A solvable relativistic hydrogen-like system

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based on [Caron-Huot & JMH, hep-th/1404.2922, hep-th/1408.0296, and work to appear]
(and slides based on talk by S. Caron-Huot)

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Introduction

- Kepler problem and hydrogen atom are important classical and quantum mechanics problems that can be exactly solved

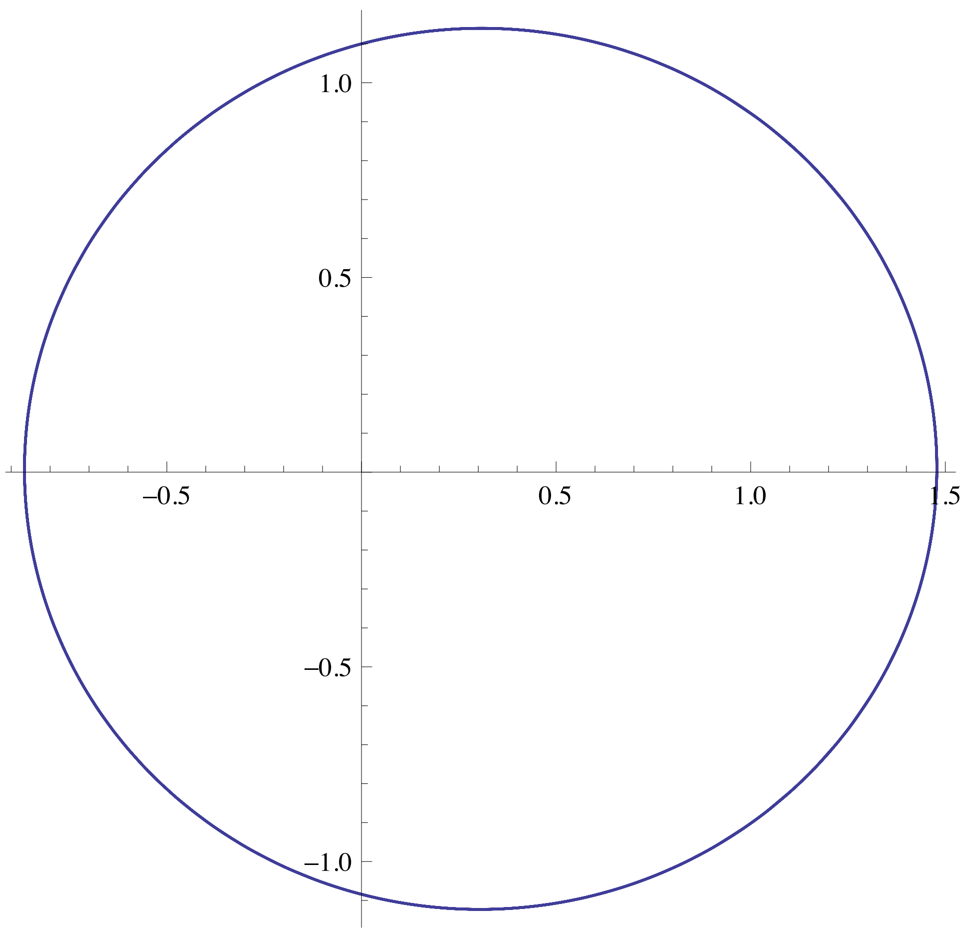
they have a hidden symmetry

- will show that N=4 super Yang-Mills is a natural QFT analogue of these systems

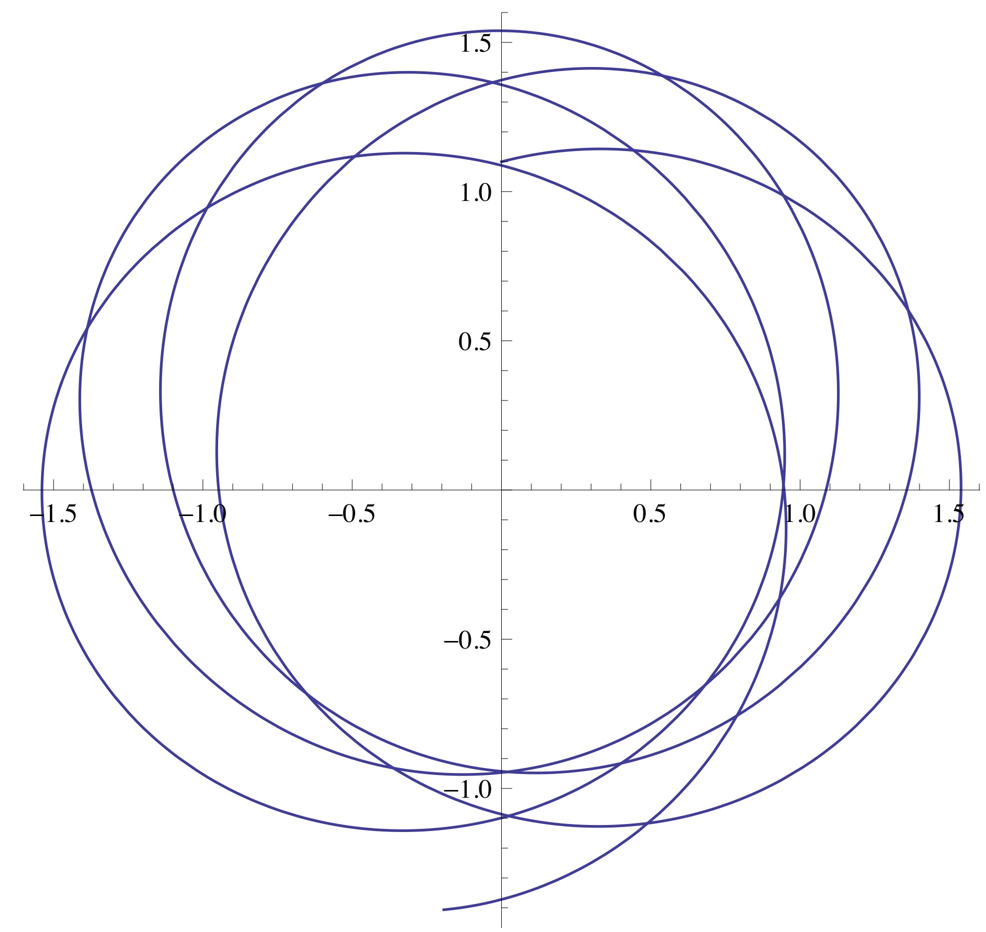
- apply hidden symmetry to calculation of bound state energies of massive W bosons in the theory

Kepler problem

$$V = 1/r$$



$$V = 1/r^{0.9}$$



- orbits do not precess
- conservation of Laplace-Runge-Lenz vector

$$\vec{A} = \frac{1}{2}\left(\vec{p} \times \vec{L} - \vec{L} \times \vec{p}\right) - \mu \frac{\lambda}{4\pi} \frac{\vec{x}}{|x|}$$

Hydrogen atom

- Hamiltonian $$H = \frac{1}{2m}p^2 - \frac{k}{r}$$

- hidden symmetry:
Laplace-Runge-Lenz-Pauli vector $$\vec{A} = \frac{1}{2}(\vec{p} \times \vec{L} - \vec{L} \times \vec{p}) - mk\frac{\vec{r}}{r}$$

