Scattering Amplitudes at Strong Coupling Beyond the Area Paradigm

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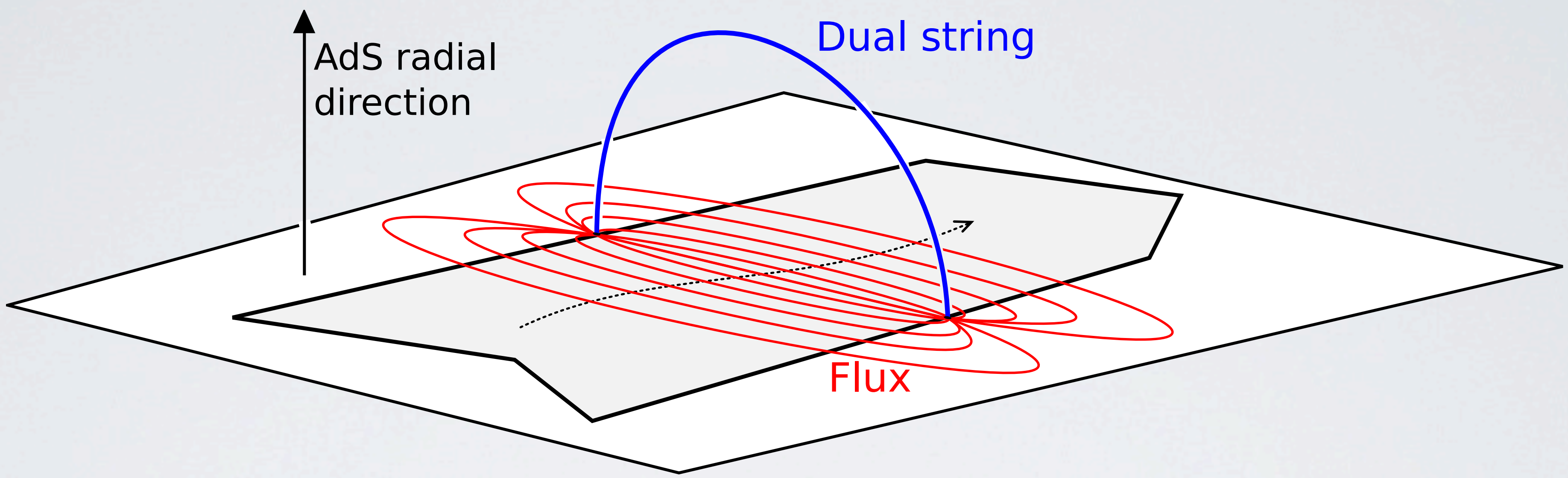
Workshop on Supersymmetric Field Theories
Nordita

based on work with Amit Sever and Pedro Vieira

Wilson loops at finite coupling in N=4 SYM

[Alday,Gaiotto,Maldacena,Sever,Vieira'10]



1+1d background : *flux tube* sourced by two parallel null lines

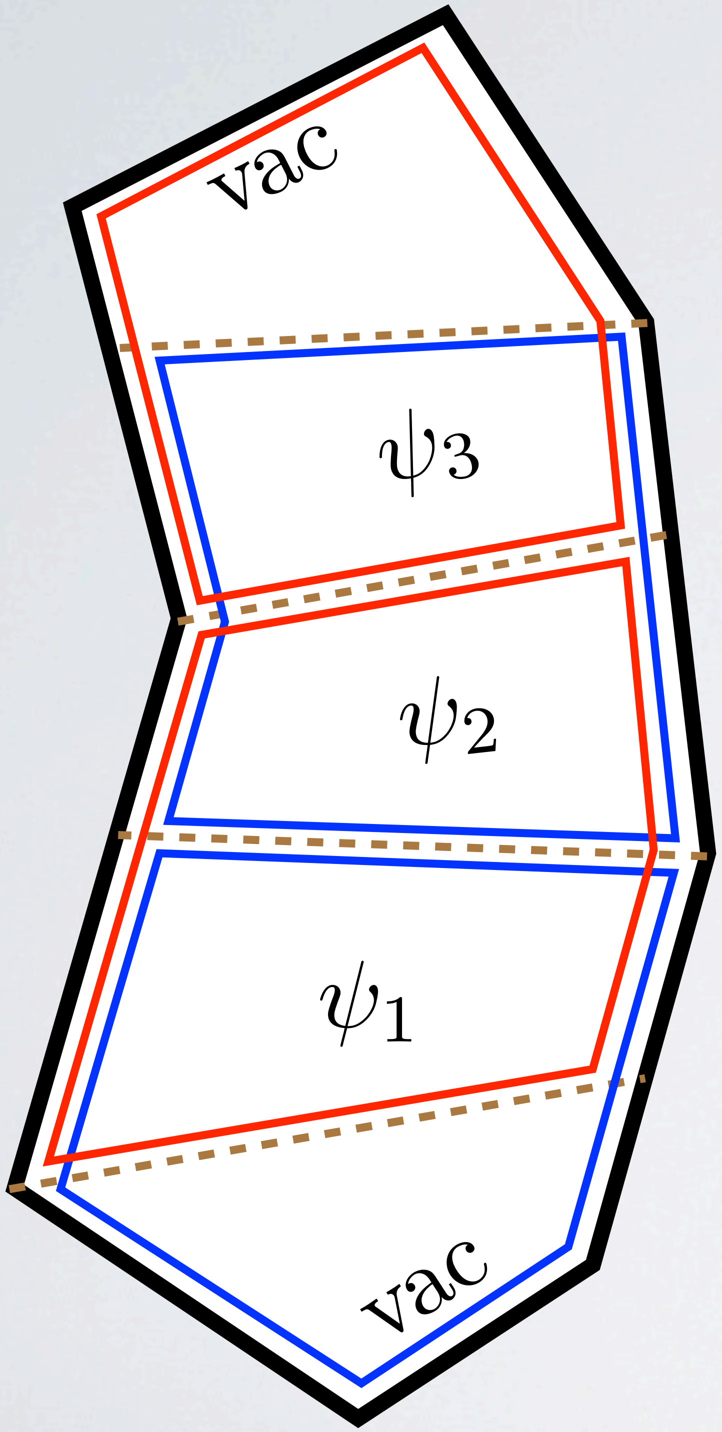
bottom&top cap excite the *flux tube* out of its ground state

→ *Sum over all flux-tube eigenstates*

$$\mathcal{W} = \sum_{\text{states } \psi} C_{\text{bot}}(\psi) \times e^{-E(\psi)\tau + ip(\psi)\sigma + im(\psi)\phi} \times C_{\text{top}}(\psi)$$

Refinement : the pentagon way

[BB,Sever,Vieira'13]



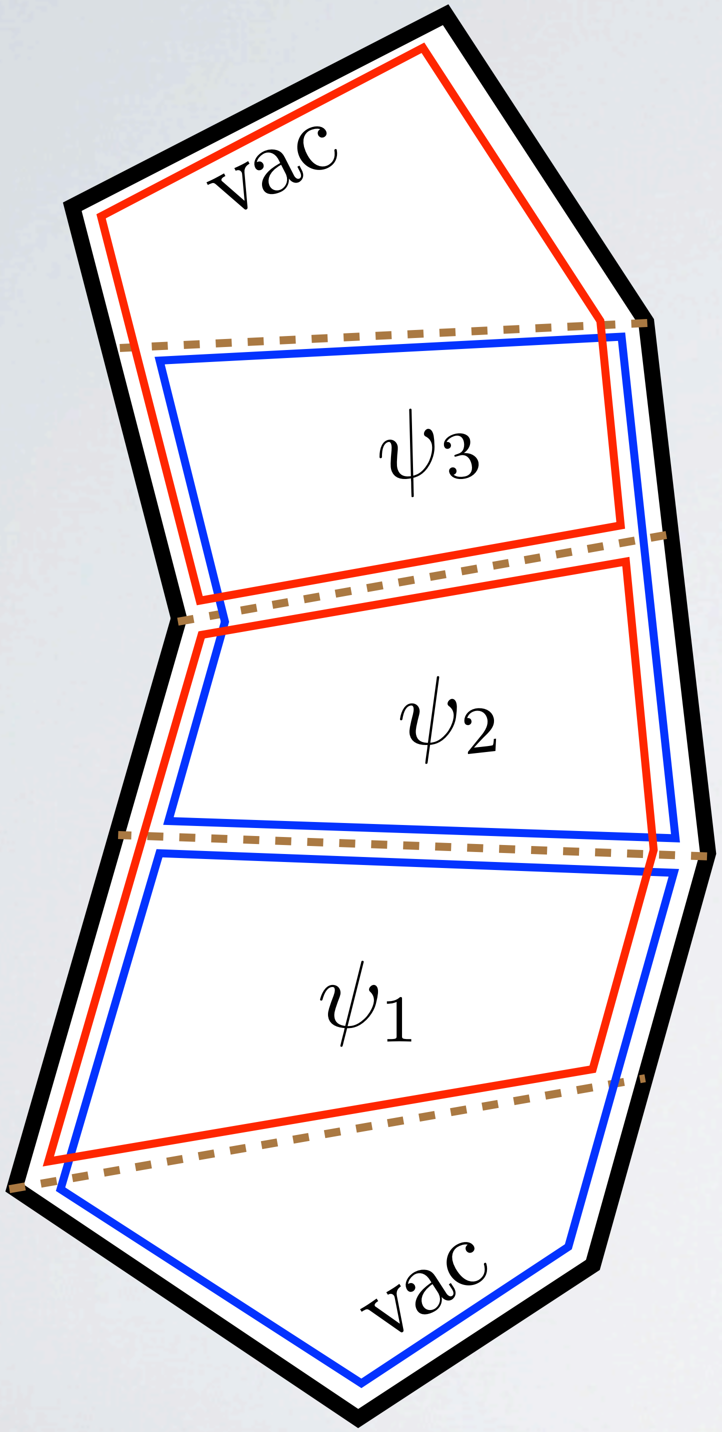
Valid at any coupling

$$= \sum_{\psi_i} \left[\prod_i e^{-E_i \tau_i + i p_i \sigma_i + i m_i \phi_i} \right] \times$$

$$P(0|\psi_1) P(\psi_1|\psi_2) P(\psi_2|\psi_3) P(\psi_3|0)$$

Refinement : the pentagon way

[BB,Sever,Vieira'13]



