

Low-Energy Behavior of Gluons and Gravitons from Gauge Invariance

Paolo Di Vecchia

Niels Bohr Institute, Copenhagen and Nordita, Stockholm

Stockholm, 13.08.2014

Workshop on “Supersymmetric Field Theories”

Foreword

- ▶ This talk is based on the work done together with
Zvi Bern, Scott Davies and Josh Nohle



“Low-Energy Behavior of Gluons and Gravitons:
from Gauge Invariance”,
arXiv:1406:6987 [hep-th].



See also the related work by
J. Broedel, M. de Leeuw, J. Plefka and M. Rosso
“Constraining subleading soft gluon and graviton theorems”,
arXiv:1406.6574 [hep-th]

Plan of the talk

- 1 Introduction
- 2 Scattering of a photon and n scalar particles
- 3 Scattering of a graviton and n scalar particles
- 4 Soft limit of n -gluon amplitude
- 5 Soft limit of n -graviton amplitude
- 6 Comments on loop corrections: gauge theory
- 7 Comments on loop corrections: gravity
- 8 What about soft theorems in string theory?
- 9 Soft theorem for dilaton
- 10 Conclusions
- 11 Outlook

Introduction

- ▶ Three kinds of symmetries with different physical consequences.
- ▶ Global unbroken symmetries as isotopic spin or $SU(3)_V$ in three-flavor QCD.
- ▶ Unique vacuum annihilated by the symmetry gener.: $Q_a|0\rangle = 0$
- ▶ Particles are classified according to multiplets of this symmetry and all particles of a multiplet have the same mass.
- ▶ If isotopic spin were an exact symmetry, the proton and the neutron would have the same mass.
- ▶ This would have happened in QCD if the lowest two quarks would have had the same mass.
- ▶ This is not the case because the mass matrix of the quarks breaks explicitly $SU(2)$ and even more $SU(3)$ flavor symmetry.

- ▶ Then, we have the **global spontaneously broken symmetries** as $SU(3)_L \times SU(3)_R$ (broken to $SU(3)_V$) symmetry in QCD for zero mass quarks.
- ▶ Degenerate vacua: $Q_a|0\rangle = |0'\rangle$.
- ▶ Not realized in the spectrum, but it implies the presence of massless particles, called **Goldstone bosons**.
- ▶ They are the **pions** in QCD with 2 flavors.
- ▶ This is one physical consequence of the spontaneous breaking.
- ▶ Another one is the existence of low-energy theorems.
- ▶ **The $\pi\pi$ scattering amplitude is fixed at low energy.**
- ▶ One gets the two scattering lengths:

$$a_0 = \frac{7m_\pi}{32\pi F_\pi^2} ; \quad a_2 = -\frac{m_\pi}{16\pi F_\pi^2}$$

explicit breaking by a mass term.

- ▶ Scattering amplitude is zero for massless pions at low energy because Goldstone bosons interact with **derivative coupling** implying **a shift symmetry**.

- ▶ Finally, we have the local gauge symmetries for massless spin 1 and spin 2 particles.
- ▶ Local gauge invariance is necessary **to reconcile the theory of relativity with quantum mechanics**.
- ▶ It allows a **fully relativistic description**, but **eliminating, at the same time, the presence of negative norm states in the spectrum of physical states**.
- ▶ Although described by A_μ and $G_{\mu\nu}$, both photons and gravitons have only two physical degrees of freedom in $d=4$,
- ▶ and respectively

$$d - 2 \quad \text{and} \quad \frac{(d - 2)(d - 1)}{2} - 1$$

in d space-time dimensions.

- ▶ Another consequence of gauge invariance is charge conservation that, however, follows from the global part of the gauge group.
- ▶ **Yet another physical consequence** of local gauge invariance is the existence of **low-energy theorems for photons and gravitons**:
[F. Low, 1958; S. Weinberg, 1964]

- ▶ Let us consider Compton scattering on spinless particles.
- ▶ The scattering amplitude $M_{\mu\nu}$ is gauge invariant:

$$k_1^\mu M_{\mu\nu} = k_2^\nu M_{\mu\nu} = 0$$

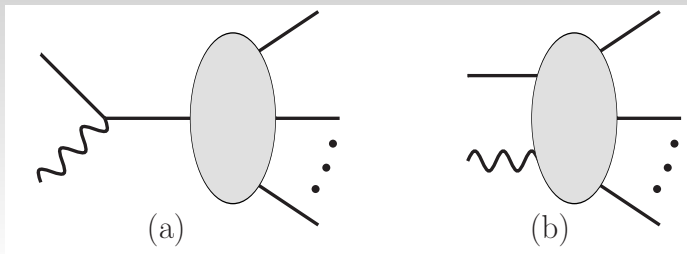
- ▶ The previous conditions determine the scattering amplitude for zero frequency photons and one gets the Thompson cross-section:

$$\sigma_T = \frac{8\pi}{3} \left(\frac{e^2}{mc^2} \right)^2 = \frac{8\pi}{3} r_{cl}$$

where r_{cl} is the classical radius of a point particle of mass m and charge e .

- ▶ The interest on the soft theorems was recently revived by [Cachazo and Strominger, arXiv:1404.1491[hep-th]].
- ▶ They study the behavior of the n -graviton amplitude when the momentum q of one graviton becomes soft ($q \sim 0$).
- ▶ They suggest a universal formula for the subleading term $O(q^0)$.
- ▶ The leading term $O(q^{-1})$ was shown to be universal by Weinberg in the sixties.
- ▶ In a previous paper Strominger et al derived the Weinberg universal behavior from the Ward identities of the BMS transformations.
- ▶ They speculate that also the next to the leading term follows from the BMS transformations.
- ▶ In the following we show that the first three leading terms are a direct consequence of gauge invariance.