- conserved quantity in quantum mechanics

$$[H, L_i] = 0 \qquad [H, A_i] = 0$$

$$[A_i, A_i] = -i\hbar\epsilon_{ijk}L_k\frac{2}{m}H$$

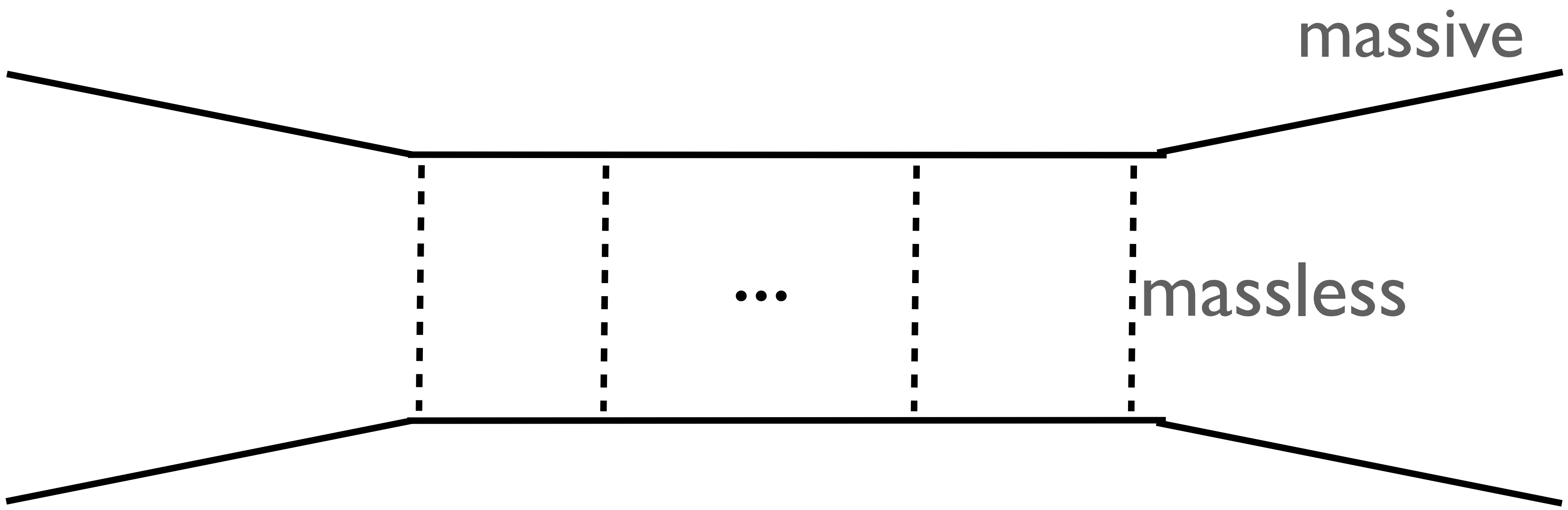
- operator algebra allows to find spectrum

$$E_n = -\frac{mk^2}{2\hbar^2}\frac{1}{n^2} \qquad n = 1, 2, \ldots$$

- degeneracy n^2

extension to a relativistic QFT

- Wick and Cutcosky considered the following model:



- This is the ladder approximation to $ep \to ep$, ignoring the spin of the photon

- In the non-relativistic limit, this reduces to the hydrogen Hamiltonian

SO(4) symmetry of Wick-Cutcosky model

- this model possesses an exact SO(4) symmetry, even away from the NR limit
- consider the Bethe-Salpether equation

$$\psi(p) = \int \frac{-4i\lambda\, m_1 m_3\, \psi(q)\, d^4q/(2\pi)^4}{(p-q)^2\left[(q-y_1)^2 + m_1^2\right]\left[(y_3-q)^2 + m_3^2\right]}$$

- not obvious in this form, but there is a conformal symmetry in momentum space
- spectrum only depends on cross-ratio

$$u = \frac{4m_1 m_3}{-s + (m_1 - m_3)^2}$$

[Wick, 1954]

- apart from obvious symmetries, this contains the Laplace-Runge-Lenz vector! [Caron-Huot, JMH]

Beyond the ladder approximation

- Unfortunately, ladder approximation is not a consistent QFT
- e.g. misses multi-particle effects, and therefore has problems with unitarity
- is there a 4-dimensional QFT that has this hidden symmetry?

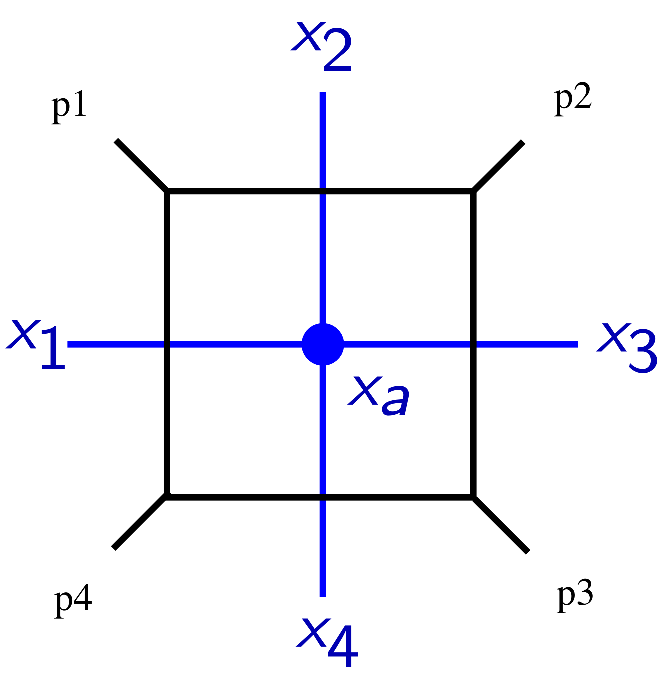
N=4 super Yang-Mills

fast-forward from 1950's to 2000's

N=4 SYM has dual conformal symmetry

[Drummond, JMH, Smirnov, Sokatchev; Alday, Maldacena; Drummond, JMH, Korchemsky, Sokatchev; ...]