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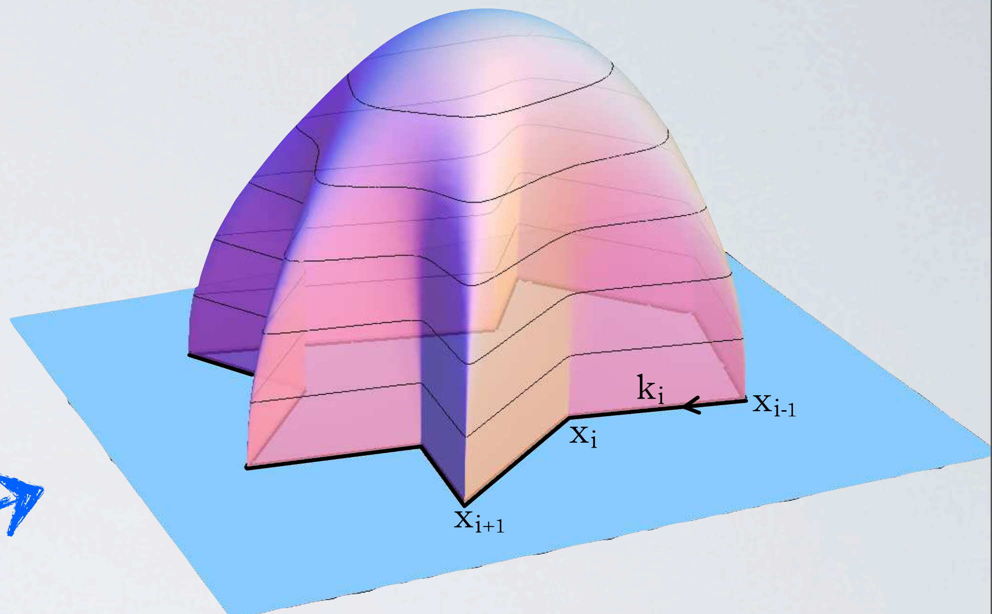
$$P(0|\psi_1) P(\psi_1|\psi_2) P(\psi_2|\psi_3) P(\psi_3|0)$$

To compute amplitudes we need

- The spectrum of flux-tube states ψ
- All the pentagon transitions $P(\psi_1|\psi_2)$

Beyond the area paradigm

Simplest case : hexagon (n = 6) WL



classical

$$\mathcal{W}_{n=6} = f_6\, \lambda^{-\frac{7}{288}}\, e^{\frac{\sqrt{\lambda}}{144} - \frac{\sqrt{\lambda}}{2\pi} A_{n=6}} (1 + O(1/\sqrt{\lambda}))$$

minimal area in AdS_5

quantum

[Alday,Gaiotto,Maldacena'09]
[Alday,Maldacena,Sever,Vieira'10]

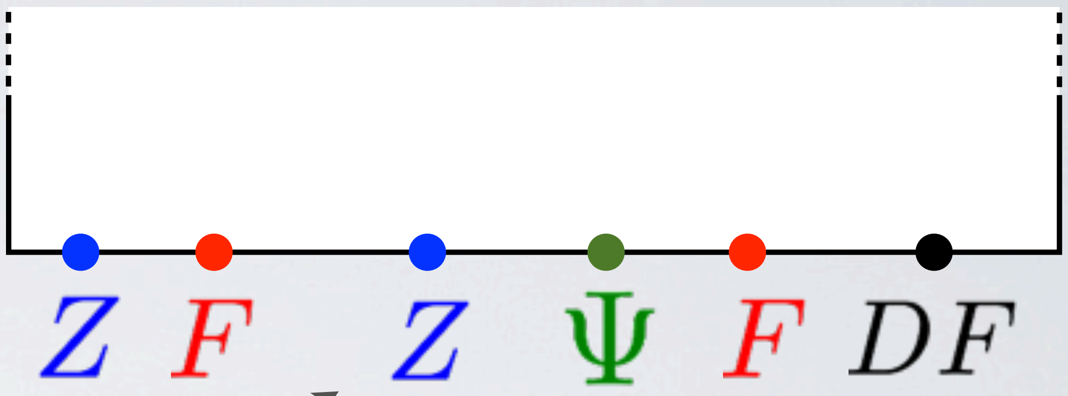
Pre-factor

$$f_6 = \frac{1.04}{(\sigma^2 + \tau^2)^{1/72}} + O(e^{-\sqrt{2}\tau})$$

[BB,Sever,Vieira'14]

The flux-tube eigenstates

$\psi = N$ particles state



(Adjoint) field insertions along a light-ray :
create/annihilate state on the flux tube

Spectral data

$$E = E(u_1) + E(u_2) + \ldots + E(u_N)$$

rapidity

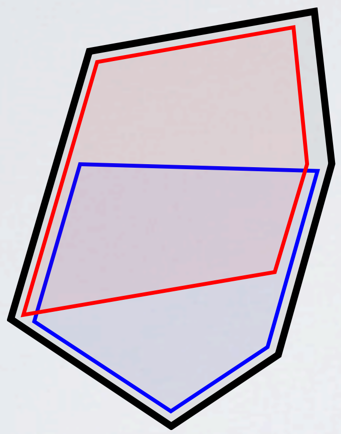
$$p = p(u_1) + \cdots + p(u_N)$$

$$E(u) = \text{twist} + g^2 \ldots$$

$$p(u) = 2u + g^2 \ldots$$

can be found using integrability

Pentagon/OPE series for hexagon

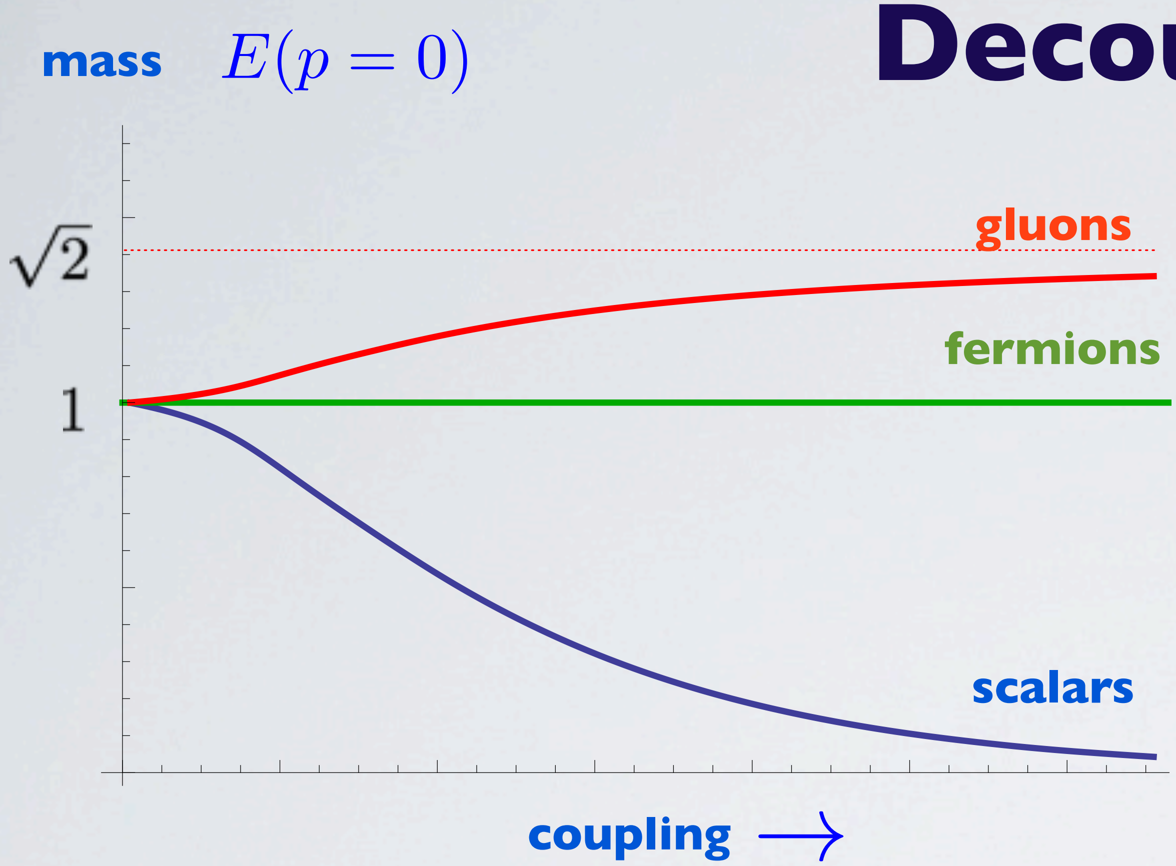
$\mathcal{W}_{\rm hex} =$  $= \sum_{\mathbf{a}} \int d\mathbf{u}\, P_{\mathbf{a}}(0|\mathbf{u})\, e^{-E(\mathbf{u})\tau + ip(\mathbf{u})\sigma + im\phi} P_{\mathbf{a}}(\bar{\mathbf{u}}|0)$

i.e. in collinear limit

Lightest states dominate at large τ

What are they?