One photon and n scalar particles



- ▶ The scattering amplitude $M_\mu(q; k_1 \dots k_n)$, involving one photon and n scalar particles, consists of two pieces:

$$A_n^\mu(q; k_1, \dots, k_n) = \sum_{i=1}^n e_i \frac{k_i^\mu}{k_i \cdot q} T_n(k_1, \dots, k_i + q, \dots, k_n) + N_n^\mu(q; k_1, \dots, k_n).$$

- ▶ and must be gauge invariant for any value of q :

$$q_\mu A_n^\mu = \sum_{i=1}^n e_i T_n(k_1, \dots, k_i + q, \dots, k_n) + q_\mu N_n^\mu(q; k_1, \dots, k_n) \equiv 0$$

- ▶ Expanding around $q = 0$, we have

$$0 = \sum_{i=1}^n e_i \left[T_n(k_1, \dots, k_i, \dots, k_n) + q_\mu \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \dots, k_i, \dots, k_n) \right] \\ + q_\mu N_n^\mu(q = 0; k_1, \dots, k_n) + \mathcal{O}(q^2).$$

- ▶ At leading order, this equation is

$$\sum_{i=1}^n e_i = 0,$$

which is simply a statement of charge conservation
[\[Weinberg, 1964\]](#)

- ▶ At the next order, we have

$$q_\mu N_n^\mu(0; k_1, \dots, k_n) = - \sum_{i=1}^n e_i q_\mu \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \dots, k_n).$$

- ▶ This equation tells us that $N_n^\mu(0; k_1, \dots, k_n)$ is entirely determined in terms of T_n up to potential pieces that are separately gauge invariant.
- ▶ However, it is easy to see that the only expressions local in q that vanish under the gauge-invariance condition $q_\mu E^\mu = 0$ are of the form,

$$E^\mu = (B_1 \cdot q) B_2^\mu - (B_2 \cdot q) B_1^\mu,$$

where B_1^μ and B_2^μ are arbitrary vectors (local in q) constructed with the momenta of the scalar particles.

- ▶ The explicit factor of the soft momentum q in each term means that they are suppressed in the soft limit and do not contribute to $N_n^\mu(0; k_1, \dots, k_n)$.
- ▶ We can therefore remove the q_μ leaving

$$N_n^\mu(0; k_1, \dots, k_n) = - \sum_{i=1}^n e_i \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \dots, k_n),$$

thereby determining $N_n^\mu(0; k_1, \dots, k_n)$ as a function of the amplitude without the photon.

- ▶ Inserting this into the original expression yields

$$A_n^\mu(q; k_1, \dots, k_n) = \sum_{i=1}^n \frac{e_i}{k_i \cdot q} [k_i^\mu - iq_\nu J_i^{\mu\nu}] T_n(k_1, \dots, k_n) + \mathcal{O}(q),$$

where

$$J_i^{\mu\nu} \equiv i \left(k_i^\mu \frac{\partial}{\partial k_{i\nu}} - k_i^\nu \frac{\partial}{\partial k_{i\mu}} \right),$$

is the orbital angular-momentum operator and $T_n(k_1, \dots, k_n)$ is the scattering amplitude involving n scalar particles (and no photon).

- ▶ The amplitude with a soft photon with momentum q is entirely determined in terms of the amplitude without the photon up to $\mathcal{O}(q^0)$.
- ▶ This goes under the name of F. Low's low-energy theorem.

- ▶ Low's theorem is unchanged at loop level for the simple reason that even at loop level, all diagrams containing a pole in the soft momentum are of the form shown, with loops appearing only in the blob and not correcting the external vertex.
- ▶ Can we get **any further information** at higher orders in the soft expansion?
- ▶ One order further in the expansion, we find the extra condition,

$$\frac{1}{2} \sum_{i=1}^n e_i q_\mu q_\nu \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\nu}} T_n(k_1, \dots, k_n) + q_\mu q_\nu \frac{\partial N_n^\mu}{\partial q_\nu}(0; k_1, \dots, k_n) = 0.$$

- ▶ This implies

$$\sum_{i=1}^n e_i \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\nu}} T_n(k_1, \dots, k_n) + \left[\frac{\partial N_n^\mu}{\partial q_\nu} + \frac{\partial N_n^\nu}{\partial q_\mu} \right] (0; k_1, \dots, k_n) = 0,$$

- ▶ Gauge invariance **determines only the symmetric part** of the quantity $\frac{\partial N_n^\nu}{\partial q_\mu}(0; k_1, \dots, k_n)$.
- ▶ The antisymmetric part **is not fixed by gauge invariance**.
- ▶ Indeed, this corresponds exactly to the gauge invariant terms considered above.
- ▶ Then, up to this order, we have

$$\begin{aligned}
 & A_n^\mu(q; k_1, \dots, k_n) \\
 &= \sum_{i=1}^n \frac{e_i}{k_i \cdot q} \left[k_i^\mu - i q_\nu J_i^{\mu\nu} \left(1 + \frac{1}{2} q_\rho \frac{\partial}{\partial k_{i\rho}} \right) \right] T_n(k_1, \dots, k_n) \\
 &+ \frac{1}{2} q_\nu \left[\frac{\partial N_n^\mu}{\partial q_\nu} - \frac{\partial N_n^\nu}{\partial q_\mu} \right] (0; k_1, \dots, k_n) + O(q^2).
 \end{aligned}$$

- ▶ It is straightforward to see that one gets zero by saturating the previous expression with q_μ .

- ▶ In order to write our universal expression in terms of the amplitude, we contract $A_n^\mu(q; k_1, \dots, k_n)$ with the photon polarization $\varepsilon_{q\mu}$.
- ▶ Finally, we have the soft-photon limit of the single-photon, n -scalar amplitude:

$$A_n(q; k_1, \dots, k_n) \rightarrow \left[S^{(0)} + S^{(1)} \right] T_n(k_1, \dots, k_n) + \mathcal{O}(q),$$

where

$$S^{(0)} \equiv \sum_{i=1}^n e_i \frac{k_i \cdot \varepsilon_q}{k_i \cdot q},$$

$$S^{(1)} \equiv -i \sum_{i=1}^n e_i \frac{\varepsilon_{q\mu} q_\nu J_i^{\mu\nu}}{k_i \cdot q},$$

where $J_i^{\mu\nu}$ is the angular momentum.

One graviton and n scalar particles

- ▶ In the case of a graviton scattering on n scalar particles, one can write

$$M_n^{\mu\nu}(q; k_1, \dots, k_n) = \sum_{i=1}^n \frac{k_i^\mu k_i^\nu}{k_i \cdot q} T_n(k_1, \dots, k_i + q, \dots, k_n) + N_n^{\mu\nu}(q; k_1, \dots, k_n),$$

- ▶ $N_n^{\mu\nu}(q; k_1, \dots, k_n)$ is symmetric under the exchange of μ and ν .
- ▶ For simplicity, we have set the gravitational coupling constant to unity.
- ▶ On-shell gauge invariance implies

$$0 = q_\mu M_n^{\mu\nu}(q; k_1, \dots, k_n) = \sum_{i=1}^n k_i^\nu T_n(k_1, \dots, k_i + q, \dots, k_n) + q_\mu N_n^{\mu\nu}(q; k_1, \dots, k_n).$$