in massless sector:



$$\boxed{p_i^\mu = x_i^\mu - x_{i+1}^\mu}$$

$$= x_{13}^2 x_{24}^2 \int \frac{d^D x_a}{x_{1a}^2 x_{2a}^2 x_{3a}^2 x_{4a}^2}$$

invariant under SO(4,2) in dual space $\quad x^\mu \to x^\mu / x^2$

- this symmetry is at the heart of many developments
  - duality Wilson loops/scattering amplitudes
  - integrability of N=4 SYM theory
  - and other recent developments
- we have just argued that it is a natural generalization of the hydrogen atom's SO(4), itself inherited from the Kepler problem
- let us return to massive particles

introducing massive particles

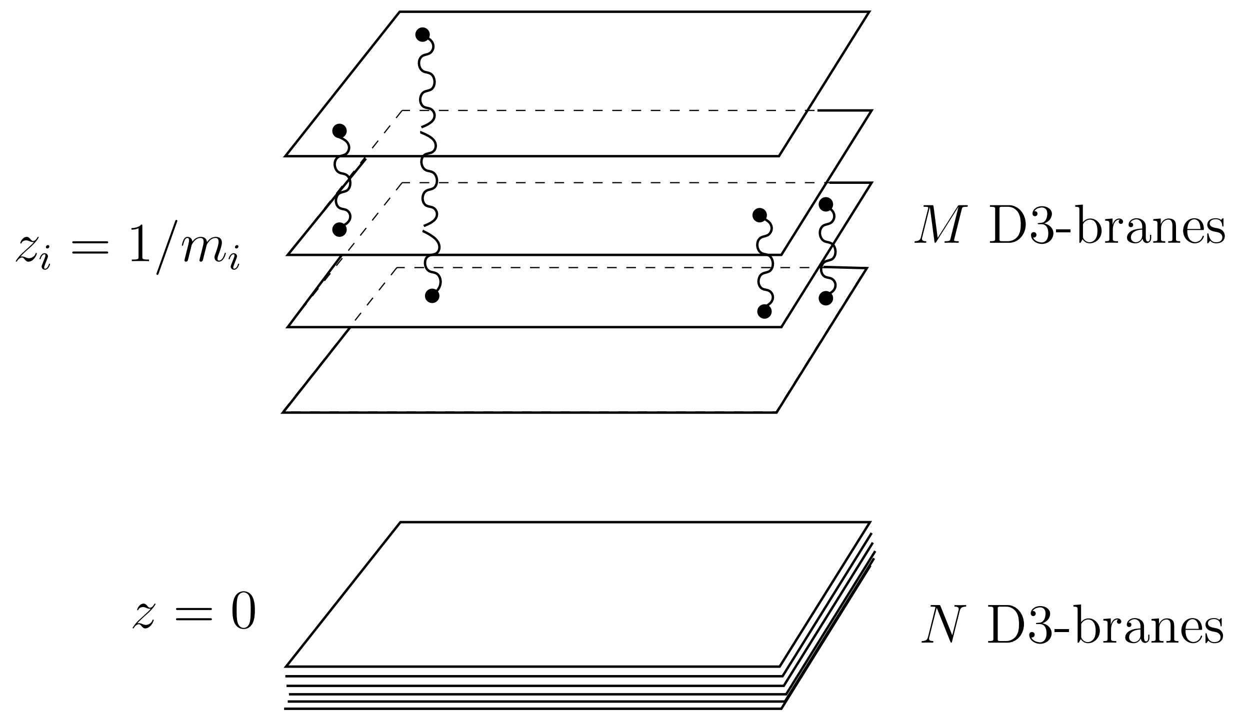
gauge theory

string theory

Higgs mechanism

$$\Phi \longrightarrow \langle \Phi \rangle + \varphi$$

$$U(N+M) \longrightarrow U(N) \times U(M)$$



(a)