Decoupling limit



Scalar mass is *exponentially small* at strong coupling

$$m = \frac{2^{1/4}}{\Gamma(5/4)} \lambda^{1/8} e^{-\frac{\sqrt{\lambda}}{4}} (1 + O(1/\sqrt{\lambda})) \ll 1$$

For $\tau \gg 1$ all heavy flux tube excitations decouple

Low energy effective theory :
(relativistic) O(6) sigma model

[Alday,Maldacena'07]

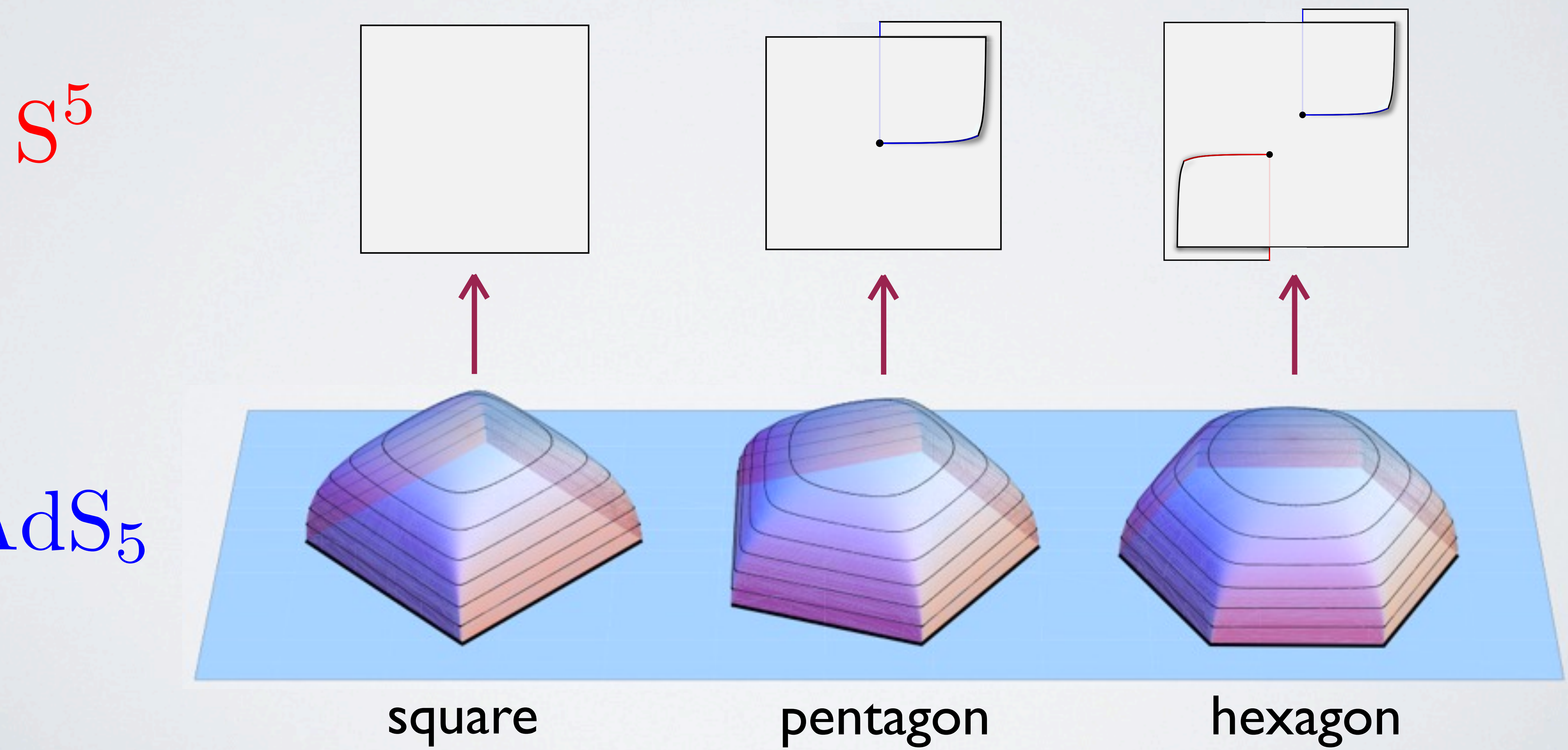
$$\mathcal{L}_{\text{eff}} = \frac{\sqrt{\lambda}}{4\pi} \partial X \cdot \partial X \qquad \text{with} \qquad X^2 = \sum_{i=1}^{6} X_i^2 = 1$$

Minimal surface from low-energy viewpoint

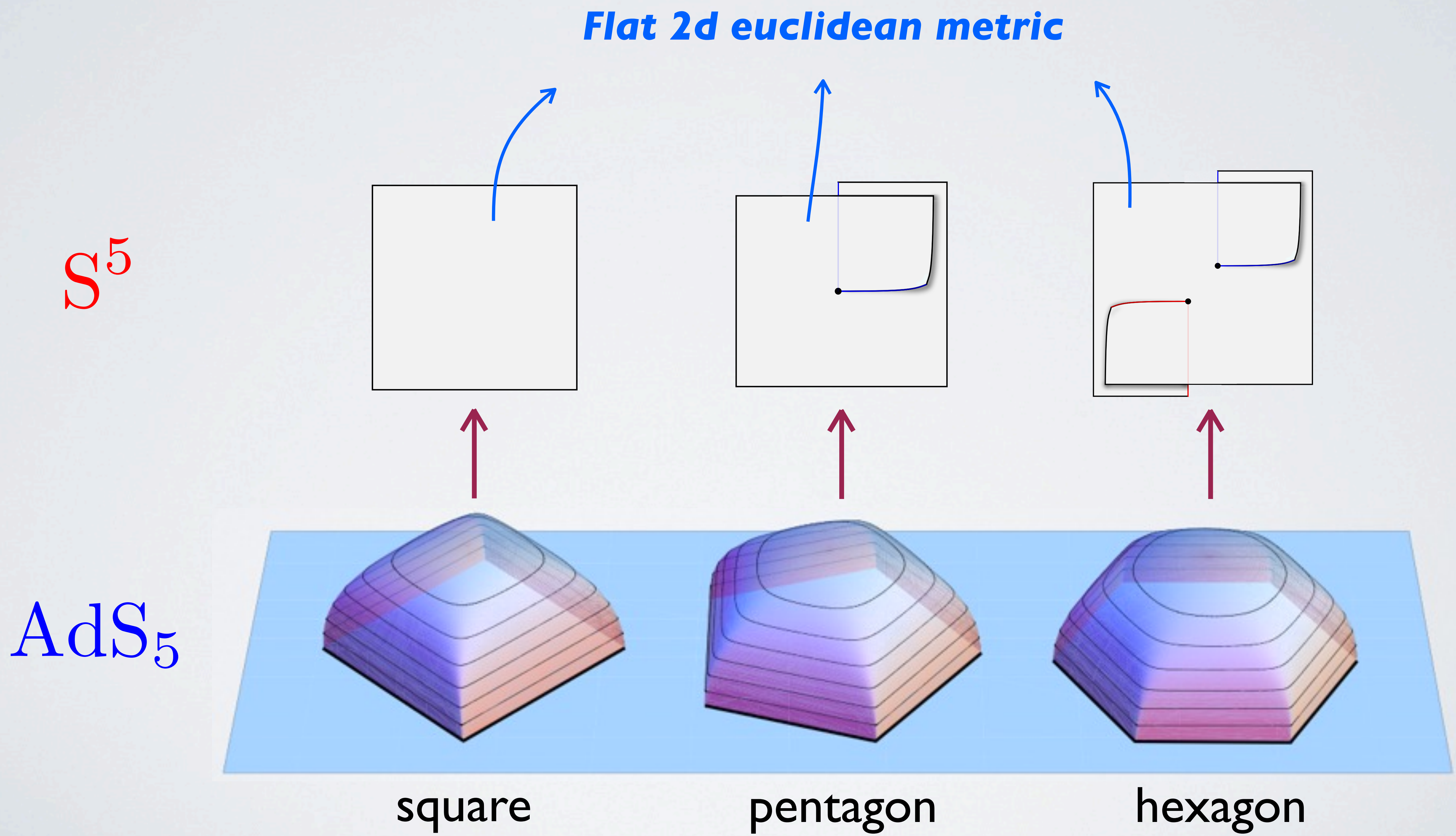
$$S_{\text{sphere}} = \frac{\sqrt{\lambda}}{4\pi}\int d^2z\,\sqrt{g}g^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta X$$