- ▶ At leading order in q , we then have

$$\sum_{i=1}^n k_i^\mu = 0,$$

- ▶ It is satisfied due to **momentum conservation**.
- ▶ If there had been different couplings to the different particles, it would have prevented this from vanishing in general.
- ▶ **This shows that gravitons have universal coupling** [Weinberg, 1964]).
- ▶ At first order in q , one gets

$$\sum_{i=1}^n k_i^\nu \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \dots, k_n) + N_n^{\mu\nu}(0; k_1, \dots, k_n) = 0,$$

- ▶ while at second order in q , it gives

$$\sum_{i=1}^n k_i^\nu \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\rho}} T_n(k_1, \dots, k_n) + \left[\frac{\partial N_n^{\mu\nu}}{\partial q_\rho} + \frac{\partial N_n^{\rho\nu}}{\partial q_\mu} \right] (0; k_1, \dots, k_n) = 0.$$

- ▶ As for the photon, this is true up to gauge-invariant contributions to $N_n^{\mu\nu}$.
- ▶ However, the requirement of locality prevents us from writing any expression that is local in q and not sufficiently suppressed in q .
- ▶ Using the previous equations, we write the expression for a soft graviton as

$$\begin{aligned}
 & M_n^{\mu\nu}(q; k_1 \dots k_n) \\
 &= \sum_{i=1}^n \frac{k_i^\nu}{k_i \cdot q} \left[k_i^\mu - i q_\rho J_i^{\mu\rho} \left(1 + \frac{1}{2} q_\sigma \frac{\partial}{\partial k_{i\sigma}} \right) \right] T_n(k_1, \dots, k_n) \\
 &+ \frac{1}{2} q_\rho \left[\frac{\partial N_n^{\mu\nu}}{\partial q_\rho} - \frac{\partial N_n^{\rho\nu}}{\partial q_\mu} \right] (0; k_1, \dots, k_n) + \mathcal{O}(q^2).
 \end{aligned}$$

- ▶ This is essentially the same as for the photon except that there is a second Lorentz index in the graviton case.
- ▶ Unlike the case of the photon, the antisymmetric quantity in the second line of the previous equation can also be determined from the amplitude $T_n(k_1, \dots, k_n)$ without the graviton.

- ▶ Saturating the previous expression with q^μ we get of course zero.
- ▶ If we instead saturate it with q^ν , we get

$$\begin{aligned}
 & q_\nu M_n^{\mu\nu}(q; k_1, \dots, k_n) \\
 &= \frac{1}{2} q_\rho q_\sigma \left\{ \sum_{i=1}^n \left(k_i^\mu \frac{\partial}{\partial k_{i\rho}} - k_i^\rho \frac{\partial}{\partial k_{i\mu}} \right) \frac{\partial}{\partial k_{i\sigma}} T_n(k_1, \dots, k_n) \right. \\
 & \left. + \left[\frac{\partial N_n^{\mu\sigma}}{\partial q_\rho} - \frac{\partial N_n^{\rho\sigma}}{\partial q_\mu} \right] (0; k_1, \dots, k_n) \right\} = 0,
 \end{aligned}$$

- ▶ The vanishing follows from the equation above (implied by gauge invariance), remembering that $N_n^{\mu\nu}$ is a symmetric matrix.
- ▶ Therefore the amplitude is gauge invariant.

- ▶ The same equation allows us to write the relation ,

$$-i \sum_{i=1}^n J_i^{\mu\rho} \frac{\partial}{\partial k_{i\sigma}} T_n(k_1, \dots, k_n) = \left[\frac{\partial N_n^{\rho\sigma}}{\partial q_\mu} - \frac{\partial N_n^{\mu\sigma}}{\partial q_\rho} \right] (0; k_1, \dots, k_n),$$

which fixes the antisymmetric part of the derivative of $N_n^{\mu\nu}$ in terms of the amplitude $T_n(k_1, \dots, k_n)$ without the graviton.

- ▶ Using the previous equation, we can then rewrite the terms of $\mathcal{O}(q)$ as follows:

$$\begin{aligned}
 & M_n^{\mu\nu}(q; k_1, \dots, k_n) \Big|_{\mathcal{O}(q)} \\
 &= -\frac{i}{2} \sum_{i=1}^n \frac{q_\rho q_\sigma}{k_i \cdot q} \left[k_i^\nu J_i^{\mu\rho} \frac{\partial}{\partial k_{i\sigma}} - k_i^\sigma J_i^{\mu\rho} \frac{\partial}{\partial k_{i\nu}} \right] T_n(k_1, \dots, k_n) \\
 &= -\frac{i}{2} \sum_{i=1}^n \frac{q_\rho q_\sigma}{k_i \cdot q} \left[J_i^{\mu\rho} k_i^\nu \frac{\partial}{\partial k_{i\sigma}} - (J_i^{\mu\rho} k_{i\nu}) \frac{\partial}{\partial k_{i\sigma}} \right. \\
 &\quad \left. - J_i^{\mu\rho} k_i^\sigma \frac{\partial}{\partial k_{i\nu}} + (J_i^{\mu\rho} k_i^\sigma) \frac{\partial}{\partial k_{i\nu}} \right] T_n(k_1, \dots, k_n) \\
 &= \frac{1}{2} \sum_{i=1}^n \frac{1}{k_i \cdot q} \left[\left((k_i \cdot q) (\eta^{\mu\nu} q^\sigma - q^\mu \eta^{\nu\sigma}) - k_i^\mu q^\nu q^\sigma \right) \frac{\partial}{\partial k_i^\sigma} \right. \\
 &\quad \left. - q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma} \right] T_n(k_1, \dots, k_n).
 \end{aligned}$$

- ▶ Finally, we wish to write our soft-limit expression in terms of the amplitude, so we contract with the physical polarization tensor of the soft graviton, $\varepsilon_{q\mu\nu}$.
- ▶ We see that the physical-state conditions set to zero the terms that are proportional to $\eta^{\mu\nu}$, q^μ and q^ν .
- ▶ We are then left with the following expression for the graviton soft limit of a single-graviton, n -scalar amplitude:

$$M_n(q; k_1, \dots, k_n) \rightarrow \left[S^{(0)} + S^{(1)} + S^{(2)} \right] T_n(k_1, \dots, k_n) + \mathcal{O}(q^2),$$

