Localization and quantum AdS_4/CFT$_3$ holography

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Based on: `arXiv:1406.0505` - Atish Dabholkar, N.D. and João Gomes

Supersymmetric Field Theories

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Localization on S^3

[Kapustin, Willett, Yaakov]

- Consider any $\mathcal{N} = 4$ super Chern-Simons matter theory on S^3.
- Add to the action a Q-exact term of the form $t\,Q(\Psi Q \Psi)$.
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$$\int \prod_{a=1}^{N} \frac{d\nu_a^{(i)}}{2\pi}\, e^{\frac{ik^{(i)}}{4\pi}(\nu_a^{(i)})^2} \prod_{a<b} \left(2\sinh\frac{\nu_a^{(i)} - \nu_b^{(i)}}{2}\right)^2$$

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- In the case of ABJM theory we have a two node circular quiver with no fundamental matter, only bifundamentals, so one finds

$$Z = \frac{1}{N!^2} \int \prod_{a=1}^{N} \frac{d\nu_a}{2\pi} \frac{d\mu_a}{2\pi} \, e^{\frac{ik}{4\pi} \sum_a (\nu_a^2 - \mu_a^2)} \, \frac{\prod_{a<b} \left(2 \sinh \frac{\nu_a - \nu_b}{2}\right)^2 \left(2 \sinh \frac{\mu_a - \mu_b}{2}\right)^2}{\prod_{a,b} \left(2 \cosh \frac{\nu_a - \mu_b}{2}\right)^2}$$

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- For a more general circular quiver

$$Z=\frac{1}{N!^r}\int\prod_{i=1}^{r}\prod_{a=1}^{N}\frac{d\nu_a^{(i)}}{2\pi}\,e^{\sum_{i=1}^{r}\frac{ik^{(i)}}{4\pi}(\nu_a^{(i)})^2}\prod_{i=1}^{r}\frac{\prod_{a<b}\left(2\sinh\frac{\nu_a^{(i)}-\nu_b^{(i)}}{2}\right)^2}{\prod_{a,b}2\cosh\frac{\nu_a^{(i)}-\nu_b^{(i+1)}}{2}}$$

Solving the matrix model

- The ABJM matrix model is very similar to that of pure Chern-Simons on a $S^3/\mathbb{Z}_2$, a lens space and can be solved exactly. [Aganagic,Klemm Mariño,Vafa] [Halmagyi Yasnov] [Drukker Mariño,Putrov]

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- The leading behavior of the free energy at large N and $\lambda = N/k$ can be written as

$$F = \frac{F_0}{g_s^2} = -\frac{\pi\sqrt{2}3}{k}^2 \hat{\lambda}^{3/2} = -\frac{\pi\sqrt{2}}{3}\sqrt{k}\,\hat{N}^{3/2}$$

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Can we do any better?

- In the case of ABJM we know (recursively) the full all-genus partition function.
- This should be captured by the full quantum string theory partition function on $AdS_4 \times \mathbb{CP}^3$.
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If this is to work, we expect the same answer for any theory with an AdS_4 dual and enough SUSY!

Outline

- Introduction and motivation
- Localization in 3d theories on S^3
- Solving the matrix model and the comparison to supergravity
- Beyond the large N, λ limit:
  - Universal Airy function behavior
- Localization in AdS_4 supergravity
  - The theory
  - The localization solution
  - The action
- Summary

Universal Airy function behavior

- The genus expansion of ABJM theory satisfies a holomorphic anomaly equation.
- Ignoring the instanton terms in the planar free energy F_0 the solution to this equation is remarkably simple [Fuji, Hirano, Moriyama]

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- By use of the Cauchy identity the matrix model can be written as

