Large N phase transitions in Supersymmetric gauge theories with massive matter

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Based on:

J.R and K. Zarembo, arxiv:1309.1004, 1312.1214, 1302.6968

A. Barranco and J.R, arxiv:1401.3672

J.R., G. Silva and M. Tierz, arxiv:1407.4794

Supersymmetric Field Theories – Nordita, Stockholm 13-16 August 2014

Plan

Study of exact results in gauge theories both at large *N* and finite *N*

"Exact" = all order in coupling, including both perturbative and non-perturbative contributions

1. Four-dimensional $\mathcal{N}$= 2 SQCD and $\mathcal{N} = 2^*$ SYM at large *N*
2. $\mathcal{N}$= 2 *U*(*N*) CS-matter theory at large *N*
3. $\mathcal{N}$= 2 *U*(*N*) CS-matter theory with finite *N*

Localization

I . Exact partition function for $\mathcal{N}=2$ supersymmetric YM theories on $\mathbf{S^4}$, with arbitrary matter content . [Pestun, 0712.2824]

II. Exact partition function for $\mathcal{N}=2$ supersymmetric CS –matter theories on $\mathbf{S^3}$ [Kapustin, Willett and Yaakov, 0909.4559]

$\mathcal{N}=2$ SYM theories in four dimensions:

Partition function localizes to a matrix integral over Coulomb moduli $\langle \Phi \rangle = diag(a_1, \ldots, a_N)$

VEV of scalar of vector multiplet

$$Z(g) = \int d^{N-1}a \prod_{i<j} (a_i - a_j)^2 \, \mathrm{e}^{-S_{cl}(a)} \, z_{1-loop}(a) \quad \left| z_{inst}(a; g^2) \right|^2$$

$$S_{cl} = \frac{1}{4g^2} \int_{S^4} d^4x \sqrt{g} \, \mathrm{R} \, \mathrm{tr}\Phi^2 = \frac{8\pi^2}{g^2} \sum_i a_i^2$$

$Z = Z(g)$

Exact g dependence

This a complicated integral which we still need to compute in order to understand the underlying physics.

$\mathcal{N}$= 4 Super Yang-Mills theory on S^4

- Instantons do not contribute.

- 1-loop corrections cancel

Gaussian matrix model: $$Z = \int d^{N-1}a \prod_{i<j} (a_i - a_j)^2 \, \mathrm{e}^{-\frac{8\pi^2}{g^2}\sum_i a_i^2}$$

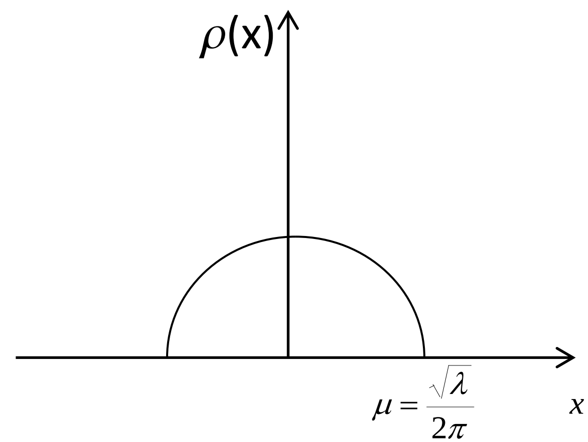
At large *N* the integral is dominated by a saddle-point.

Introducing the eigenvalue density:

$$\rho(x) = \frac{1}{N}\sum_{i=1}^{N} \delta(x - a_i)$$

the saddle-point equation reads

$$\fint_{-\mu}^{\mu} dy\, \rho(y)\, \frac{1}{x-y} = \frac{8\pi^2}{\lambda} x \quad \Rightarrow \quad \rho(x) = \frac{8\pi}{\lambda}\sqrt{\frac{\lambda}{4\pi^2} - x^2}$$



Eigenvalues are distributed in a semicircle (Wigner's law)

$\mathcal{N}$= 4 SYM : Wilson loop

[Erickson, Semenoff and Zarembo, 0003055] [Drukker, Gross, 0010274]

$$W(C) = \int_{-\mu}^{\mu} dx\, \rho(x)\, e^{2\pi x} = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \xrightarrow[\lambda \to \infty]{} \sqrt{\frac{2}{\pi}}\, \lambda^{-3/4}\, e^{\sqrt{\lambda}}$$

Reproduced by holography

Expanding the Bessel function at small λ one gets the perturbative series that must reproduce the Feynman diagram calculations

$$W(C) = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \xrightarrow[\lambda << 1]{} 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \ldots$$

There is a smooth dependence with λ all the way from 0 to infinity.
No phase transition between the perturbative $\lambda << 1$ regime and the strong coupling $\lambda >> 1$ regime (described by AdS/CFT duality).