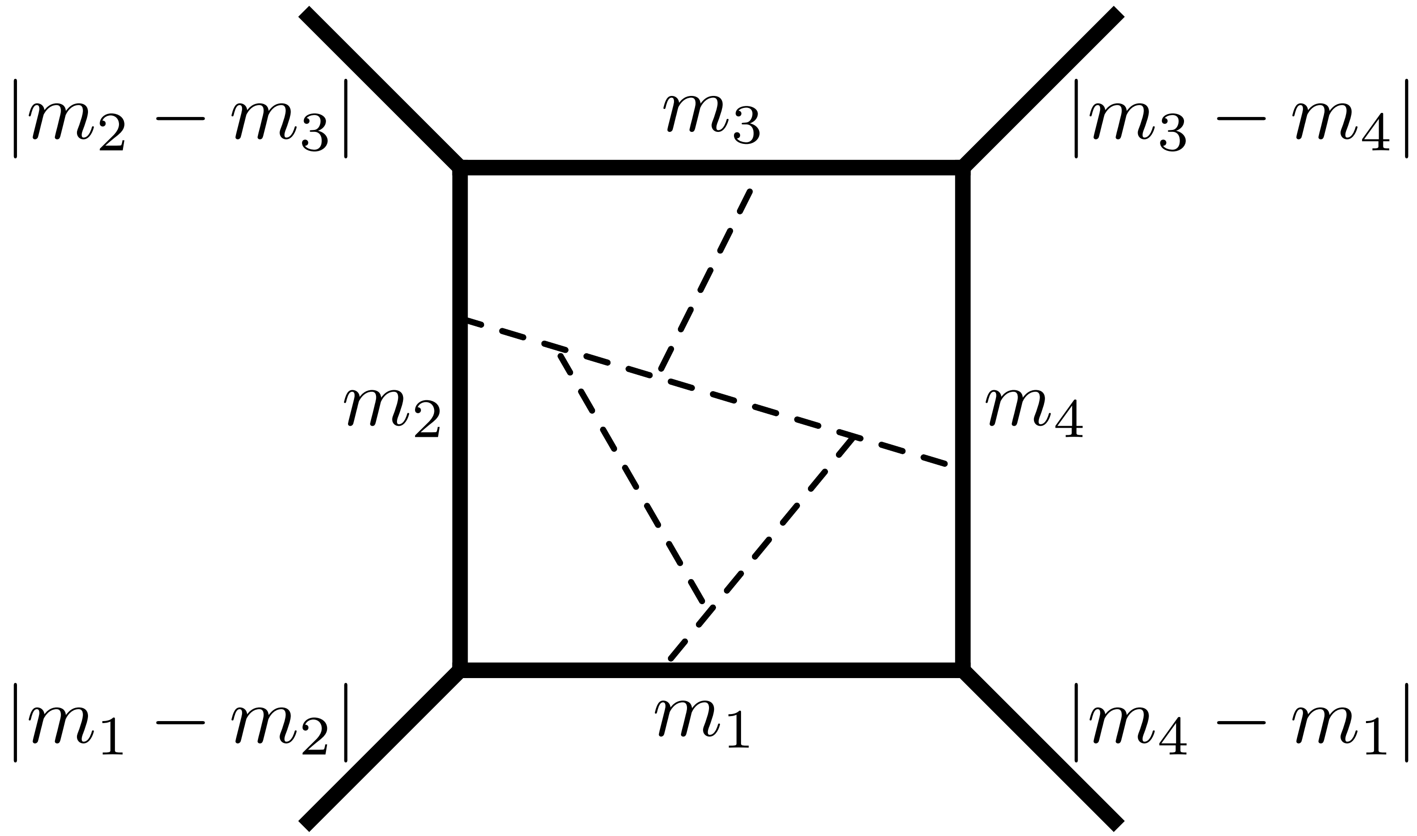
- e.g. four-particle scattering

$$U(N+4) \longrightarrow U(N) \times U(4)$$

consider scattering of U(4) fields in large N limit

- infrared finite
- preserves dual conformal symmetry

- four-particle scattering (planar)



- dual conformal symmetry (planar) [Alday, JMH, Plefka,Schuster]

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu \qquad p_i^2 = -(m_i - m_{i+1})^2$$

$$y_i^A \to \frac{y_i^A}{y_i^2} \qquad y_i^A = (x_i^\mu, m_i)$$

isometries of AdS_5 space
Poincare coordinates

[proof of DCS for loop integrands: Dennen, Huang, 6d; Caron-Huot, O`Connell, 10d]