- ▶ where

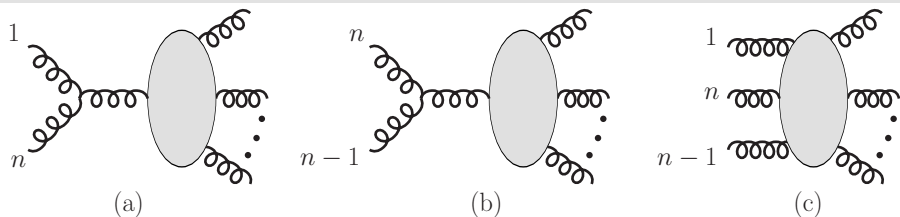
$$S^{(0)} \equiv \sum_{i=1}^n \frac{\varepsilon_{\mu\nu} k_i^\mu k_i^\nu}{k_i \cdot q},$$

$$S^{(1)} \equiv -i \sum_{i=1}^n \frac{\varepsilon_{\mu\nu} k_i^\mu q_\rho J_i^{\nu\rho}}{k_i \cdot q},$$

$$S^{(2)} \equiv -\frac{1}{2} \sum_{i=1}^n \frac{\varepsilon_{\mu\nu} q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{k_i \cdot q}.$$

- ▶ These soft factors follow from gauge invariance and agree with those computed by Cachazo and Strominger.
- ▶ We have also looked at higher-order terms and found that gauge invariance does not fully determine them in terms of derivatives acting on $T_n(k_1, \dots, k_n)$.

Soft limit of n -gluon amplitude



- ▶ We consider a tree-level color-ordered amplitude where gluon n becomes soft with $q \equiv k_n$.
- ▶ Being the amplitude color-ordered, we have to consider only two poles.

- ▶ We get

$$\begin{aligned}
 & A_n^{\mu; \mu_1 \dots \mu_{n-1}}(q; k_1, \dots, k_{n-1}) \\
 &= \frac{\delta_\rho^{\mu_1} k_1^\mu + \eta^{\mu \mu_1} q_\rho - \delta_\rho^\mu q^{\mu_1}}{\sqrt{2}(k_1 \cdot q)} A_{n-1}^{\rho \mu_2 \dots \mu_{n-1}}(k_1 + q, k_2, \dots, k_{n-1}) \\
 &- \frac{\delta_\rho^{\mu_{n-1}} k_{n-1}^\mu + \eta^{\mu \mu_{n-1}} q_\rho - \delta_\rho^\mu q^{\mu_{n-1}}}{\sqrt{2}(k_{n-1} \cdot q)} A_{n-1}^{\mu_1 \dots \mu_{n-2} \rho}(k_1, \dots, k_{n-2}, k_{n-1} + q) \\
 &+ N_n^{\mu; \mu_1 \dots \mu_{n-1}}(q; k_1, \dots, k_{n-1}).
 \end{aligned}$$

- ▶ We have dropped terms from the three-gluon vertex that vanish when saturated with the external-gluon polarization vectors in addition to using the current-conservation conditions,

$$\begin{aligned}
 & (k_1 + q)_\rho A_{n-1}^{\rho \mu_2 \dots \mu_{n-1}}(k_1 + q, k_2, \dots, k_{n-1}) = 0, \\
 & (k_{n-1} + q)_\rho A_{n-1}^{\mu_1 \dots \mu_{n-2} \rho}(k_1, \dots, k_{n-2}, k_{n-1} + q) = 0,
 \end{aligned}$$

which are valid once we contract with the polarization vectors carrying the μ_j indices.

- ▶ By introducing the spin-one angular-momentum operator,

$$(\Sigma_i^{\mu\sigma})^{\mu_i\rho} \equiv i(\eta^{\mu\mu_i}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\mu_i\sigma}),$$

we can write the total amplitude as

$$\begin{aligned} & A_n^{\mu;\mu_1\cdots\mu_{n-1}}(q; k_1, \dots, k_{n-1}) \\ &= \frac{\delta_\rho^{\mu_1} k_1^\mu - iq_\sigma (\Sigma_1^{\mu\sigma})^{\mu_1\rho}}{\sqrt{2}(k_1 \cdot q)} A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1 + q, k_2, \dots, k_{n-1}) \\ &- \frac{\delta_\rho^{\mu_{n-1}} k_{n-1}^\mu - iq_\sigma (\Sigma_{n-1}^{\mu\sigma})^{\mu_{n-1}\rho}}{\sqrt{2}(k_{n-1} \cdot q)} A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1, \dots, k_{n-2}, k_{n-1} + q) \\ &+ N_n^{\mu;\mu_1\cdots\mu_{n-1}}(q; k_1, \dots, k_{n-1}). \end{aligned}$$

- ▶ Notice that the spin-one terms independently vanish when contracted with q_μ .

- ▶ On-shell gauge invariance requires

$$\begin{aligned}
 0 &= q_\mu A_n^{\mu; \mu_1 \dots \mu_{n-1}}(q; k_1, \dots, k_{n-1}) \\
 &= \frac{1}{\sqrt{2}} A_{n-1}^{\mu_1 \mu_2 \dots \mu_{n-1}}(k_1 + q, k_2, \dots, k_{n-1}) \\
 &\quad - \frac{1}{\sqrt{2}} A_{n-1}^{\mu_1 \dots \mu_{n-2} \mu_{n-1}}(k_1, \dots, k_{n-2}, k_{n-1} + q) \\
 &\quad + q_\mu N_n^{\mu; \mu_1 \dots \mu_{n-1}}(q; k_1, \dots, k_{n-1}).
 \end{aligned}$$

- ▶ For $q = 0$, this is automatically satisfied.
- ▶ At the next order in q , we obtain

$$\begin{aligned}
 &-\frac{1}{\sqrt{2}} \left[\frac{\partial}{\partial k_{1\mu}} - \frac{\partial}{\partial k_{n-1\mu}} \right] A_{n-1}^{\mu_1 \dots \mu_{n-1}}(k_1, k_2 \dots k_{n-1}) \\
 &= N_n^{\mu; \mu_1 \dots \mu_{n-1}}(0; k_1, \dots, k_{n-1}).
 \end{aligned}$$

- ▶ Similar to the photon case, we ignore local gauge-invariant terms in $N_n^{\mu; \mu_1 \dots \mu_{n-1}}$ because they are necessarily of a higher order in q .
- ▶ Thus, $N_n^{\mu; \mu_1 \dots \mu_{n-1}}(0; k_1, \dots, k_{n-1})$ is determined in terms of **the amplitude without the soft gluon**.