What about $\mathcal{N}$= 2 SYM theories?

Is the interpolation between weak and strong coupling still smooth?
Could there be a quantum phase transition at some value of λ ?

1. $\mathcal{N} = 2$ SQCD with $2N_f$ massive hypermultiplets

J.R and K. Zarembo, arxiv:1309.1004

We assume $N_f < N$, in which case the theory is **asymptotically free**.
The partition function computed by localization is given by [Pestun, 0712.2824]

$$Z = \int d^{N-1}a \quad \frac{\prod_{i<j} (a_i - a_j)^2 H^2(a_i - a_j)}{\prod_i H(a_i + M)^{N_f} H(a_i - M)^{N_f}} \ \mathrm{e}^{-\frac{8\pi^2 N}{\lambda}\sum_i a_i^2} \quad \left| z_{inst}(a; g^2) \right|^2$$

Dynamically generated scale

$$\Lambda R = \mathrm{e}^{-\frac{4\pi^2}{\lambda(1-\zeta)}} \quad , \qquad \zeta \equiv \frac{N_f}{N}$$

The one-loop factor is expressed in terms of a single function $H(x) \equiv \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right)^n \mathrm{e}^{-\frac{x^2}{n}}$

At large N, instantons do not contribute.
The integral is determined by the saddle-point at the solution of the integral equation

$$\fint_{-\mu}^{\mu} dy\ \rho(y) \left(\frac{1}{x-y} - K(x-y) \right) = -4(1-\zeta)(\log \Lambda R)\ x - \zeta\ K(x+M) - \zeta\ K(x-M)$$

μ: width of eigenvalue distribution, $-\mu < x < \mu$

Here $$K(x) \equiv -\frac{H'(x)}{H(x)}$$

In the decompactification limit, $K(xR) \xrightarrow[R\to\infty]{} 2xR\ln|xR|$

The saddle-point equation simplifies.

$$\fint_{-\mu}^{\mu} dy\ \frac{\rho(y)}{x-y} = \frac{\zeta}{x+M} + \frac{\zeta}{x-M}$$

The RHS has poles at $x = \pm M$ which may or may not lie within the eigenvalue distribution.
They represent points in moduli space where the hypermultiplet are massless.

Massless hypermultiplets:
The masses of the hypermultiplets are not just *M,* but get a contribution from the vacuum condensate. They are equal to $|M + a|$ and $|M - a|$.

The solution to the saddle-point equations changes discontinuously when the pole at *x = M crosses the endpoint at x = μ.*

The model thus has two phases:

1. The weak-coupling phase with $\mu <$ M, in which all hypermultiplets are heavy.

2. The strong-coupling phase at $\mu >$ M, where massless hypermultiplets contribute to the saddle-point.

1. Strong coupling phase $\mu > M$.

The poles sit within the eigenvalue distribution.

$$\rho(x) = \frac{1-\zeta}{\pi\sqrt{\mu^2 - x^2}} + \frac{\zeta}{2}\delta(x+M) + \frac{\zeta}{2}\delta(x-M)$$

$$\mu = 2\Lambda$$

The phase transition thus occurs when $\mu = M$, i.e. at $M_c = 2\,\Lambda$

2. Weak coupling phase $\mu < M$.

The poles sit outside the eigenvalue distribution.