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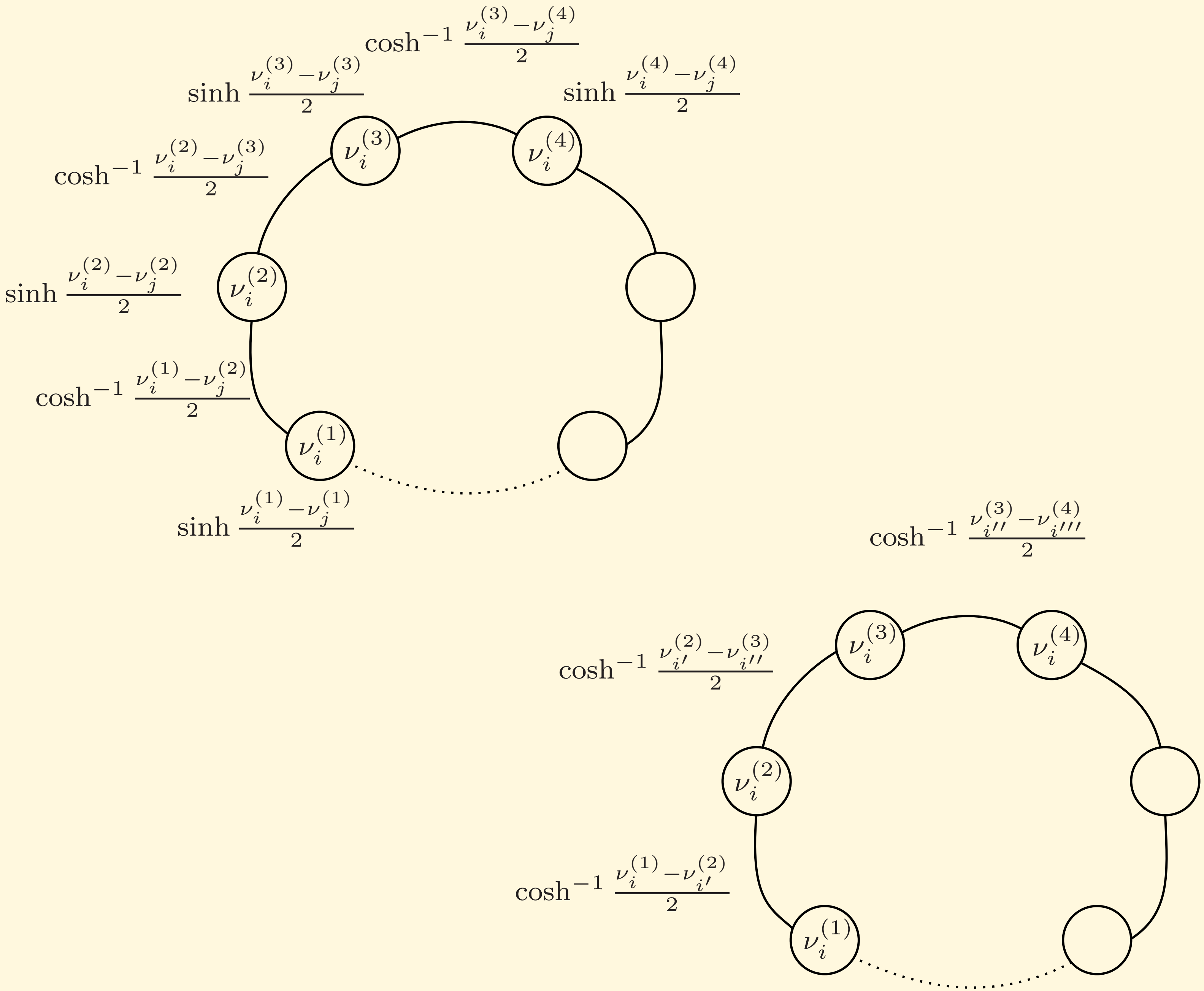
$$= \int \prod_{i=1}^{r}\prod_{a=1}^{N}\frac{d\nu_a^{(i)}}{2\pi}\, e^{\sum_{i=1}^r \frac{ik^{(i)}}{4\pi}(\nu_a^{(i)})^2} \sum_{\sigma^{(i)}\in S_N}\prod_{i=1}^{r}(-1)^{\sigma^{(i)}}\frac{1}{\prod_a 2\cosh\frac{\nu_a^{(i)}-\nu_{\sigma_{i+1}(a)}^{(i+1)}}{2}}$$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- The resulting expressions are simpler if we switch to the grand-canonical partition function $z = e^{\mu}$

$$\Xi(z) = 1 + \sum_{N=1}^{\infty} z^N Z(N) = \det(1 + z\rho) = \prod_n (1 + e^{\mu - E_n})$$