- ▶ With this, the total expression becomes

$$\begin{aligned}
 & A_n^{\mu; \mu_1 \dots \mu_{n-1}}(q; k_1 \dots k_{n-1}) \\
 &= \left(\frac{k_1^\mu}{\sqrt{2}(k_1 \cdot q)} - \frac{k_{n-1}^\mu}{\sqrt{2}(k_{n-1} \cdot q)} \right) A_{n-1}^{\mu_1 \dots \mu_{n-1}}(k_1, \dots, k_{n-1}) \\
 &\quad - i \frac{q_\sigma (J_1^{\mu\sigma})^{\mu_1 \rho}}{\sqrt{2}(k_1 \cdot q)} A_{n-1}^{\rho \mu_2 \dots \mu_{n-1}}(k_1, \dots, k_{n-1}) \\
 &\quad + i \frac{q_\sigma (J_{n-1}^{\mu\sigma})^{\mu_{n-1} \rho}}{\sqrt{2}(k_{n-1} \cdot q)} A_{n-1}^{\mu_1 \dots \mu_{n-2} \rho}(k_1, \dots, k_{n-1}) + \mathcal{O}(q),
 \end{aligned}$$

where

$$(J_i^{\mu\sigma})^{\mu_i \rho} \equiv L_i^{\mu\sigma} \eta^{\mu_i \rho} + (\Sigma_i^{\mu\sigma})^{\mu_i \rho},$$

with

$$L_i^{\mu\sigma} \equiv i \left(k_i^\mu \frac{\partial}{\partial k_{i\sigma}} - k_i^\sigma \frac{\partial}{\partial k_{i\mu}} \right) ; \quad (\Sigma_i^{\mu\sigma})^{\mu_i \rho} \equiv i (\eta^{\mu\mu_i} \eta^{\rho\sigma} - \eta^{\mu\rho} \eta^{\mu_i\sigma})$$

- ▶ In order to write the final result in terms of full amplitudes, we contract with external polarization vectors.
- ▶ We must pass polarization vectors $\varepsilon_{1\mu_1}$ and $\varepsilon_{n-1\mu_{n-1}}$ through the spin-one angular-momentum operator such that they will contract with the ρ index of, respectively, $A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1, \dots, k_{n-1})$ and $A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1, \dots, k_{n-1})$.
- ▶ It is convenient write the spin angular-momentum operator as

$$\varepsilon_{i\mu_i}(\Sigma_i^{\mu\sigma})^{\mu_i}{}_{\rho} A^{\rho} = i \left(\varepsilon_i^{\mu} \frac{\partial}{\partial \varepsilon_{i\sigma}} - \varepsilon_i^{\sigma} \frac{\partial}{\partial \varepsilon_{i\mu}} \right) \varepsilon_{i\rho} A^{\rho} .$$

- ▶ We may therefore write

$$A_n(q; k_1, \dots, k_{n-1}) \rightarrow \left[S_n^{(0)} + S_n^{(1)} \right] A_{n-1}(k_1, \dots, k_{n-1}) + \mathcal{O}(q),$$

where

$$S_n^{(0)} \equiv \frac{k_1 \cdot \varepsilon_n}{\sqrt{2}(k_1 \cdot q)} - \frac{k_{n-1} \cdot \varepsilon_n}{\sqrt{2}(k_{n-1} \cdot q)},$$

$$S_n^{(1)} \equiv -i\varepsilon_{n\mu} q_\sigma \left(\frac{J_1^{\mu\sigma}}{\sqrt{2}(k_1 \cdot q)} - \frac{J_{n-1}^{\mu\sigma}}{\sqrt{2}(k_{n-1} \cdot q)} \right).$$

- ▶ Here

$$J_i^{\mu\sigma} \equiv L_i^{\mu\sigma} + \Sigma_i^{\mu\sigma},$$

where

$$L_i^{\mu\nu} \equiv i \left(k_i^\mu \frac{\partial}{\partial k_{i\nu}} - k_i^\nu \frac{\partial}{\partial k_{i\mu}} \right), \quad \Sigma_i^{\mu\sigma} \equiv i \left(\varepsilon_i^\mu \frac{\partial}{\partial \varepsilon_{i\sigma}} - \varepsilon_i^\sigma \frac{\partial}{\partial \varepsilon_{i\mu}} \right).$$

Soft limit of n -graviton amplitude

- ▶ As before the amplitude is the sum of two pieces:

$$\begin{aligned} & M_n^{\mu\nu; \mu_1\nu_1 \dots \mu_{n-1}\nu_{n-1}}(q; k_1, \dots, k_{n-1}) \\ &= \sum_{i=1}^{n-1} \frac{1}{k_i \cdot q} \left[k_i^\mu \eta^{\mu_i\alpha} - i q_\rho (\Sigma_i^{\mu\rho})^{\mu_i\alpha} \right] \left[k_i^\nu \eta^{\nu_i\beta} - i q_\sigma (\Sigma_i^{\mu\sigma})^{\nu_i\beta} \right] \\ &\quad \times M_{n-1}^{\mu_1\nu_1 \dots \mu_{n-1}\nu_{n-1}}(k_1, \dots, k_i + q, \dots, k_{n-1}) \\ &\quad + N_n^{\mu\nu; \mu_1\nu_1 \dots \mu_{n-1}\nu_{n-1}}(q; k_1, \dots, k_{n-1}), \end{aligned}$$

where

$$(\Sigma_i^{\mu\rho})^{\mu_i\alpha} \equiv i (\eta^{\mu\mu_i} \eta^{\alpha\rho} - \eta^{\mu\alpha} \eta^{\mu_i\rho}).$$

- ▶ On-shell gauge invariance implies

$$\begin{aligned}
 0 &= q_\mu M_n^{\mu\nu; \mu_1\nu_1 \dots \mu_{n-1}\nu_{n-1}}(q; k_1, \dots, k_{n-1}) \\
 &= \sum_{i=1}^{n-1} \left[k_i^\nu \eta^{\nu_i\beta} - i q_\rho (\Sigma_i^{\nu\rho})^{\nu_i\beta} \right] M_{n-1}^{\mu_1\nu_1 \dots \mu_i \dots \mu_{n-1}\nu_{n-1}}(k_1, \dots, k_i + q, \dots, k_{n-1}) \\
 &\quad + q_\mu N_n^{\mu\nu; \mu_1\nu_1 \dots \mu_{n-1}\nu_{n-1}}(q; k_1, \dots, k_{n-1}).
 \end{aligned}$$