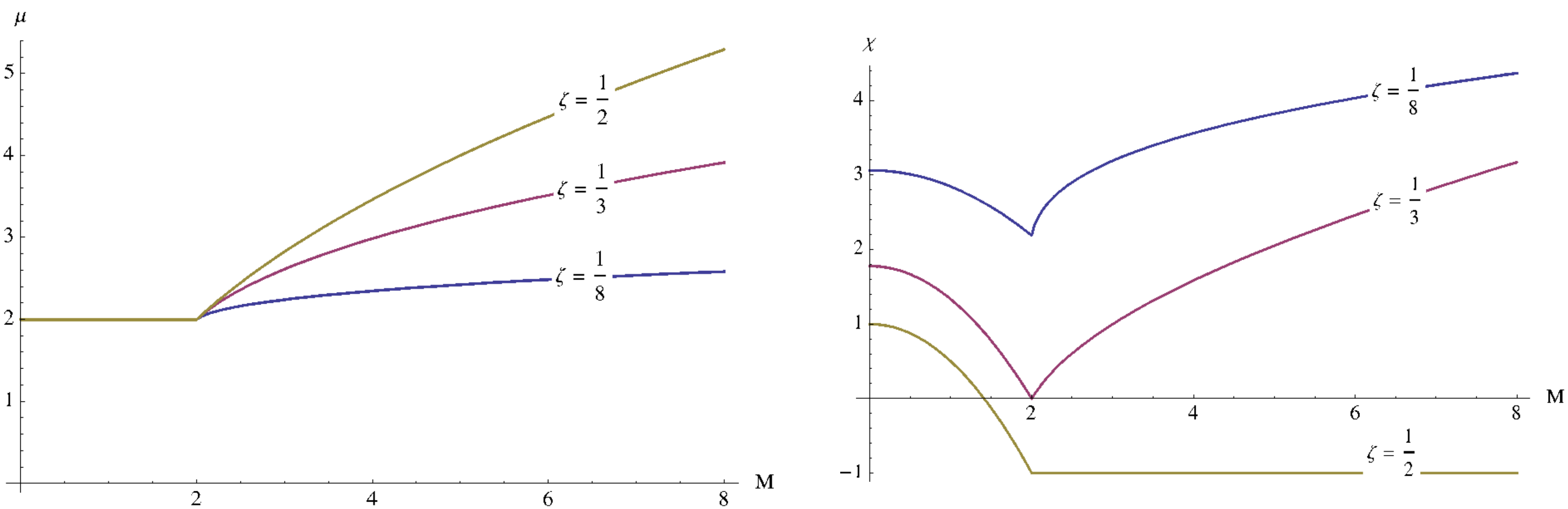
$$\rho(x) = \frac{1-\zeta}{\pi\sqrt{\mu^2 - x^2}} + \zeta\, M\sqrt{M^2 - \mu^2}\,\frac{1}{\pi\sqrt{\mu^2 - x^2}}\,\frac{1}{M^2 - x^2}$$

Free energy

Computing

$$\frac{\partial F}{\partial \log \Lambda} = -\langle x^2 \rangle = -\int_{-\mu}^{\mu} dx\, \rho(x)\, x^2$$

in each phase we find a discontinuity in the third derivative of *F*
Thus the transition is **third order**.



The width of the eigenvalue distribution μ and susceptibility $\chi = -\frac{\partial^2 F}{\partial \Lambda^2}$ as functions of the quark mass for different values of ζ.
Wilson loop W= exp(2 $\pi\mu$) is thus discontinuous in the first derivative.

Phase transitions with adjoint matter: $\mathcal{N}$= 2* SU(N) Super Yang-Mills theory

$\mathcal{N}$= 2* SYM is $\mathcal{N}$= 4 theory with mass term preserving $\mathcal{N}$= 2 supersymmetry.
This means a mass term for the hypermultiplet, i.e. the same mass for **4** scalars and **2** fermions.

LIGHT STATES AT CRITICAL COUPLINGS

The mass spectrum in the scalar VEV background is given by $m_{ij}^{\mathrm{v}} = |a_i - a_j|$, $m_{ij}^{\mathrm{h}} = |a_i - a_j \pm M|$

Recall $-\mu < a_i < \mu$.