sphere embedding coordinates

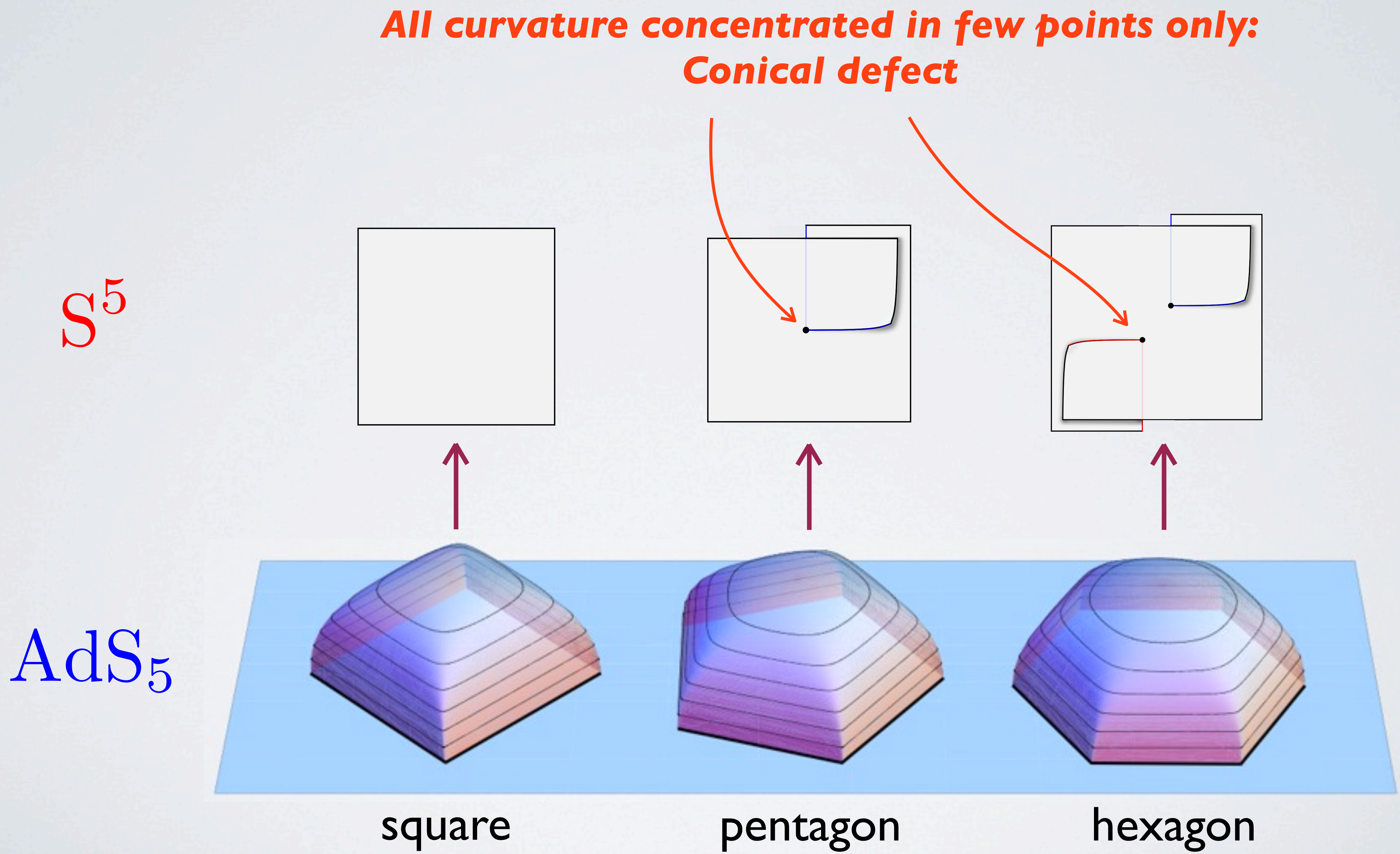
induced metric



Minimal surface from low-energy viewpoint

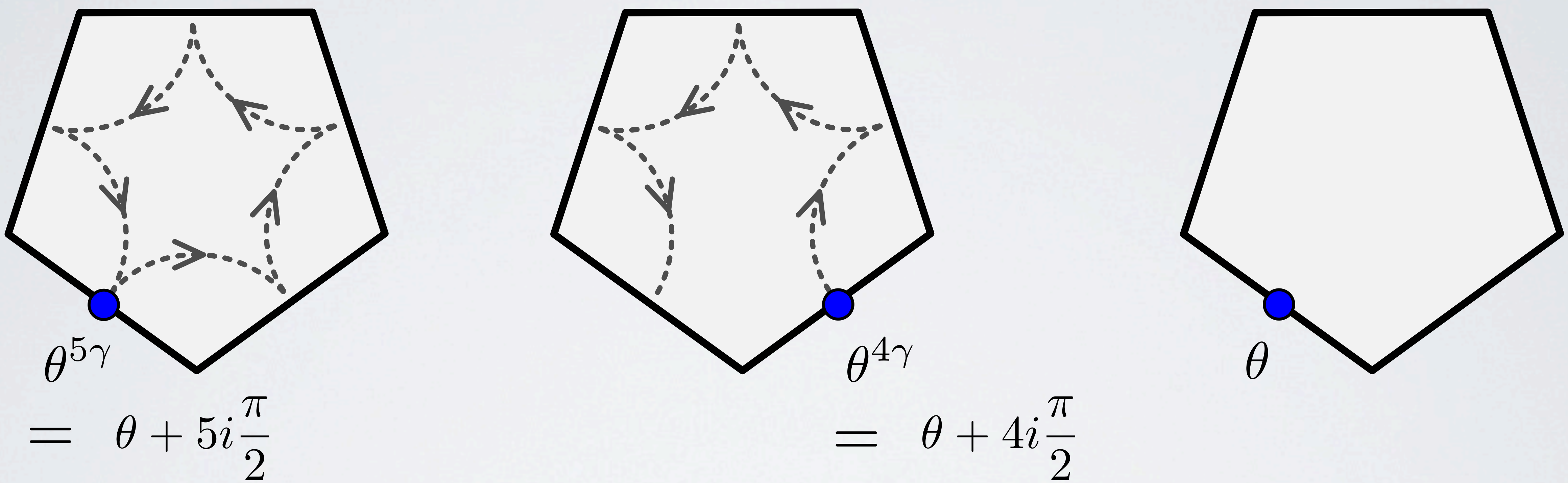


Minimal surface from low-energy viewpoint



Monodromy

One can go around the pentagon with 5 mirror rotations

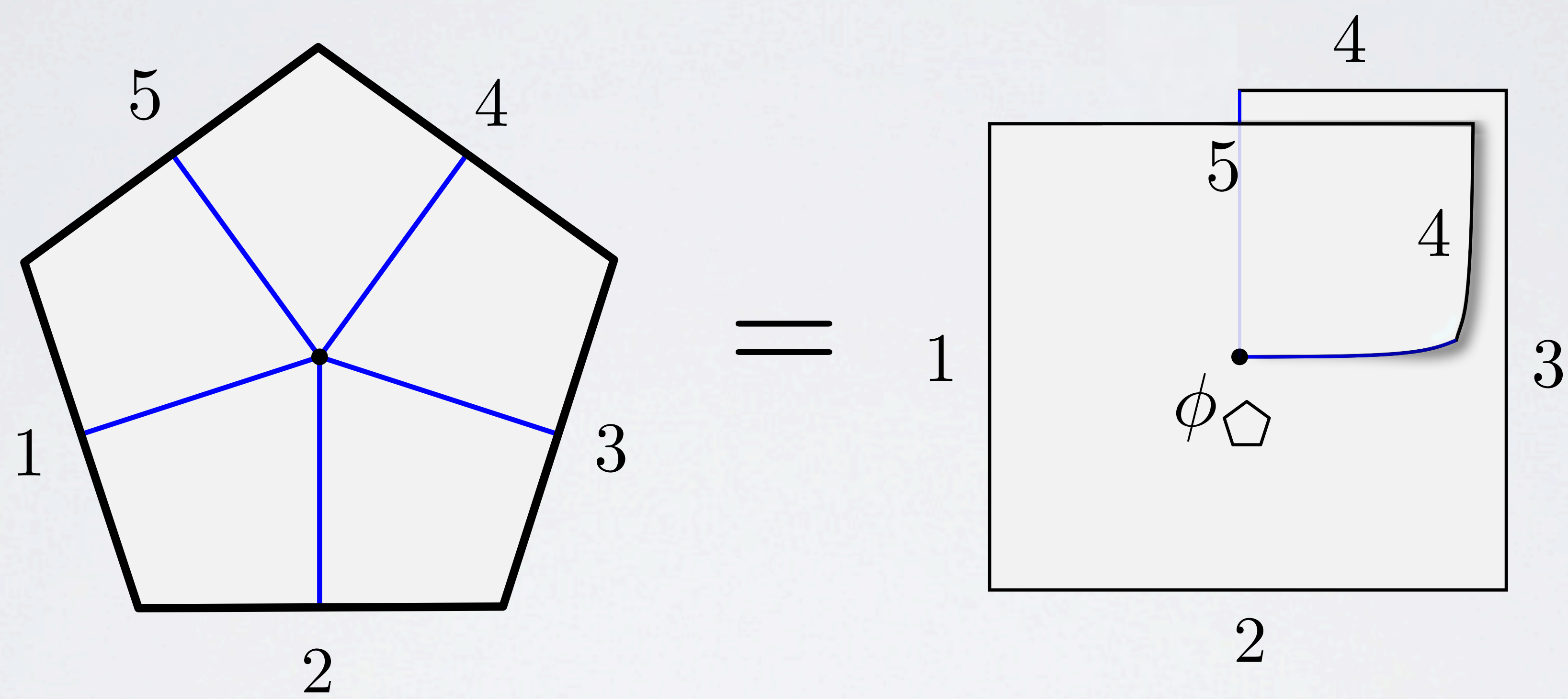


This is one more than for a square

$$E \xrightarrow{\gamma} ip \longrightarrow -E \longrightarrow -ip \longrightarrow E$$