- ▶ Proceeding as before we end up getting

$$\begin{aligned}
 &M_n^{\mu\nu; \mu_1\nu_1 \dots \mu_{n-1}\nu_{n-1}}(q; k_1, \dots, k_{n-1}) \\
 &= \sum_{i=1}^{n-1} \frac{1}{k_i \cdot q} \left\{ k_i^\mu k_i^\nu \eta^{\mu_i\alpha} \eta^{\nu_i\beta} \right. \\
 &\quad - \frac{i}{2} q_\rho \left[k_i^\mu \eta^{\mu_i\alpha} \left[L_i^{\nu\rho} \eta^{\nu_i\beta} + 2(\Sigma_i^{\nu\rho})^{\nu_i\beta} \right] + k_i^\nu \eta^{\nu_i\beta} \left[L_i^{\mu\rho} \eta^{\mu_i\alpha} + 2(\Sigma_i^{\mu\rho})^{\mu_i\alpha} \right] \right. \\
 &\quad \left. - \frac{1}{2} q_\rho q_\sigma \left[\left[L_i^{\mu\rho} \eta^{\mu_i\alpha} + 2(\Sigma_i^{\mu\rho})^{\mu_i\alpha} \right] \left[L_i^{\nu\sigma} \eta^{\nu_i\beta} + 2(\Sigma_i^{\nu\sigma})^{\nu_i\beta} \right] - 2(\Sigma_i^{\mu\rho})^{\mu_i\alpha} (\Sigma_i^{\nu\sigma})^{\nu_i\beta} \right] \right\} \\
 &\quad \times M_{n-1}^{\mu_1\nu_1 \dots \alpha\beta \dots \mu_{n-1}\nu_{n-1}}(k_1, \dots, k_i, \dots, k_{n-1}) + \mathcal{O}(q^2).
 \end{aligned}$$

- ▶ In order to write our expression in terms of amplitudes, we saturate with graviton polarization tensors using $\varepsilon_{\mu\nu} \rightarrow \varepsilon_\mu \varepsilon_\nu$ where ε_μ are spin-one polarization vectors.
- ▶ As we did for the case with gluons, we must pass the polarization vectors through the spin-one operators.



$$M_n(q; k_1, \dots, k_{n-1}) = \left[S_n^{(0)} + S_n^{(1)} + S_n^{(2)} \right] M_{n-1}(k_1, \dots, k_{n-1}) + \mathcal{O}(q^2)$$

where

$$S_n^{(0)} \equiv \sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} k_i^\mu k_i^\nu}{k_i \cdot q},$$

$$S_n^{(1)} \equiv -i \sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} k_i^\mu q_\rho J_i^{\nu\rho}}{k_i \cdot q},$$

$$S_n^{(2)} \equiv -\frac{1}{2} \sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{k_i \cdot q}.$$

► Here

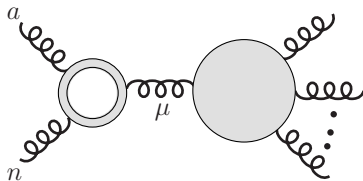
$$J_i^{\mu\sigma} \equiv L_i^{\mu\sigma} + \Sigma_i^{\mu\sigma} ,$$

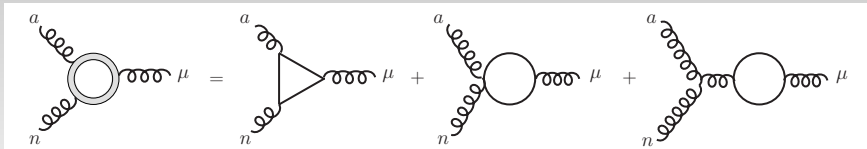
with

$$L_i^{\mu\sigma} \equiv i \left(k_i^\mu \frac{\partial}{\partial k_{i\sigma}} - k_i^\sigma \frac{\partial}{\partial k_{i\mu}} \right) , \quad \Sigma_i^{\mu\sigma} \equiv i \left(\varepsilon_i^\mu \frac{\partial}{\partial \varepsilon_{i\sigma}} - \varepsilon_i^\sigma \frac{\partial}{\partial \varepsilon_{i\mu}} \right) .$$

Comments on loop corrections: gauge theory

- ▶ At one-loop the amplitude will have in general IR and UV divergences.
- ▶ We are not giving here a complete study of them.
- ▶ The one-loop contributions have been classified into the factorizing ones and the non-factorizing ones.
- ▶ We will concentrate here to the factorizing ones.
- ▶ They modify the vertex present in the pole term.
- ▶ For the gauge theory they are of the type shown in the figure.





- ▶ They have been computed in QCD and are given by:

$$D^{\mu, \text{fact}} = \frac{i}{\sqrt{2}} \frac{1}{3} \frac{1}{(4\pi)^2} \left(1 - \frac{n_f}{N_c} + \frac{n_s}{N_c} \right) (q - k_a)^\mu \left[(\epsilon_n \cdot \epsilon_a) - \frac{(q \cdot \epsilon_a)(k_a \cdot \epsilon_n)}{(k_a \cdot q)} \right]$$

[Z. Bern, V. Del Duca, C.R. Schmidt, 1998]

[Z. Bern, V. Del Duca, W.B. Kilgore, C.R. Schmidt, 1999]

- ▶ It is both IR and UV finite and the limit $\epsilon \rightarrow 0$ has been taken.
- ▶ It is non-local because of the pole in (qk_a) .
- ▶ It is gauge invariant under the substitution $\epsilon_q \rightarrow q$.
- ▶ It does not contribute to the leading soft behavior.

- ▶ Attaching to it the rest of the amplitude

$$D_{\mu}^{\text{fact}} \frac{-i}{2q \cdot k_a} \mathcal{J}^{\mu},$$

- ▶ \mathcal{J}^{μ} is a conserved current:

$$(q + k_a)_{\mu} \mathcal{J}^{\mu} = 0,$$

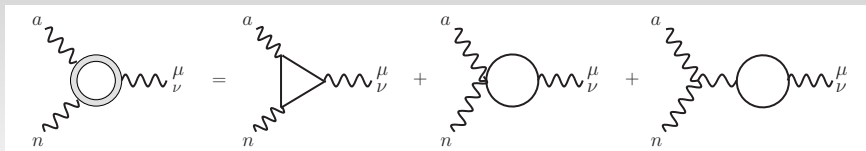
assuming that all the remaining legs are contracted with on-shell polarizations.

- ▶ We can trade k_a with q and we get immediately:

$$D_{\mu}^{\text{fact}} \frac{-i}{2q \cdot k_a} \mathcal{J}^{\mu} = \mathcal{O}(q^0),$$

- ▶ No leading $\mathcal{O}(\frac{1}{q})$ correction from the factorizing contribution to the one-loop soft functions.