- If $\mu < M/2$, the hypermultiplet mass cannot vanish for any a_i

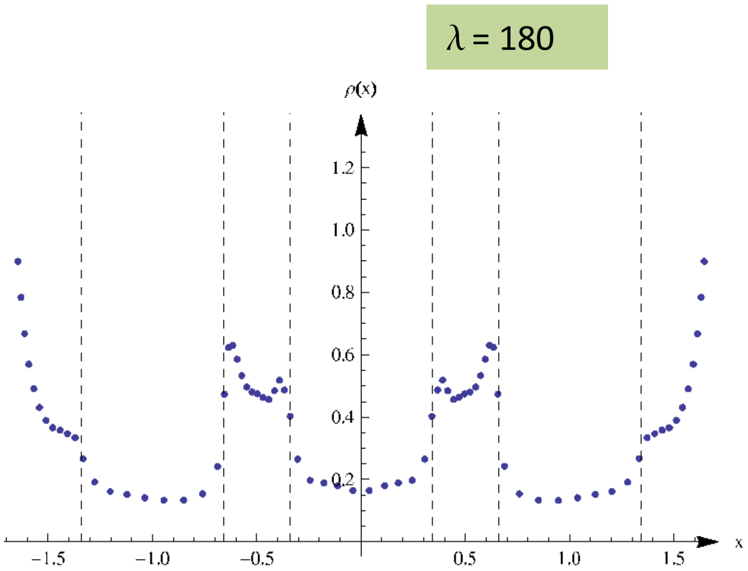
- However, when $\mu > M/2$, there can be light hypermultiplets for states with $a_i - a_j \approx \pm M$

extra light states contribute to the saddle-point whenever μ crosses $n\, M/2$, n = 1, 2, 3, …

At which coupling these resonances occur?
For n >> 1 we can use the strong coupling analytic formula for μ. Infinite number of phase transitions at

$$\mu = \frac{\sqrt{\lambda}}{2\pi} M \quad \to \quad \frac{\sqrt{\lambda}}{2\pi} = \frac{n}{2}$$

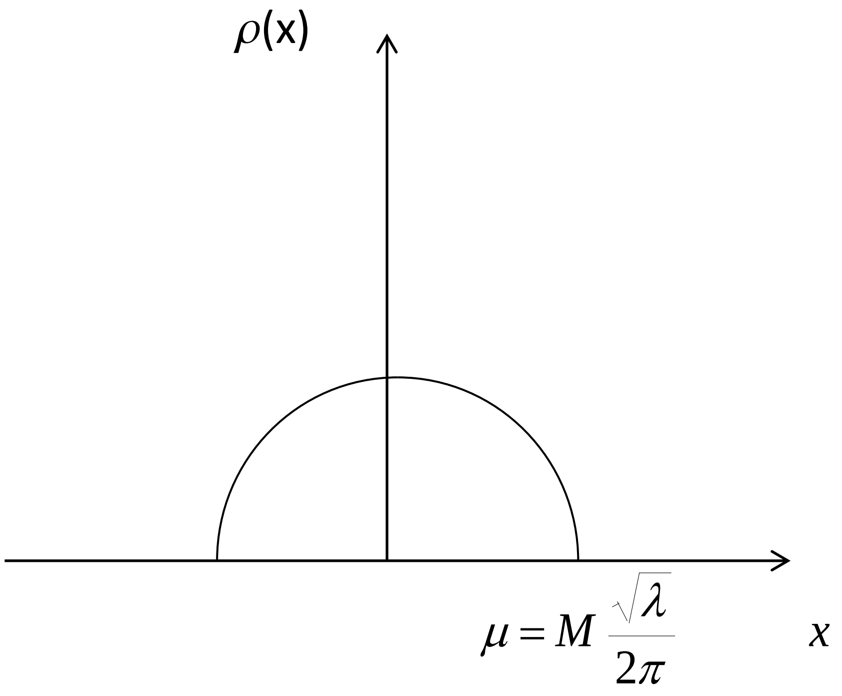
$$\lambda_1 \approx 35.4 \qquad \lambda_2 \approx 83 \qquad \lambda_3 \approx 155 \quad \dots \qquad \lambda_n \approx n^2 \pi^2$$



As λ is increased, the theory goes through new phases where more cusps are formed pairwise, whenever μ crosses nM/2, n = 1,2,3,…