Pentagon as twist operator

In short, a pentagon = 5 quadrants glued together



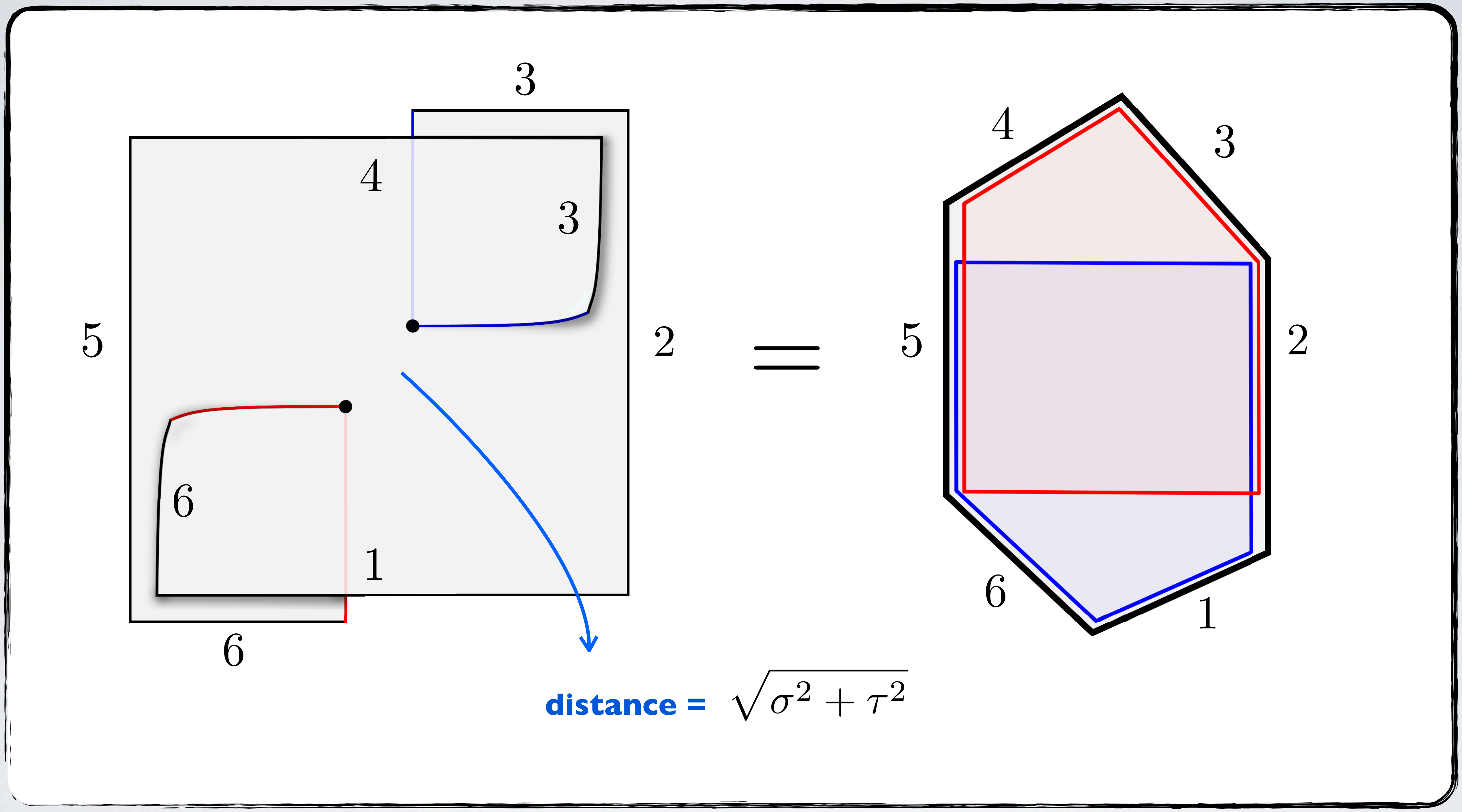
Geometrical picture :

excess angle $= \dfrac{\pi}{2}$

Hamiltonian picture:

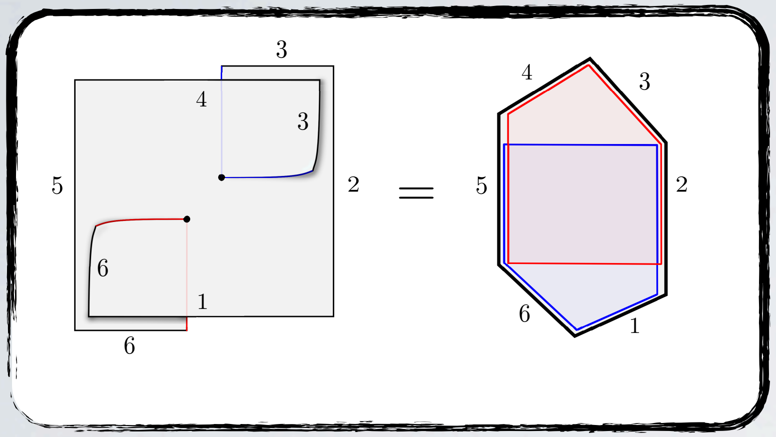
twist operator $\phi_{\pentagon}$

Hexagon as a correlator of twist operators



$$\mathcal{W}_6 = \langle 0|\, \phi_{\pentagon}(\tau,\sigma)\phi_{\pentagon}(0,0)\,|0\rangle$$

Hexagon as a correlator of twist operators



corrections from heavy modes irrelevant in collinear limit

$$\mathcal{W}_6 = \langle 0 | \, \phi_{\pentagon}(\tau,\sigma)\phi_{\pentagon}(0,0) \, |0\rangle \; + O(e^{-\sqrt{2}\tau})$$

Probes the physics of the O(6) sigma model :

$$\mathcal{W}_{O(6)}(z) \qquad z = m\sqrt{\sigma^2+\tau^2}$$

Large distance $\quad z \gg 1 \quad \mathcal{W}_{O(6)} = 1 + O(e^{-2z})$

Short distance $\quad z \ll 1 \quad \mathcal{W}_{O(6)} = \; ?$

OPE as form factor expansion

Insert complete basis of states

See [Cardy,Castro-Alvaredo,Doyon'07] for similar considerations for computing entanglement entropy in integrable QFT

$$\mathcal{W}_{O(6)} = \sum_N \frac{1}{N!} \int \frac{d\theta_1 \dots d\theta_N}{(2\pi)^N} \langle 0| \phi_\pentagon |\theta_1, \dots, \theta_N\rangle \langle \theta_1, \dots, \theta_N| \phi_\pentagon |0\rangle \, e^{-z \sum_i \cosh\theta_i}$$

Pentagon transition = form factor of twist operator

$$P(0|\theta_1, \dots, \theta_N) = \langle \theta_1, \dots, \theta_N| \phi_\pentagon |0\rangle$$

Normalization

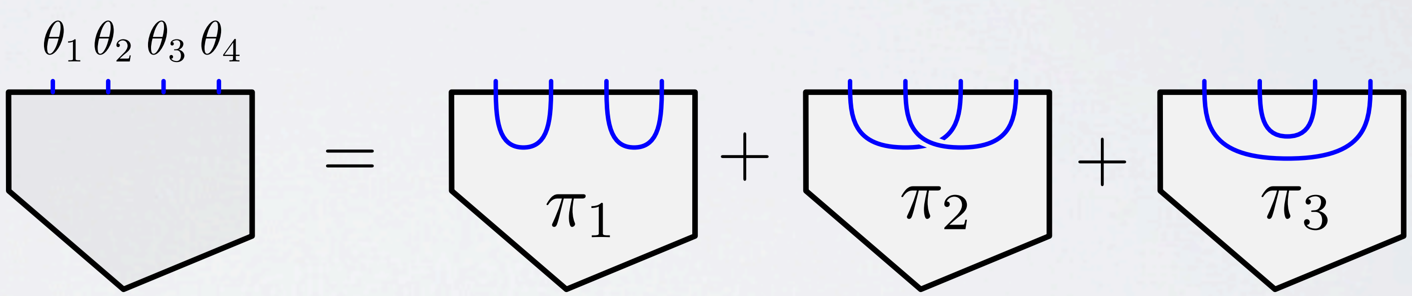
$\langle 0| \phi_\pentagon |0\rangle = 1$ which enforces that $\mathcal{W}_{O(6)} \to 1$ $z \to \infty$

Hexagon beyond 2pt approximation

$$\mathcal{W}_6 = 1 + \frac{1}{2}\int \frac{d\theta_1 d\theta_2}{(2\pi)^2} |P(0|\theta_1,\theta_2)|^2 e^{-m\tau(\cosh\theta_1+\cosh\theta_2)+im\sigma(\sinh\theta_1+\sinh\theta_2)} + \dots$$

multi-particle states

Multi-particle transitions



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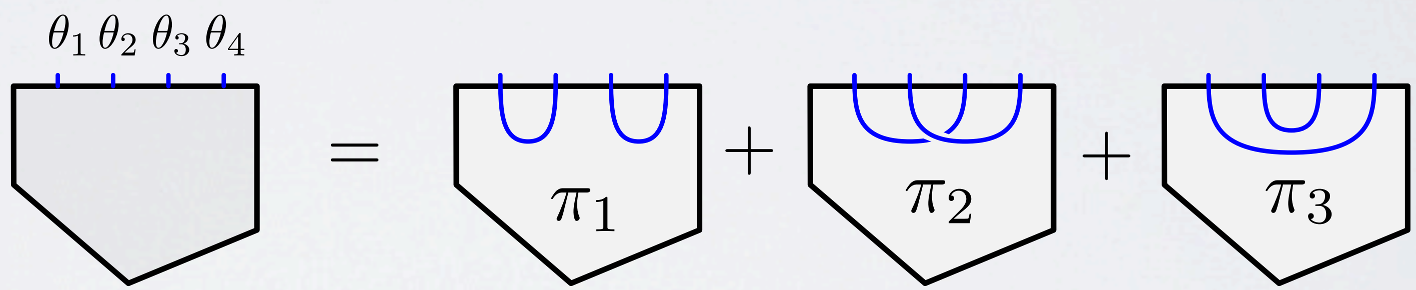
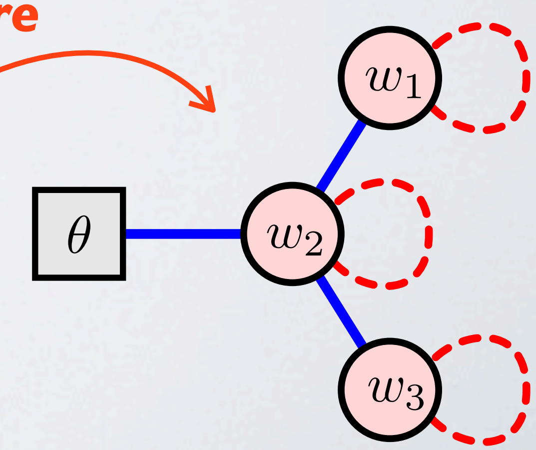
Multi-particle transitions