Comments on loop corrections: gravity



- ▶ A similar calculation can be done for the gravity case.
- ▶ We consider only the case in which **scalar fields** circulate in the loop.
- ▶ The result of this calculation is:

$$\mathcal{D}^{\mu\nu, \text{fact}, s} = \frac{i}{(4\pi)^2} \left(\frac{\kappa}{2}\right)^3 \frac{1}{30} \left[(\varepsilon_n \cdot \varepsilon_a) - \frac{(\mathbf{q} \cdot \varepsilon_a)(\mathbf{k}_a \cdot \varepsilon_n)}{(\mathbf{q} \cdot \mathbf{k}_a)} \right] \\ \times \left((\mathbf{q} \cdot \varepsilon_a)(\mathbf{k}_a \cdot \varepsilon_n) - (\varepsilon_n \cdot \varepsilon_a)(\mathbf{q} \cdot \mathbf{k}_a) \right) k_a^\mu k_a^\nu + \mathcal{O}(q^2),$$

- ▶ As in the gauge-theory case, the diagrams $\mathcal{D}^{\mu\nu,\text{fact},s}$ contract into a conserved current:

$$(k_a + q)^\mu \mathcal{J}_{\mu\nu} = f(k_i, \epsilon_i)(k_a + q)_\nu, \quad (k_a + q)^\nu \mathcal{J}_{\mu\nu} = f(k_i, \epsilon_i)(k_a + q)_\mu.$$

- ▶ This means

$$\begin{aligned} k_a^\mu k_a^\nu \mathcal{J}_{\mu\nu} &= (k_a + q)^\mu (k_a + q)^\nu \mathcal{J}_{\mu\nu} + \mathcal{O}(q) \\ &= f(k_i, \epsilon_i)(k_a + q)^2 + \mathcal{O}(q) = 2f(k_i, \epsilon_i)q \cdot k_a + \mathcal{O}(q) = \mathcal{O}(q) \end{aligned}$$

- ▶ We therefore have

$$\mathcal{D}^{\mu\nu,\text{fact},s} \frac{i}{2q \cdot k_a} \mathcal{J}_{\mu\nu} = \mathcal{O}(q).$$

- ▶ No modification of the two first leading terms.
- ▶ As in QCD, we expect that the contribution of other particles circulating in the loop will not modify this result.

What about soft theorems in string theory?

- ▶ In superstring the soft theorems have been investigated by B.U.W. Schwab, arXiv:1406.4172 and M. Bianchi, Song He, Yu-tin Huang and Congkao Wen, arXiv:1406.5155.
- ▶ Here we give just few examples in the bosonic string.
- ▶ One gluon and three tachyons:

$$A_{\mu}(p_1, p_2, q, p_3) \sim \sqrt{2\alpha'} \frac{\Gamma(1 + 2\alpha' p_3 q) \Gamma(1 + 2\alpha' p_2 q)}{\Gamma(1 + 2\alpha' (p_2 + p_3) q)} \\ \times \left(\frac{p_{2\mu}}{2\alpha' p_3 q} - \frac{p_{3\mu}}{2\alpha' p_2 q} \right)$$

- ▶ One graviton(dilaton) and three tachyons ($p_1 + p_2 + p_3 = -q$):

$$A_{\mu\nu}(p_1, p_2, p_3, q) \sim \left(\frac{p_{1\mu} p_{1\nu}}{p_1 q} + \frac{p_{2\mu} p_{2\nu}}{p_2 q} + \frac{p_{3\mu} p_{3\nu}}{p_3 q} \right) \\ \times \frac{\Gamma(1 + \frac{\alpha'}{2} p_1 q) \Gamma(1 + \frac{\alpha'}{2} p_2 q) \Gamma(1 + \frac{\alpha'}{2} p_3 q)}{\Gamma(1 - \frac{\alpha'}{2} p_1 q) \Gamma(1 - \frac{\alpha'}{2} p_2 q) \Gamma(1 - \frac{\alpha'}{2} p_3 q)}$$

► One gluon and 4 tachyons

With [R. Marotta]

$$\begin{aligned} A_\mu(p_1, p_2, p_3, q, p_4) &\sim \int_0^1 dz_3 (1 - z_3)^{2\alpha' p_2 p_3} z_3^{2\alpha' p_3 p_4} \\ &\times \int_0^{z_3} dz_4 (1 - z_4)^{2\alpha' p_2 q} (z_3 - z_4)^{2\alpha' p_3 q} z_4^{2\alpha' p_4 q} \\ &\times \left[\frac{p_{2\mu}}{1 - z_4} + \frac{p_{3\mu}}{z_3 - z_4} - \frac{p_{4\mu}}{z_4} \right] \end{aligned}$$

- It is gauge invariant: $q^\mu A_\mu = 0$.
- The last two lines are equal to ($z_4 = z_3 t$)

$$\begin{aligned} &z_3^{2\alpha'(p_3+p_4)q} \int_0^1 dt (1 - t)^{2\alpha' p_3 q} t^{2\alpha' p_4 q} (1 - z_3 t)^{2\alpha' p_2 q} \\ &\times \left[\frac{z_3 p_{2\mu}}{1 - z_3 t} + \frac{p_{3\mu}}{1 - t} - \frac{p_{4\mu}}{t} \right] \end{aligned}$$

- ▶ They are equal to

$$\begin{aligned}
 & z_3^{2\alpha'(p_3+p_4)q} \left[\frac{\Gamma(1+2\alpha'p_4q)\Gamma(2\alpha'p_3q)}{\Gamma(2+2\alpha'(p_3+p_4)q)} z_3 \right. \\
 & \times {}_2F_1(1-2\alpha'p_2q, 1+2\alpha'p_4q; 2+2\alpha'(p_3+p_4)q; z_3) \\
 & + \frac{\Gamma(2\alpha'p_4q+1)\Gamma(1+2\alpha'p_3q)}{\Gamma(1+2\alpha'(p_3+p_4)q)} \left(-\frac{p_{4\mu}}{2\alpha'p_4q} \right. \\
 & \times {}_2F_1(-2\alpha'p_2q, 2\alpha'p_4q; 1+2\alpha'(p_3+p_4)q; z_3) \\
 & + \frac{p_{3\mu}}{2\alpha'p_3q} (1-z_3)^{2\alpha'p_2q} \\
 & \left. \times {}_2F_1(-2\alpha'p_2q, 2\alpha'p_3q; 2\alpha'(p_3+p_4)q+1; -\frac{z_3}{1-z_3}) \right)
 \end{aligned}$$

- ▶ In the soft limit up to the order q^0 we can forget the ratio of Γ -functions, we can approximate the last two ${}_2F_1$ with 1 and the first one with: ${}_2F_1(1, 1; 2; z_3)z_3 = -\log(1-z_3)$.

