled{s}\, T^{(0)}}_{\text{rational imaginary}} + \underbrace{\frac{1}{16\pi}\left(\frac{1}{m^2}\widetilde{T}^{(0)} \underset{\leftarrow}{\textcircled{t}}\, T^{(0)} + \frac{1}{m'^2} T^{(0)} \underset{\rightarrow}{\textcircled{t}}\, \widetilde{T}^{(0)}\right)}_{\text{rational real}}$$

Unitarity-cut result and the Yang-Baxter equation

$$\mathbb{S}_{12}\mathbb{S}_{13}\mathbb{S}_{23} = \mathbb{S}_{23}\mathbb{S}_{13}\mathbb{S}_{12}\ ,$$



- $[\mathbb{T}^{(0)}_{12},\mathbb{T}^{(0)}_{13}] + [\mathbb{T}^{(0)}_{12},\mathbb{T}^{(0)}_{23}] + [\mathbb{T}^{(0)}_{13},\mathbb{T}^{(0)}_{23}] = 0$ Classical Yang-Baxter
- $[\mathbb{T}^{(0)}_{12},\mathbb{T}^{(1)}_{13}] + [\mathbb{T}^{(0)}_{12},\mathbb{T}^{(1)}_{23}] + [\mathbb{T}^{(0)}_{13},\mathbb{T}^{(1)}_{23}] - [\mathbb{T}^{(0)}_{13},\mathbb{T}^{(1)}_{12}] - [\mathbb{T}^{(0)}_{23},\mathbb{T}^{(1)}_{12}] - [\mathbb{T}^{(0)}_{23},\mathbb{T}^{(1)}_{13}] = \mathbb{T}^{(0)}_{23}\mathbb{T}^{(0)}_{13}\mathbb{T}^{(0)}_{12} - \mathbb{T}^{(0)}_{12}\mathbb{T}^{(0)}_{13}\mathbb{T}^{(0)}_{23}$

$$T^{(1)} = \underbrace{\frac{\theta}{2\pi}(T^{(0)} \textcircled{u}\, T^{(0)} - T^{(0)} \textcircled{s}\, T^{(0)})}_{\text{logarithmic terms}} + \underbrace{\frac{i}{2}T^{(0)} \textcircled{s}\, T^{(0)}}_{\text{rational imaginary}} + \underbrace{\frac{1}{16\pi}(\frac{1}{m^2}\widetilde{T}^{(0)} \underset{\leftarrow}{\textcircled{t}}\, T^{(0)} + \frac{1}{m'^2}T^{(0)} \underset{\rightarrow}{\textcircled{t}}\, \widetilde{T}^{(0)})}_{\text{rational real}}$$

$\sim \mathbb{1}$ $\sim \mathbb{1}$

Yang-Baxter is automatically satisfied by the unitarity-constructed one-loop S-matrix.

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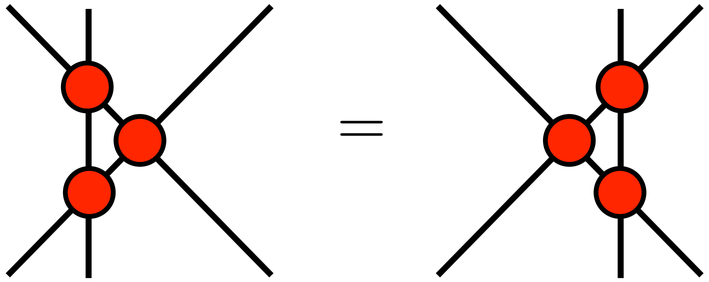

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