General formula:

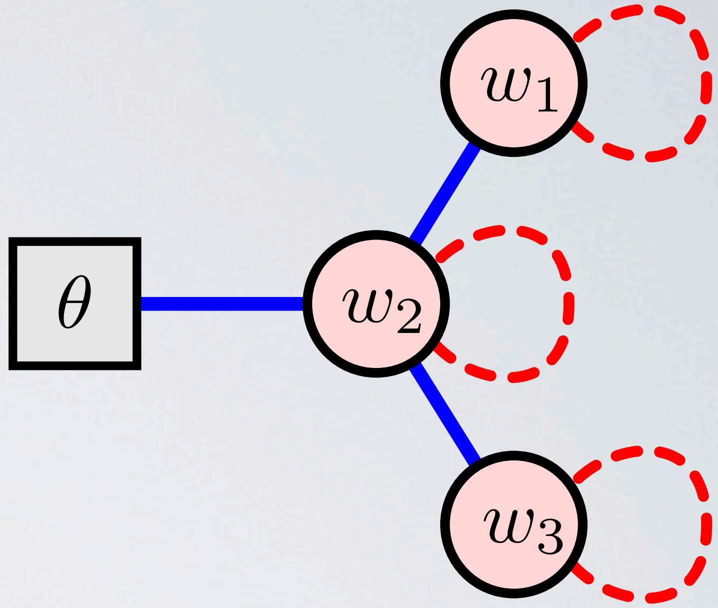
$$\text{integrand} = \prod_{i<j} \frac{1}{P(\theta_i|\theta_j)P(\theta_j|\theta_i)} \times \text{rational}$$

All group/algebraic structure inthere



Interlude on matrix part

Integral representation of rational part



$$r = \frac{1}{K_1!K_2!K_3!} \int \prod_i \frac{dw_{1,i}}{2\pi} \prod_i \frac{dw_{2,i}}{2\pi} \prod_i \frac{dw_{3,i}}{2\pi}$$

$$\times \frac{\prod_{i<j} g(w_{1,i} - w_{1,i}) \prod_{i<j} g(w_{2,i} - w_{2,i}) \prod_{i<j} g(w_{3,i} - w_{3,i})}{\prod_{i,j} f(w_{2,i} - \frac{2}{\pi}\theta_j) \prod_{i,j} f(w_{1,i} - w_{2,j}) \prod_{i,j} f(w_{3,i} - w_{2,j})}$$

$$g(x) = x^2(x^2+1) \qquad f(x) = x^2 + \frac{1}{4}$$

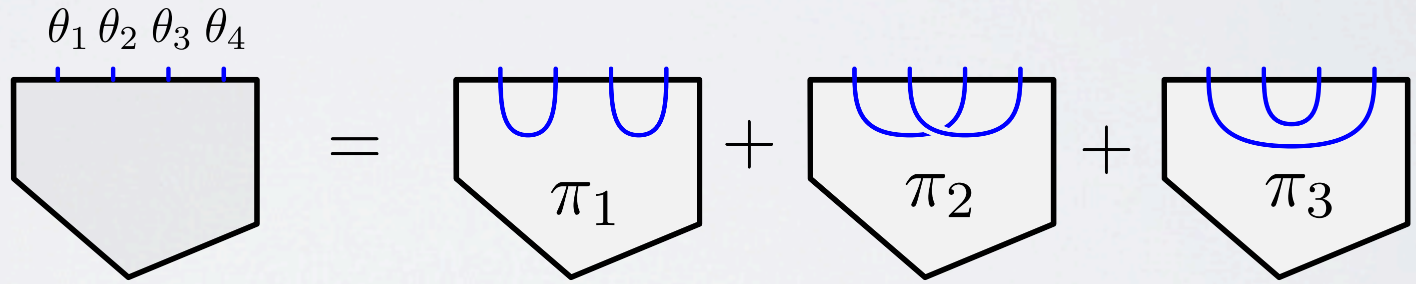
Here (for singlet states) $\quad K_2 = K_\theta \qquad K_1 = K_3 = \frac{1}{2} K_\theta$

Hexagon beyond 2pt approximation

$$\mathcal{W}_6 = 1 + \frac{1}{2}\int \frac{d\theta_1 d\theta_2}{(2\pi)^2}|P(0|\theta_1,\theta_2)|^2 e^{-m\tau(\cosh\theta_1+\cosh\theta_2)+im\sigma(\sinh\theta_1+\sinh\theta_2)} + \dots$$

multi-particle states

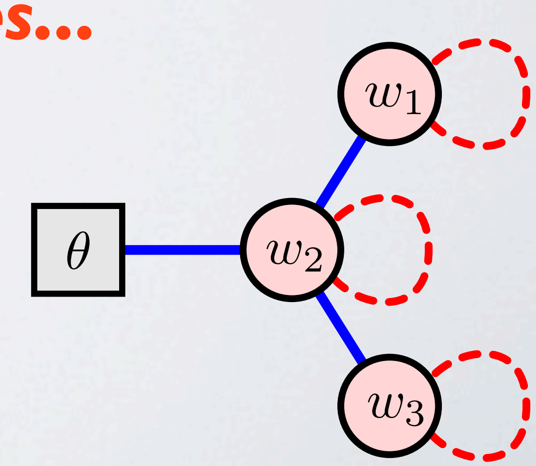
Multi-particle transitions



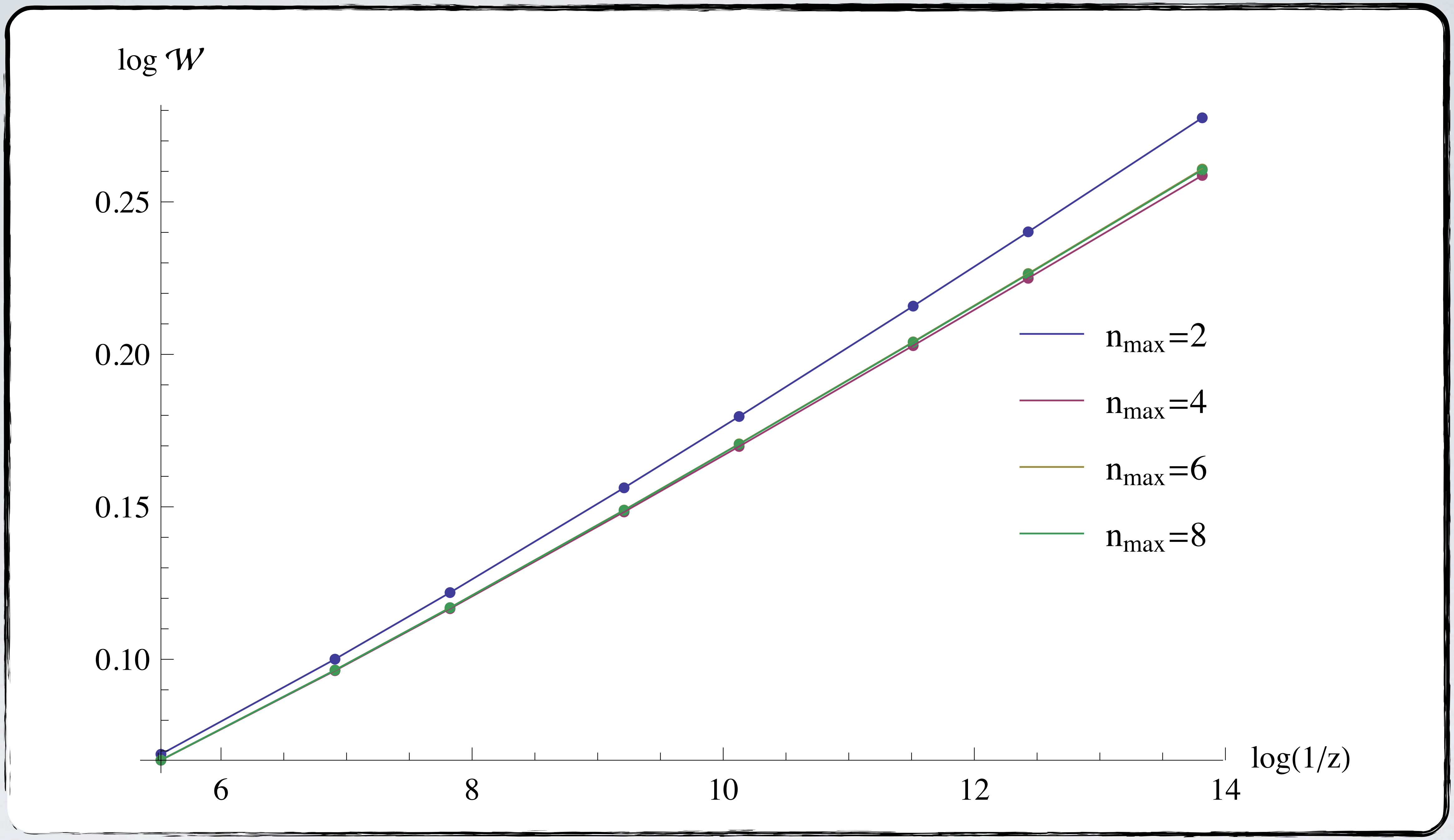
We have all ingredients!
We can plot the form factor series...