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- It can be shown that classically this hamiltonian $H = -\log\rho$ is

$$H(p,q) = \sum_{i=1}^{r} \log\left[2\cosh\frac{p - \sum_{j=1}^{i}\frac{k^{(j)}}{k}q}{2}\right]$$

- For large p and q this is simply

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- and the density of states is given by the derivative of the area of the polygon enclosed by $H = E$

$$\rho(E) = \frac{dn}{dE} \qquad n(E) = \frac{c}{k} E^2 + n_0 (1 - e^{-E})$$

- The grand potential is then

$$J(\mu) = \int_0^\infty dE\,\rho(E)\log(1+e^{\mu-E}) = -\frac{2c}{k}\,\mathrm{Li}_3(-e^{\mu}) + n_0\mu(1+e^{-\mu})\log(1+e^{-\mu}) - n_0$$

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- Now the canonical partition function can be derived from the canonical potential by

$$Z(N) = \frac{1}{2\pi i}\int d\mu\, e^{J(\mu)-N\mu} = \left(\frac{c}{k}\right)^{-1/3} e^A\,\mathrm{Ai}\left[\left(\frac{c}{k}\right)^{-1/3}\left(N - \frac{\pi^2 c}{3k} - n_0\right)\right],$$

c and n_0 can be evaluated for any particular model. A depends more intimately on the instanton corrections and can be evaluated perturbatively.

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- I will describe only some of these steps for asymptotically AdS_4 - those which we have succeeded in doing.

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- In it we embed the dimensional reduction of M-theory on a Sasaki-Einstein space (or IIA on a 6d manifold) with two vector multiplets (one is a compensator field for conformal transformations in the superconformal formulation). [Gauntlett,Kim Varela,Waldram]
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  We didn't do that.
- We showed that the AdS_4 background is supersymmetric and fixed the metric and spinor to that.

- The bosonic action for the vector multiplet is

$$S_{\text{vec}} = \int d^4x \sqrt{g} \Big[N_{IJ} \bar{X}^I X^J \left(\frac{R}{6} + D \right) + N_{IJ} \partial \bar{X}^I \partial X^J - \frac{1}{8} N_{IJ} Y^{ijI} Y^J_{ij} + \Big]$$

- The BPS equations for the vector multiplet are

$$\delta \Omega^i_+ = -i \not{\partial} X \xi^i_- - \frac{1}{2} Y^i_j \xi^j_+ + X \eta^i_+ = 0$$
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where Y is the auxiliary field and ξ and η are Killing spinors which we take to be related by

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- Writing $X=iJ+H$ we find that to preserve *all* SUSYs we need

$$J=\text{const}\,,\qquad H=0\,,\qquad Y^1{}_1=-y^2{}_2=-2i\frac{J}{L}\,,\qquad Y^1{}_2=Y^2{}_1=0$$

where L is the AdS radius and the constant J gets related to each-other by the identification of the coefficient of R with the Newton constant (together with the hypers).

- They hypermultiplet Lagrangian is

$$\mathcal{L}_{\rm hyp} = \Big[- D_\mu A^i{}_\beta D^\mu A_i{}^\alpha - \frac{1}{6} R A^i{}_\beta A_i{}^\alpha + \frac{1}{2} D A^i{}_\beta A_i{}^\alpha + F^i{}_\beta F_i{}^\alpha + 4g^2 A^i{}_\beta \bar{X}^\alpha_\gamma X^\gamma_\delta A_i{}^\delta + g A^i{}_\beta Y^{jk\,\alpha}_\gamma A_k^\gamma \epsilon_{ij} \Big] d_\alpha^\beta + \text{fermionic terms}$$