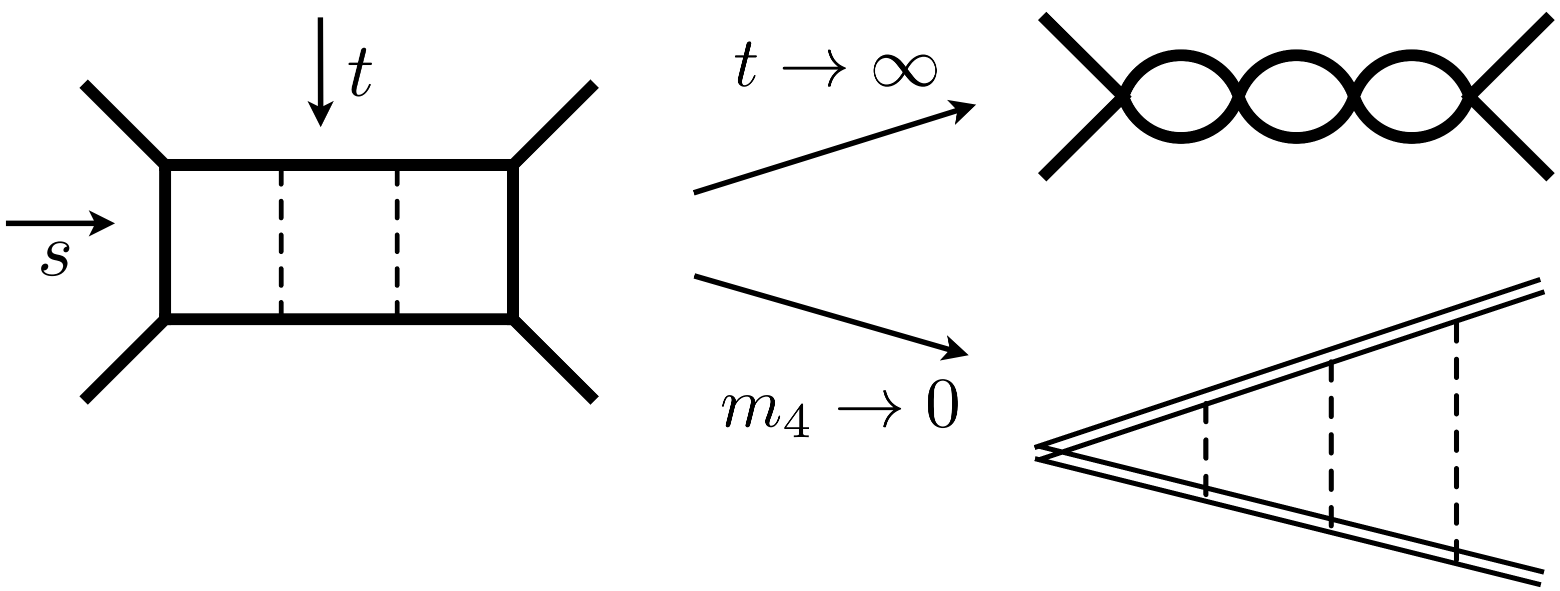
uses of the symmetry

$$M_4(s,t;m_1,m_2,m_3,m_4) = M(u,v)$$ [Alday, JMH, Plefka,Schuster]

$$u = \frac{4m_1m_3}{-s+(m_1-m_3)^2}\,, \quad v = \frac{4m_2m_4}{-t+(m_2-m_4)^2}$$

- equivalence of two physical pictures as $v \to 0$

[JMH, Naculich Plefka, Schnitzer]



$$M \sim_{t\to\infty} t^{(j(s)+1)}$$

$$M \sim_{m_4\to 0} (m_4)^{\Gamma_{\rm cusp}(\phi)}$$

$$\boxed{j(s)+1 = -\Gamma_{\rm cusp}(\phi) \quad \text{where} \quad s = 4m^2\sin^2\frac{\phi}{2}\,.}$$

$$m_i = m$$

bound state energy of pair of W bosons

- Regge theory: extract spectrum from

$$\Gamma_{\text{cusp}}(\phi_n) = -n \qquad\qquad E_n = 2m \sin\frac{\phi_n}{2}$$

n integer

- obtain bound state energy from cusp anomalous dimension

$$\Gamma_{\text{cusp}}(\phi) = -\frac{\lambda}{8\pi^2}\,\phi\,\tan\frac{\phi}{2} + \mathcal{O}(\lambda^2)\,.$$

$$\Gamma_{\text{cusp}}(\pi - \delta) \approx -\frac{\lambda}{4\pi\delta}$$

we find $$E_n - 2m = -\frac{\lambda^2 m}{64\pi^2 n^2} + O(\lambda^3)\,.$$

as expected from $$H = \frac{p^2}{2\mu} - \frac{\lambda}{4\pi}\frac{1}{|x|}$$

higher orders

- resummation required (ultrasoft effects) [systematic EFT, e.g. Pineda 2007]