General formula:

$$\text{integrand} = \prod_{i<j} \frac{1}{P(\theta_i|\theta_j)P(\theta_j|\theta_i)} \times \text{rational}$$



Numerical analysis



Plot of the truncated OPE/form factor series representation for $\log \mathcal{W}_{O(6)}$

Short distance analysis

Short distance OPE (valid for $z \ll 1$)

$$\phi_{\pentagon}(\tau,\sigma)\phi_{\pentagon}(0,0) \sim \frac{\log(1/z)^{B}}{z^{A}}\phi_{\hexagon}(0,0)$$

3-point function

Critical exponent A

$$A = 2\Delta_{\pentagon} - \Delta_{\hexagon} = 2\Delta_{5/4} - \Delta_{3/2}$$

with Δ_k the scaling dimension of the twist operator ϕ_k

$$\Delta_k = \frac{c}{12}\left(k - \frac{1}{k}\right)$$

$$\begin{cases} c = \text{ central charge} \\ 2\pi(k-1) = \text{ excess angle for } \phi_k \end{cases}$$

[Knizhnik'87]
[Lunin,Mathur'00]
[Calabrese,Cardy'04]

Short distance analysis

Short distance OPE (valid for $z \ll 1$)

$$\phi_{\pentagon}(\tau,\sigma)\phi_{\pentagon}(0,0) \sim \frac{\log(1/z)^B}{z^A}\phi_{\hexagon}(0,0)$$

3-point function

Critical exponent A

$$A = \frac{1}{36} \qquad \text{since in our case } c = 5$$

Critical exponent B from one-loop anomalous dimensions

$$B = -\frac{3}{2}A = -\frac{1}{24}$$

Short distance analysis

For $z \ll 1$

include subleading RG logs

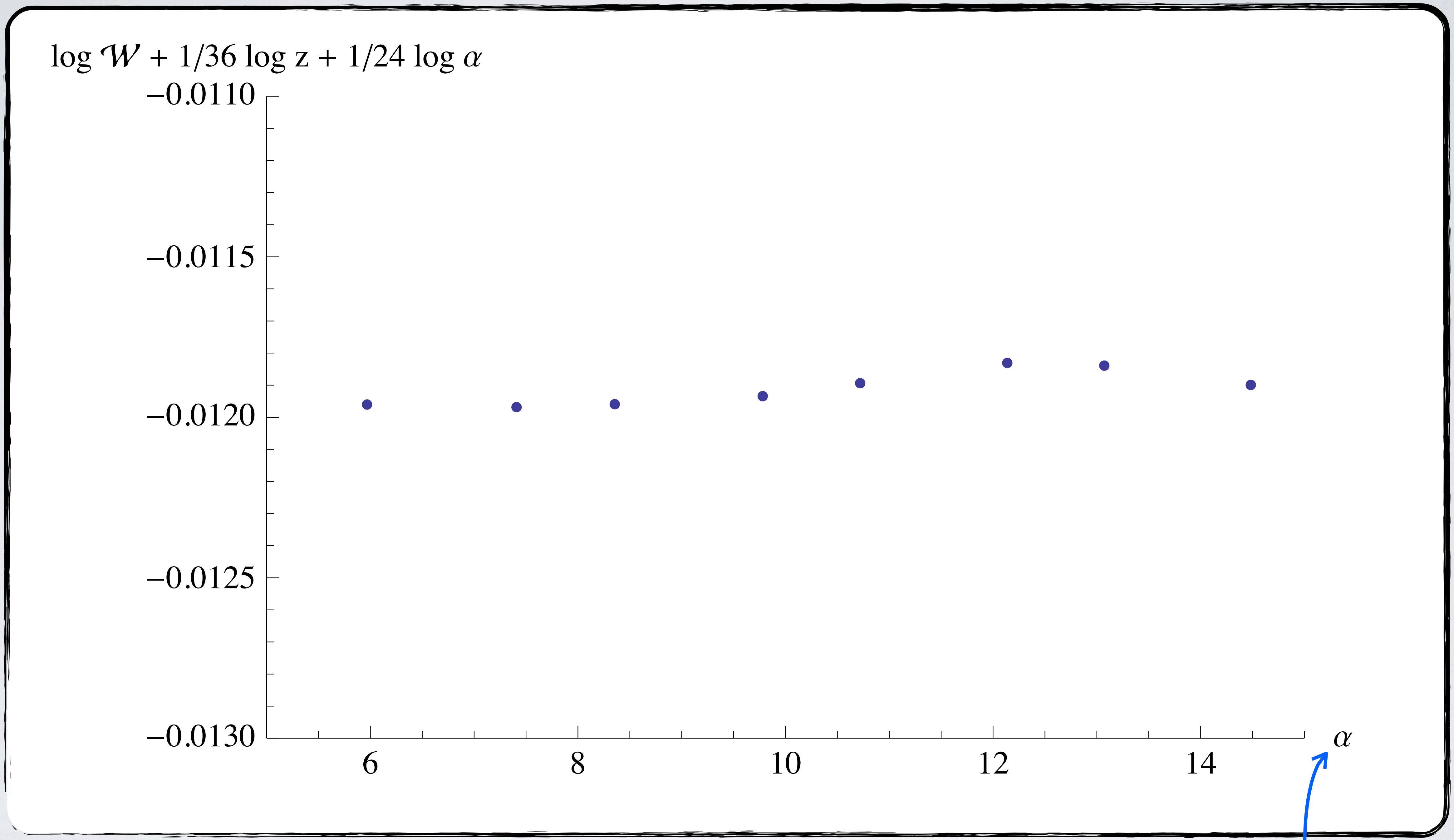
$$\mathcal{W}_{O(6)} = \frac{C}{z^{1/36} \log\left(1/z\right)^{1/24}} + \ldots$$

Constant C is fixed in the IR by

$$\mathcal{W}_{O(6)} \to 1 \qquad \text{when} \qquad z \to \infty$$

and is thus *non perturbative*

Numerical analysis



$$\log C = -0.01$$

running coupling

$$\alpha = \log(1/z) + \ldots$$

Short distance analysis

For $z \ll 1$ (i.e. $1 \ll \tau \ll e^{\sqrt{\lambda}/4}$)

$$m \simeq \frac{2^{1/4}}{\Gamma(5/4)} \lambda^{1/8} e^{-\frac{\sqrt{\lambda}}{4}}$$

$$\mathcal{W}_{O(6)} = \frac{C}{z^{1/36} \log(1/z)^{1/24}} + \dots$$

$$z = m\sqrt{\sigma^2 + \tau^2}$$

controlled by the gluons

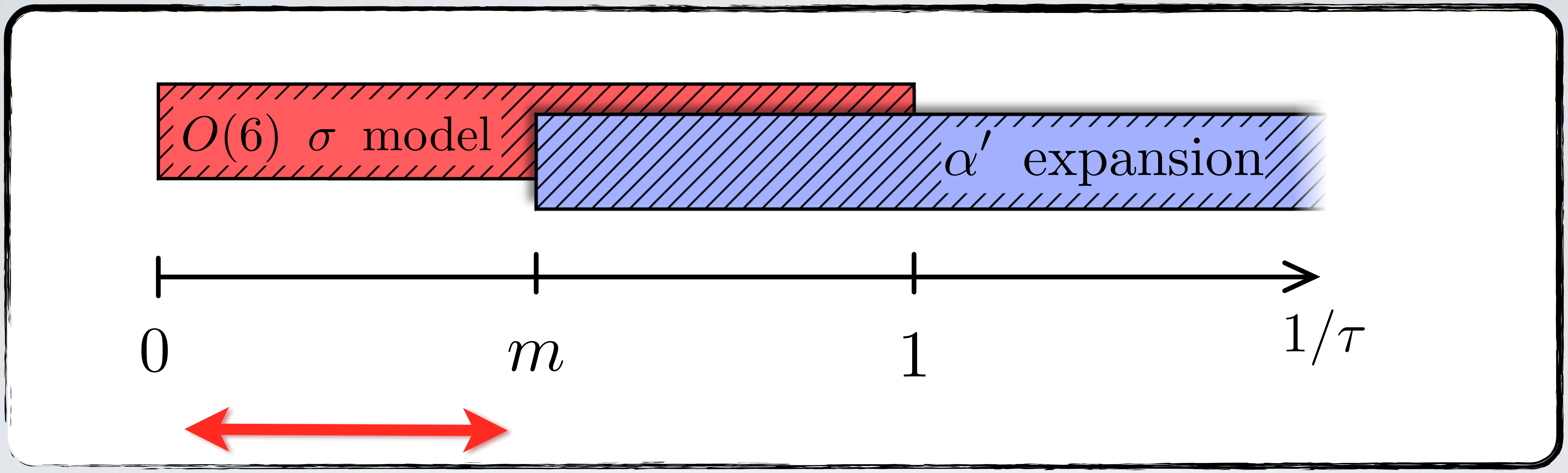
$$A_{n=6} = O(e^{-\sqrt{2}\tau})$$

$$\mathcal{W}_{n=6} = f_6 \, \lambda^{-\frac{7}{288}} \, e^{\frac{\sqrt{\lambda}}{144} - \frac{\sqrt{\lambda}}{2\pi} A_{n=6}} (1 + O(1/\sqrt{\lambda}))$$