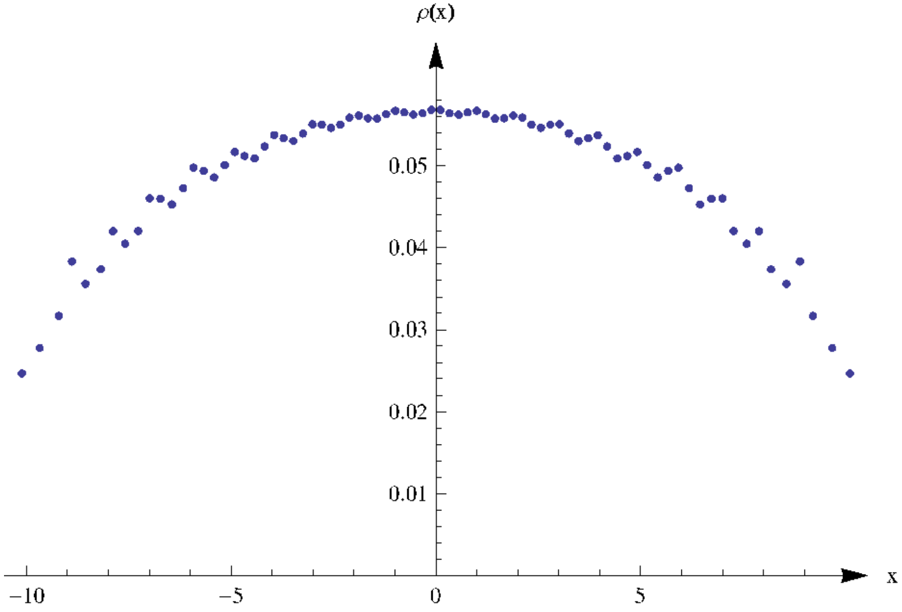
This looks different from what one expects to find at strong coupling:

$$\lambda >> 1, \quad MR >> 1$$



Getting the Wigner's semicircle distribution is necessary to match the AdS/CFT prediction.

$\lambda = 5000$



The Wigner distribution at strong coupling is the result of a coarse grain averaging over an infinite number of infinitely weak cusps.

2. Supersymmetric Chern-Simons with massive matter

Do Chern-Simons-matter theories undergo quantum phase transitions like in the analogous four-dimensional case?

Consider the $\mathcal{N} = 2$ supersymmetric $U(N)$ Chern-Simons theory with level k, coupled to a matter content given by N_f fundamentals and N_f antifundamentals chiral multiplets of mass m .

The partition function localizes to [Kapustin, Willett, Yaakov, 1003.5694]

$$Z_{N_f}^{U(N)} = \int d^N\mu \quad e^{-\frac{1}{2g}\sum_i \mu_i^2} \frac{\prod_{i<j}(4\sinh^2 \tfrac{1}{2}(\mu_i - \mu_j))}{\prod_i [4\cosh\tfrac{1}{2}(\mu_i - m)\cosh\tfrac{1}{2}(\mu_i + m)]^{N_f}} \quad , \quad g = \frac{2\pi i}{k}$$

Can this integral be computed explicitly? It depends on four parameters (g, m, N, N_f)

Consider the infinite N (planar) limit. Then the partition function can be determined by a saddle-point calculation.

Veneziano limit

$$N \to \infty \,, \qquad t \equiv gN = \text{fixed} \,, \qquad \zeta \equiv \frac{N_f}{N} = \text{fixed}$$

The saddle-point equations are then

$$\int d\nu \; \rho(\nu)\coth\tfrac{1}{2}(\mu - \nu) = \frac{\mu}{t} + \frac{\zeta}{2}\tanh\tfrac{1}{2}(\mu + m) + \frac{\zeta}{2}\tanh\tfrac{1}{2}(\mu - m)$$

This is an exactly solvable model [Barranco, J.R, 1401.3672]

Large N solution in the decompactification limit

$$\int d\nu\ \rho(\nu)\coth\tfrac{1}{2}(\mu-\nu)=\frac{\mu}{t}\quad+\frac{\zeta}{2}\tanh\tfrac{1}{2}(\mu+m)+\frac{\zeta}{2}\tanh\tfrac{1}{2}(\mu-m)$$

Repulsion *attraction to origin* *attraction to μ = +- m (combined effect: attraction to μ = 0)*

Restore R dependence and take decompactification limit

$$m\to mR\to\infty\,,\qquad \mu\to\mu R\to\infty$$

If t is fixed, then this limit just decouples matter multiplets, which get an infinite mass. The theory reduces to pure $\mathcal{N}$ = 2 Chern-Simons theory.