$$\Gamma_{\text{cusp}}(\pi-\delta) = \frac{-\lambda}{4\pi\delta}\left(1-\frac{\delta}{\pi}\right) + \frac{\lambda^2}{8\pi^3\delta}\log\frac{\epsilon_{\text{uv}}}{2\delta} - \frac{\lambda}{4\pi^2}\int_{\epsilon_{\text{uv}}}^{\infty}\frac{d\tau}{\cosh(\tau)-1}\left(e^{-\tau\frac{\lambda}{4\pi\delta}}-1\right) + \mathcal{O}(\lambda^3)\,.$$

[Correa, JMH, Maldacena, Sever, 2012]

- result for energy

$$(E_n - 2m)\,|_{\lambda^3} = \frac{-\lambda^3 m}{64\pi^4 n^2}\left[S_1(n) + \log\frac{\lambda}{2\pi n} - 1 - \frac{1}{2n}\right] \qquad S_1(n) = \sum_{k=1}^{n}\frac{1}{k}$$

- checks
  - [...] bounded for any n
  - n large correctly gives quark-antiquark potential

    [Ericksson, Semenoff, Szabo Zarembo, 1999; Pineda 2007]
  - confirmed by standard ‘Coulomb resummation’

    [Caron-Huot, JMH, to appear]

    [see Beneke, Kiyo & Schuller 1312.4791]

Strong coupling check

- cusp anomalous dimension $\Gamma_{\rm cusp}(\phi)$ at strong coupling was computed from minimal surface

[Drukker, Gross, Ooguri, 1999]

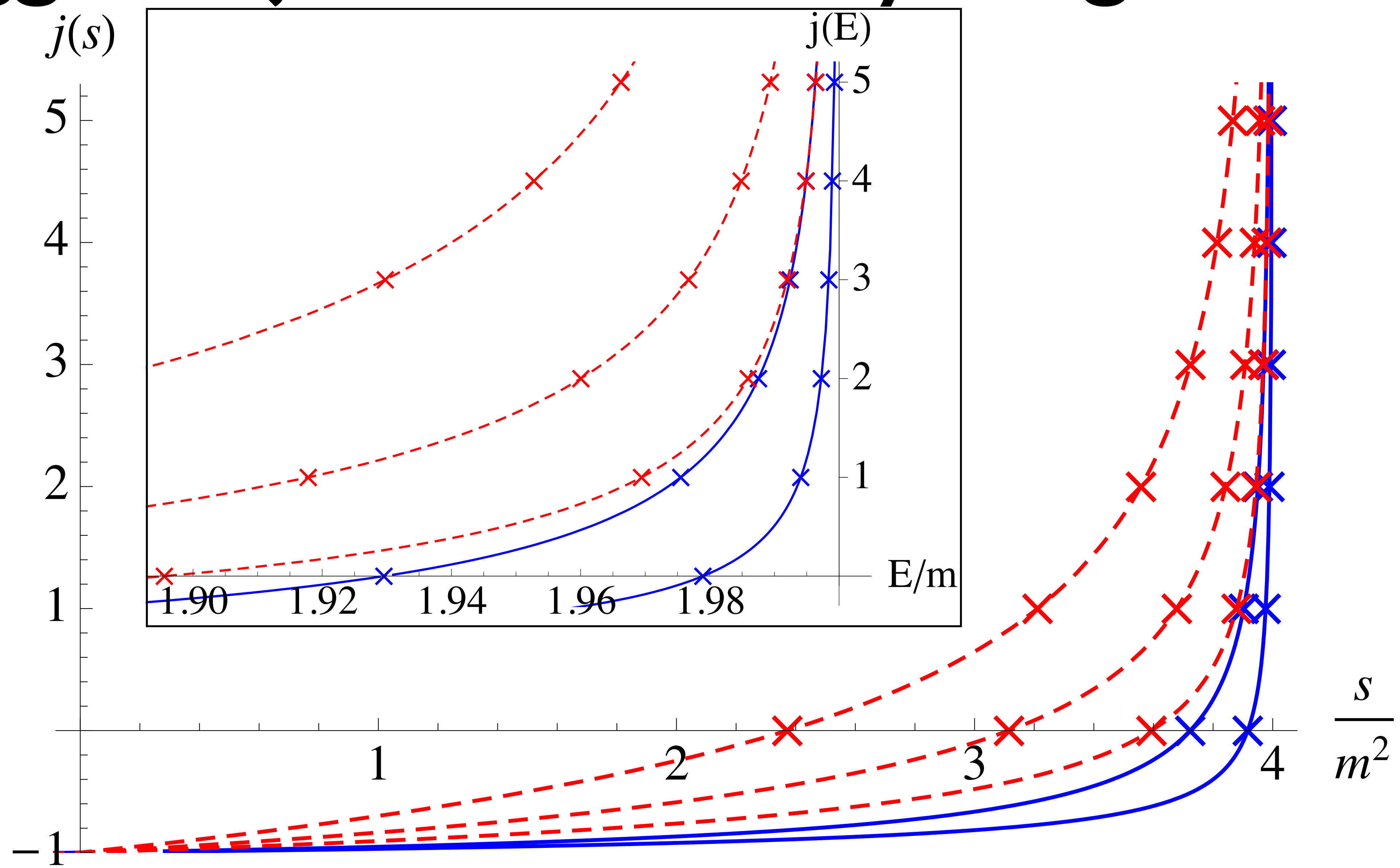
- spectrum of 'mesons' was computed at strong coupling in 2003