- The fields are fixed (assymptotically) to carry a charge under the vector X_I

$$t_I A_i{}^\alpha = P_I (i\sigma^3)^\alpha{}_\beta A_i{}^\beta$$

- The SUSY variation of the the hypers leads to

$$F_i^\alpha = -i2gA_j^\alpha \sigma_{3i}^j (H \cdot P) = 0, \qquad 2g(J \cdot P) = -\frac{1}{L}$$

- With the prepotential for our model

$$F = \sqrt{X^0 (X^1)^3}$$

the last relation is refined to

$$2gJ^0 P_0 = -\frac{1}{4L}, \qquad 2gJ^1 P_1 = -\frac{3}{4L}$$

- We take the AdS metric to be $ds^2 = L^2(d\eta^2 + \sinh^2(\eta) d\Omega_3^2)$.
- With a specific choice of a single SUSY generator (ϵ are S^3 Killing spinors)

$$\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = \begin{pmatrix} \chi_+ \times \epsilon_-^1 \\ (\sigma_3 \chi_+) \times \epsilon_-^2 \end{pmatrix} \qquad \chi_+ = \begin{pmatrix} \sinh(\eta/2) \\ -i\cosh(\eta/2) \end{pmatrix}$$

we find a more general solution

$$X = iJ + H = iJ + \frac{Jh}{\cosh(\eta)}, \qquad Y^1{}_1 = -Y^2{}_2 = -2i\frac{J}{L} + 2\frac{Jh}{L\cosh^2(\eta)}$$

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- The hypers A are constant for both the vacuum solution and the BPS instanton, but the auxiliary field is proportional to H

$$F_i^\alpha = \frac{2g}{\sqrt{8\pi G}}H^I P_I \sigma_{3i}^\alpha, \qquad A_i{}^\alpha = \frac{1}{\sqrt{8\pi G}}\delta_i^\alpha$$

Localizing action

- The action for the vector multiplet is

$$S_{\text{vec}} = \int d^4x\sqrt{g}\Big[N_{IJ}\bar{X}^I X^J \frac{R}{6} + N_{IJ}\partial\bar{X}^I\partial X^J - \frac{1}{8}N_{IJ}Y^{ijI}Y^J_{ij}\Big]_{\text{Loc. locus}}$$

$$= \Omega_3 L^2 \frac{\sqrt{J^0(J^1)^3}}{2i}\int dr(r^2-1)\Big[-\left(1+\frac{(h^0)^2}{r^2}\right)(t+\bar{t})^3 + \frac{3}{4}h^2\left(\frac{1}{t}+\frac{1}{\bar{t}}\right)\frac{r^2-1}{r^4}$$

$$+\frac{3}{2}h^1h^0(t+\bar{t})\frac{r^2-1}{r^4} - \frac{1}{4}(h^0)^2(t^3+\bar{t}^3)\frac{r^2-1}{r^4} - \frac{3}{4}\left(\frac{1}{t}+\frac{1}{\bar{t}}\right)\left(1+i\frac{h^1}{r^2}\right)^2$$

$$-\frac{3}{2}(t+\bar{t})\left(1+i\frac{h^1}{r^2}\right)\left(1+i\frac{h^0}{r^2}\right)+\frac{1}{4}(t^3+\bar{t}^3)\left(1+i\frac{h^0}{r^2}\right)^2\Big]$$

Here $N_{IJ} = \text{Im}(F_{IJ})$, $F_{IJ} = \partial_I\partial_J F$, $t = \sqrt{\frac{X^1}{J^1}/\frac{X^0}{J^0}} = \sqrt{\frac{i+h^1/r}{i+h^0/r}}$ and $r = \cosh(\eta)$.