Pre-factor

$$f_6 = \frac{1.04}{(\sigma^2 + \tau^2)^{1/72}} + O(e^{-\sqrt{2}\tau})$$

Infrared/non-perturbative regime

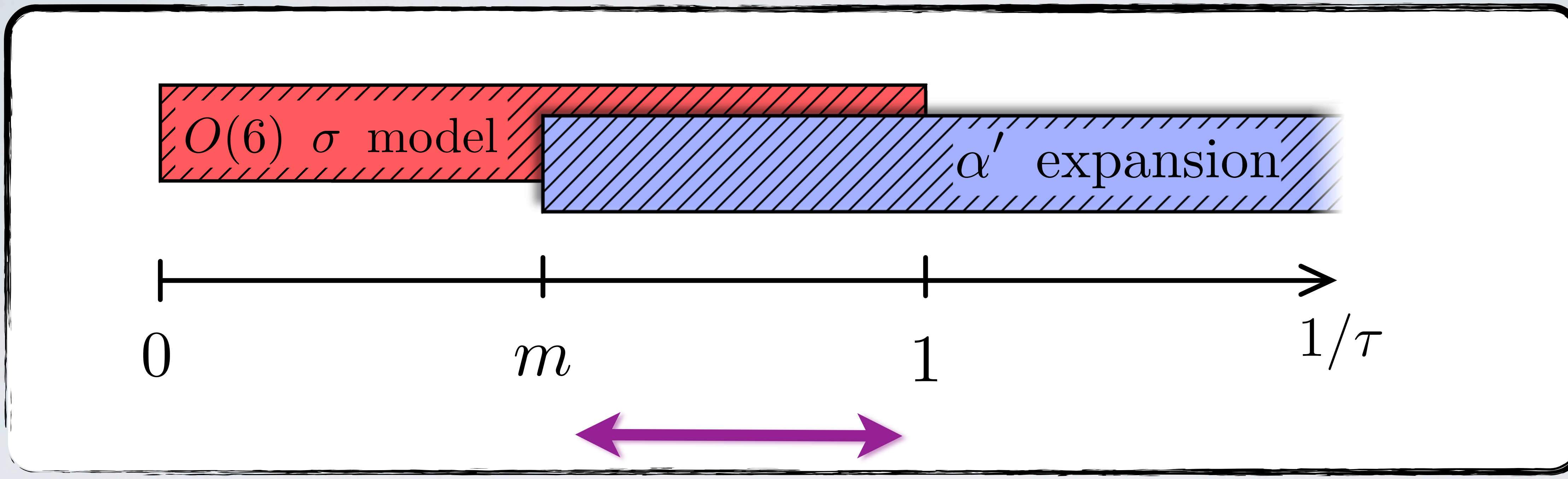


$z \gg 1$ equivalently $\tau \gg e^{\sqrt{\lambda}/4}$

Deep (infrared) collinear limit

Completely non perturbative

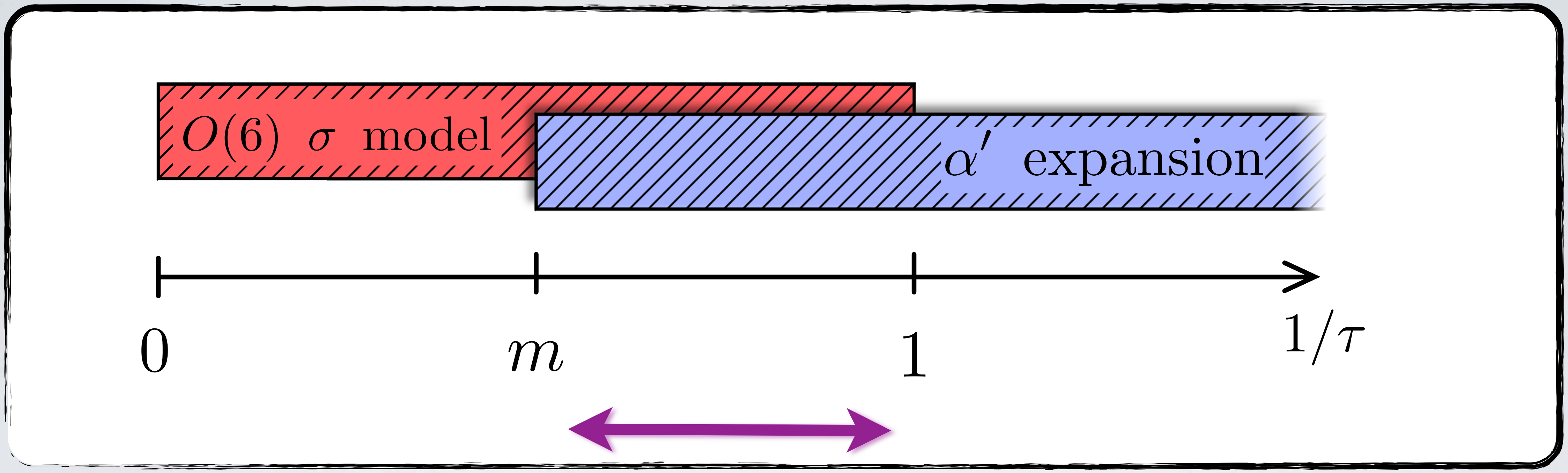
Cross over



$$z \ll 1 \qquad \text{equivalently} \qquad 1 \ll \tau \ll e^{\sqrt{\lambda}/4}$$

UV regime of O(6) model :
perturbative collinear limit

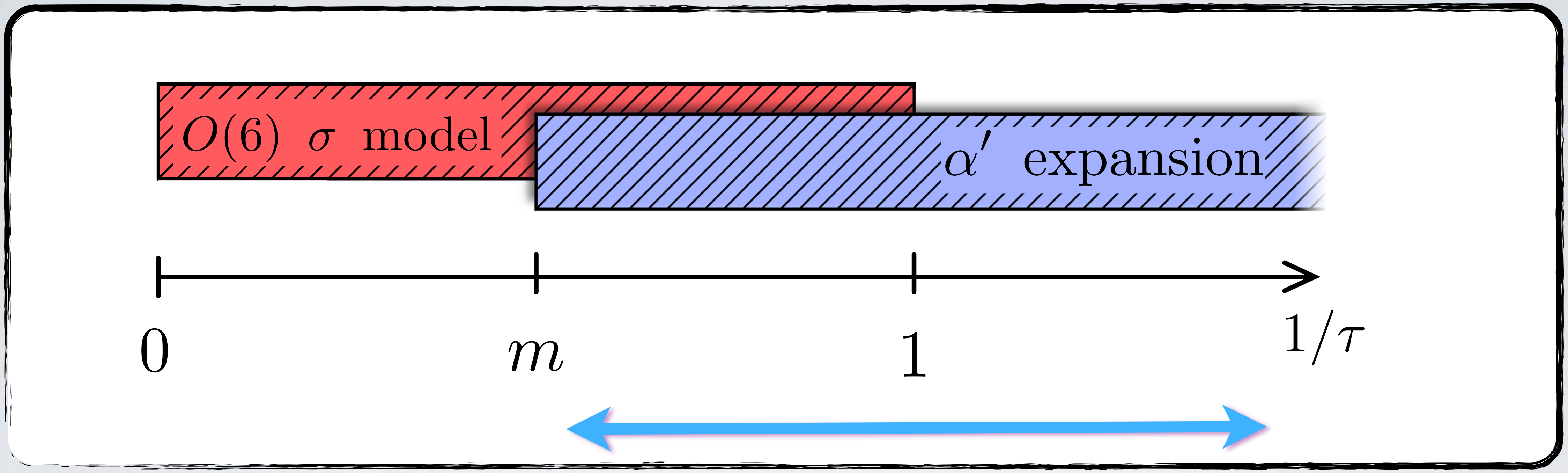
Cross over



here

we could match O(6) analysis
with
string perturbative expansion

Full stringy pre-factor



full thing :

include all heavy modes

gluons, *fermions*, ...

$$f_6 = \frac{1.04}{(\sigma^2 + \tau^2)^{1/72}} + O(e^{-\sqrt{2}\tau}) \quad + O(e^{-2\tau})$$

Conclusions

At strong coupling SA develop a non-perturbative regime in the near collinear limit

The string α' expansion breaks down for extremely large values of $\tau \sim -\log u_2 \sim e^{\sqrt{\lambda}/4}$

That's because flux tube mass gap m becomes extremely small

One should think in terms of correlators of twist operators

This fixes the collinear limit of SA at strong coupling

Outlook

Higher multiplicity (heptagon, ...)?

Next-to-MHV amplitudes?

Full one-loop pre-factor?

One-loop Thermodynamical-Bubble-Ansatz equations?

... *and many other questions...*

THANK YOU!

BACK UP

Higher multiplicity

Higher-point amplitudes correspond to higher-points correlators

$$\mathcal{W}_n = \langle 0| \, \phi_{\pentagon}(\tau_{n-4}, \sigma_{n-4}) \ldots \phi_{\pentagon}(\tau_1, \sigma_1) \, |0\rangle$$

Overall short-distance scaling is controlled by OPE

$$\underbrace{\phi_{\pentagon} \ldots \phi_{\pentagon}}_{n-4} \sim m^{-(n-4)\Delta(\frac{5}{4})+\Delta(\frac{n}{4})} \phi_{\varphi}$$

with final excess angle $\varphi = 2\pi \times \dfrac{n-4}{4}$

This leads to the addition

$$\mathcal{W}_n = e^{-\frac{\sqrt{\lambda}}{2\pi} A_n + \frac{\sqrt{\lambda}(n-4)(n-5)}{48n} + o(\sqrt{\lambda})}$$