[Kruczensky, Mateos, Myers, Winters, 2003]

- the two curves agree perfectly, once one uses the correct dictionary!

$$E_n = 2m \sin \frac{\phi_n}{2}$$

Regge trajectories of Hydrogen-like states



$\lambda = 5, 10, 10, 30, 100$ (bottom to top)

solid/blue: based on weak-coupling formulas

dashed/red: based on strong-coupling formulas

- exact spectrum should be computable from TBA for $\Gamma_{\text{cusp}}(\phi)$

[Correa, Maldacena, Sever; Drukker]

Conclusions & outlook

- dual conformal symmetry of N=4 SYM is QFT generalization of conservation of Laplace-Runge-Lenz vector in classical in quantum mechanics

  - open question: is this unique? other 4-d QFT's with such a hidden symmetry?

- application: equivalence between gluon Regge trajectory and cusp anomalous dimension in this theory

- compute bound state energies of hydrogen-like system from

$$\Gamma_{\mathrm{cusp}}(\phi_n) = -n \qquad\qquad E_n = 2m \sin \frac{\phi_n}{2}$$

  - determine exact result for bound state energy from integrability

[Correa, Maldacena, Sever; Drukker]

- we started a systematic investigation of massive four-particle amplitudes $M(u,v)$ in N=4 SYM (to 3 loops)

[Caron-Huot & JMH, hep-th/1404.2922]

- e.g. one loop $\quad M = 1 - \lambda/(16\pi^2) I_1 + \mathcal{O}(\lambda^2)$

$$I_1 = \frac{2}{\beta_{uv}}\Big\{2\log^2\left(\frac{\beta_{uv}+\beta_u}{\beta_{uv}+\beta_v}\right) + \log\left(\frac{\beta_{uv}-\beta_u}{\beta_{uv}+\beta_u}\right)\log\left(\frac{\beta_{uv}-\beta_v}{\beta_{uv}+\beta_v}\right) - \frac{\pi^2}{2}$$
$$+ \sum_{i=1,2}\left[2\,\mathrm{Li}_2\left(\frac{\beta_i - 1}{\beta_{uv}+\beta_i}\right) - 2\,\mathrm{Li}_2\left(-\frac{\beta_{uv}-\beta_i}{\beta_i+1}\right) - \log^2\left(\frac{\beta_i+1}{\beta_{uv}+\beta_i}\right)\right]\Big\}.$$

with $u = \frac{4m^2}{-s}\,, \qquad v = \frac{4m^2}{-t}\,,$

$$\beta_u = \sqrt{1+u}\,, \qquad \beta_v = \sqrt{1+v}\,, \qquad \beta_{uv} = \sqrt{1+u+v}\,.$$

- symmetry suggests SO(4) partial wave expansion

[Caron-Huot & JMH, to appear]

remarkably, only one subleading Regge trajectory (to 3 loops), corresponding to operator of dimension

$$\Gamma_1(\phi) = 1 + \lambda/(4\pi^2) + \mathcal{O}(\lambda^2)\,.$$

reproduce from integrability?

Tack så mycket!

Extra slides

(from Simon Caron-Huot's talk at Amplitudes 2014, Saclay)