- This integrates to

$$\Omega_3 L^2 \frac{\sqrt{J^0(J^1)^3}}{2i}\bigg[\frac{(r-1)^2}{r}\sqrt{\frac{1+ih^1/r}{1+ih^0/r}}(-ih^1(1-2ih^0-r)-2r(2+r)-ih^0(3+r))+ \\ +\frac{(r+1)^2}{r}\sqrt{\frac{1-ih^1/r}{1-ih^0/r}}(-ih^1(1-2ih^0+r)+2r(2-r)-ih^0(3-r))\bigg]$$

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$$\left.+\frac{(r+1)^2}{r}\sqrt{\frac{1-ih^1/r}{1-ih^0/r}}(-ih^1(1-2ih^0+r)+2r(2-r)-ih^0(3-r))\right]$$

- This diverges at large r, which requires careful regularization. With a cutoff r_0 the divergent and finite pieces are

$$S_{\text{vec}} = -\frac{\Omega_3 L^2}{32\pi G}\left[4r_0^3+\frac{r_0}{2}(3(h^1)^2+6h^1h^0-(h^0)^2-24)+2ir_0(3h^1+h^0)\right.$$
$$\left.+8(1-ih^1)^{3/2}\sqrt{1-ih^0}\right]$$

- One needs to add a Gibbons-Hawking term and a term assocciated to the intrinsic curvature of the boundary S^3. This removes the real part of the divergent terms.

- The hypermultiplet action is

$$S_{\rm hyp} = \int d^4x\sqrt{g}\left[-\frac{1}{6}RA^2 + \left[4g^2 A^i{}_\beta \bar{X}^\alpha{}_\gamma X^\gamma{}_\delta A_i{}^\delta + gA^i{}_\beta Y^{jk\alpha}_\gamma A_k{}^\gamma \epsilon_{ij} + F_i{}^\alpha F^i{}_\beta\right] d_\alpha{}^\beta\right]$$

$$= -i\Omega_3 L^4 \int_1^{r_0} dr(r^2-1)\frac{1}{r^2}\frac{g}{2\pi GL}(h^0J^0P_0 + h^1J^1P_1)$$

$$= -i\frac{\Omega_3 gL^3}{2\pi G}(r_0-2)(h^0J^0P_0 + h^1J^1P_1) + \mathcal{O}(1/r_0)$$

- P^I are the charges carried by the hypermultiplets and exactly these combinations are fixed to the values

$$8gLJ^0P_0 = -1, \qquad 8gLJ^1P_1 = -3$$

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- This is canceled by a topological term, which can be traced to an integral of F_4 over AdS_4. This gives the volume divergence (canceling GH) plus a finite piece.

- We end up with the finite action

$$S_{ren} = -\frac{\pi\sqrt{2}}{3}k^{1/2}N^{3/2}\left[(1-ih^1)^{3/2}\sqrt{1-ih^0}+\frac{i}{2}(3h^1+h^0)-2\right]$$

which was written in terms of the ABJM parameters

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- It is convenient to define new variables

$$\phi^0 := \frac{\pi}{3\sqrt{2}}\frac{N^{3/2}}{k^{1/2}}(1-ih^0), \qquad \phi^1 := \frac{\pi}{\sqrt{2}}k^{1/2}N^{1/2}(1-ih^1).$$

These are propotional to the values of the fields X^I at the center of AdS_4

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- We thus see that the localization integral looks precisely like a Laplace transform of the partition function

$$Z(\phi) = e^{F(\phi)}, \qquad \text{with} \quad F(\phi) = \frac{2\sqrt{2}}{\pi\sqrt{3}}\sqrt{\phi^0(\phi^1)^3}.$$

- We do not really know the measure for integrating over ϕ^0 and ϕ^1.
- Notice, though, that one can massage the action to the form

$$-S_{ren} = -k\left(\sqrt{\phi^0} - \frac{1}{\pi k}\sqrt{\frac{2}{3}}(\phi^1)^{3/2}\right)^2 + \frac{2}{3\pi^2 k}(\phi^1)^3 - N\phi^1$$

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This agrees with the matrix model to all orders in N.

Summary

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  - We found the localization locus.
  - We evaluated the action.
- Gaps:
  - We didn't find all BPS geometries.
  - We didn't derive a measure from first principles.
- Open questions:
  - Can we get the instanton corrections?
  - Generalize to other settings.
  - What can we learn more generally about quantum gravity from localization.

The end