- ▶ In this way we get:

$$\int_0^1 dz_3 (1 - z_3)^{2\alpha' p_2 p_3} z_3^{2\alpha' p_3 p_4} \left[-\log(1 - z_3) p_{2\mu} + z_3^{2\alpha'(p_3+p_4)q} \left(\frac{p_{3\mu}}{2\alpha' p_3 q} (1 - z_3)^{2\alpha' p_2 q} - \frac{p_{4\mu}}{2\alpha' p_4 q} \right) \right]$$

- ▶ It can be written as follows:

$$\frac{1}{2\alpha'} \left[\frac{p_{3\mu}}{p_3 q} - \frac{p_{4\mu}}{p_4 q} + \frac{q^\rho J_{\mu\rho}^{(3)}}{p_3 q} - \frac{q^\rho J_{\mu\rho}^{(4)}}{p_4 q} \right] \times \int_0^1 dz_3 (1 - z_3)^{\alpha'(p_2+p_3)^2-2} z_3^{\alpha'(p_3+p_4)^2-2}$$

- ▶ The last integral is the amplitude for four tachyons and

$$J_{\mu\rho}^{(3,4)} = p_{(3,4)\mu} \frac{\partial}{\partial p_{(3,4)\rho}} - p_{(3,4)\rho} \frac{\partial}{\partial p_{(3,4)\mu}}$$

Soft theorem for dilaton

- ▶ The soft dilaton behavior in string theory goes back to the 70s [Ademollo et al , 1975] and [Shapiro, 1975].
- ▶ The loop amplitudes in the bosonic string are divergent because of the dilaton tadpole, corresponding to a zero momentum dilaton disappearing in the vacuum.
- ▶ In the previous papers it was proposed how to get rid of these divergence renormalizing the slope of the Regge trajectory and the string coupling constant.
- ▶ The soft theorem for a dilaton can, in principle, be computed starting from the expression that we obtained for the graviton except that now **we cannot neglect** terms proportional to $\eta^{\mu\nu}$ as we did in the case of a graviton.

- ▶ For the graviton we got:

$$\begin{aligned}
 & M_n^{\mu\nu}(q; k_1 \dots k_n) \\
 &= \sum_{i=1}^n \frac{k_i^\nu}{k_i \cdot q} [k_i^\mu - iq_\rho J_i^{\mu\rho}] T_n(k_1, \dots, k_n) \\
 &+ \frac{1}{2} \sum_{i=1}^n \frac{1}{k_i \cdot q} \left[\left((k_i \cdot q)(\eta^{\mu\nu} q^\sigma - q^\mu \eta^{\nu\sigma}) - k_i^\mu q^\nu q^\sigma \right) \frac{\partial}{\partial k_i^\sigma} \right. \\
 &\quad \left. - q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma} \right] T_n(k_1, \dots, k_n).
 \end{aligned}$$

and we have neglected the terms in the third line because the graviton polarization satisfies the identities:

$$q^\mu \epsilon_{\mu\nu} = q^\nu \epsilon_{\mu\nu} = \eta^{\mu\nu} \epsilon_{\mu\nu} = 0$$

- ▶ In other words, gauge invariance imposes:

$$q_\mu M_n^{\mu\nu} = f(k_i) q^\nu \implies q_\mu (M_n^{\mu\nu} - f(k_i) \eta^{\mu\nu}) = 0$$

- ▶ The extra term with $\eta^{\mu\nu}$ is irrelevant for the graviton, but not for the dilaton.

- ▶ Let us forget for a moment this problem and, in the case of the dilaton, let us saturate $M_n^{\mu\nu}$ with the dilaton projector:

$$(\eta_{\mu\nu} - q_\mu \bar{q}_\nu - q_\nu \bar{q}_\mu) M_n^{\mu\nu} \quad ; \quad q^2 = \bar{q}^2 = 0 \quad ; \quad q\bar{q} = 1$$

- ▶ We get

$$\begin{aligned} S^{(0)} + S^{(1)} + S^{(2)} &= - \sum_{i=1}^n \frac{m_i^2 \left(1 + q^\rho \frac{\partial}{\partial k_{i\rho}} + \frac{1}{2} q_\rho q^\sigma \frac{\partial^2}{\partial k_{i\rho} \partial k_{i\sigma}} \right)}{k_i q} \\ &- \sum_{i=1}^n k_{i\mu} \frac{d}{dk_{i\mu}} + 2 \\ &- \sum_{i=1}^n k_{i\mu} q_\sigma \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\sigma}} + \frac{1}{2} (k_i q) \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\mu}} \end{aligned}$$

- ▶ We have checked the previous expression **up to order q^0** computing the amplitude involving a dilaton and n closed tachyons.

- ▶ It is given by

$$M_n^{\mu\nu} \sim \int \frac{\prod_{i=1}^n d^2 z_i}{dV_{abc}} \prod_{i<j} |z_i - z_j|^{\alpha' k_i k_j} \int d^2 z \prod_{i=1}^n |z - z_i|^{\alpha' k_i q}$$

$$\times \alpha' \sum_{i=1}^n \frac{k_i^\mu}{z - z_i} \sum_{i=1}^n \frac{k_i^\nu}{\bar{z} - \bar{z}_i}$$

- ▶ In the soft limit ($q \rightarrow 0$) we can put directly $q = 0$ in the non-diagonal terms, while we have to be more careful with the diagonal terms that provide the terms of order q^{-1} .
- ▶ We have checked that the amplitude for both the graviton and the dilaton satisfies the general low energy theorems derived above up to the order q^0 .
- ▶ No extra term proportional to $\eta^{\mu\nu}$ is needed to reproduce the previous amplitude and also the amplitude involving massless closed string states.
- ▶ For amplitudes involving N massless open strings, one needs to add a term $\eta^{\mu\nu} \frac{N-2}{4}$.

Conclusions

- ▶ We have extended Low's proof of the universality of sub-leading behavior of photons to non-abelian gauge theory and to gravity.
- ▶ On-shell gauge invariance can be used to fully determine the first sub-leading soft-gluon behavior at tree level.
- ▶ In gravity the first two subleading terms in the soft expansion can also be fully determined from on-shell gauge invariance.
- ▶ We have considered the factorizing contribution to both gauge theories and gravity.
- ▶ In non-abelian gauge theories the leading term is not affected by it, but the next to the leading is affected.
- ▶ Similarly in gravity the first two leading terms are not affected by the factorizing contribution, but the next term is affected.
- ▶ For the dilaton gauge invariant terms may appear at the order q^0 and therefore they cannot be obtained using gauge invariance as for the graviton.

Outlook

- ▶ It would be nice to have under control, together with the factorizing contribution, also the ones involving both the IR and the UV divergences at one loop.
- ▶ In gauge theory they are well established, but in gravity some more work has to be done.
- ▶ It would be very nice to extract everything from string theory in the limit of $\alpha' \rightarrow 0$.