An inspection of the saddle-point equation shows that the limit R = infinity is regular if at the same time t goes to infinity with

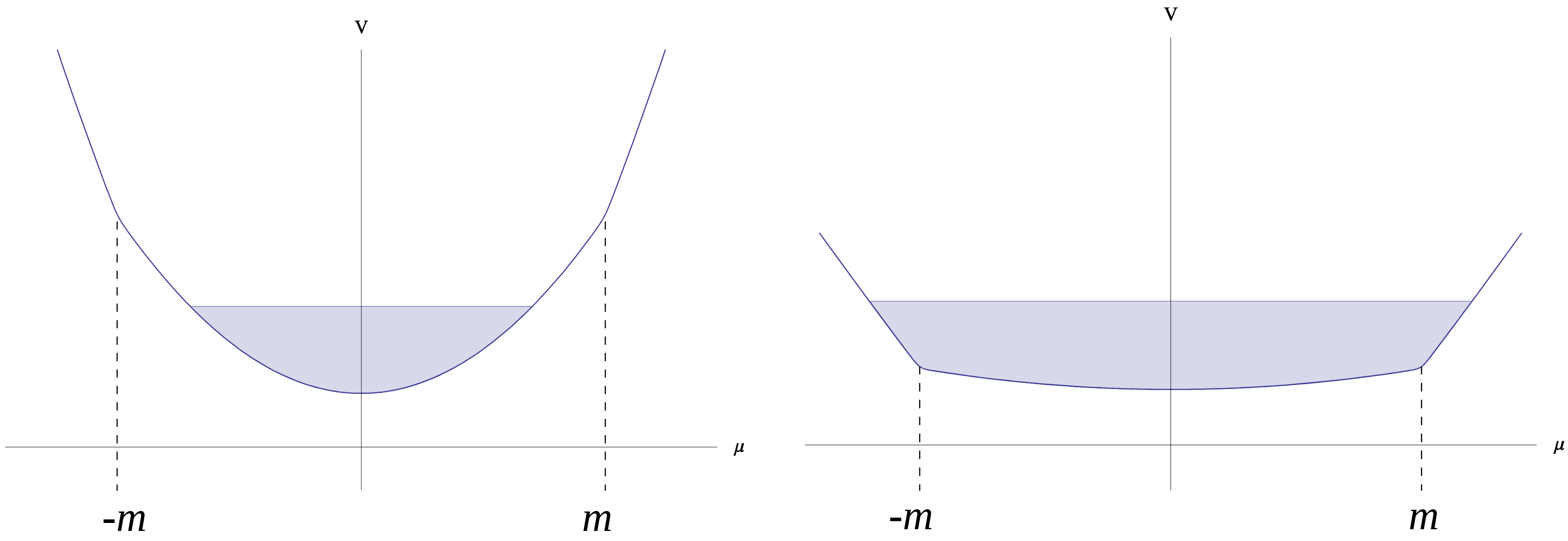
$$\lambda\equiv\frac{t}{mR}=\text{fixed}$$

Then the dependence on R completely cancels out and one obtains

$$\int d\nu\ \rho(\nu)\,\mathrm{sign}(\mu-\nu)=\frac{\mu}{\lambda m}\quad+\frac{\zeta}{2}\mathrm{sign}(\mu+m)+\frac{\zeta}{2}\mathrm{sign}(\mu-m)$$

Potential

$$V = \frac{\mu^2}{2\lambda m} + \frac{N_f}{2N}\left(|\mu + m| + |\mu - m|\right)$$



Consider the integral equation:

$$\int_{-A}^{A} d\nu\, \rho(\nu)\,\mathrm{sign}(\mu-\nu) = \frac{\mu}{\lambda m} + \frac{\zeta}{2}\mathrm{sign}(\mu+m) + \frac{\zeta}{2}\mathrm{sign}(\mu-m)$$

As the coupling λ is increased from 0, the system goes through different phases I, II, III, where:

I. $A < m$

II. $A = m$

III. $A > m$

Phase I **($\lambda < 1$):** arises when $A < m$, implying that $|\mu| < m$. Then the sign functions cancel out.

$$\rho(\mu) = \frac{1}{2m\lambda} \quad , \qquad \mu \in [-m\lambda, m\lambda]$$

The distribution expands as λ is increased, until the endpoints hit + /- m.

Phase II **($1 < \lambda < 1/(1-\zeta)$):** $A = m$

$$\rho(\mu) = \frac{1}{2m\lambda} + \frac{\lambda-1}{2\lambda}\big(\delta(\mu+m)+\delta(\mu-m)\big) \quad , \qquad \mu \in [-m, m]$$

Phase III **($\lambda > 1/(1-\zeta)$):** arises when $A > m$.

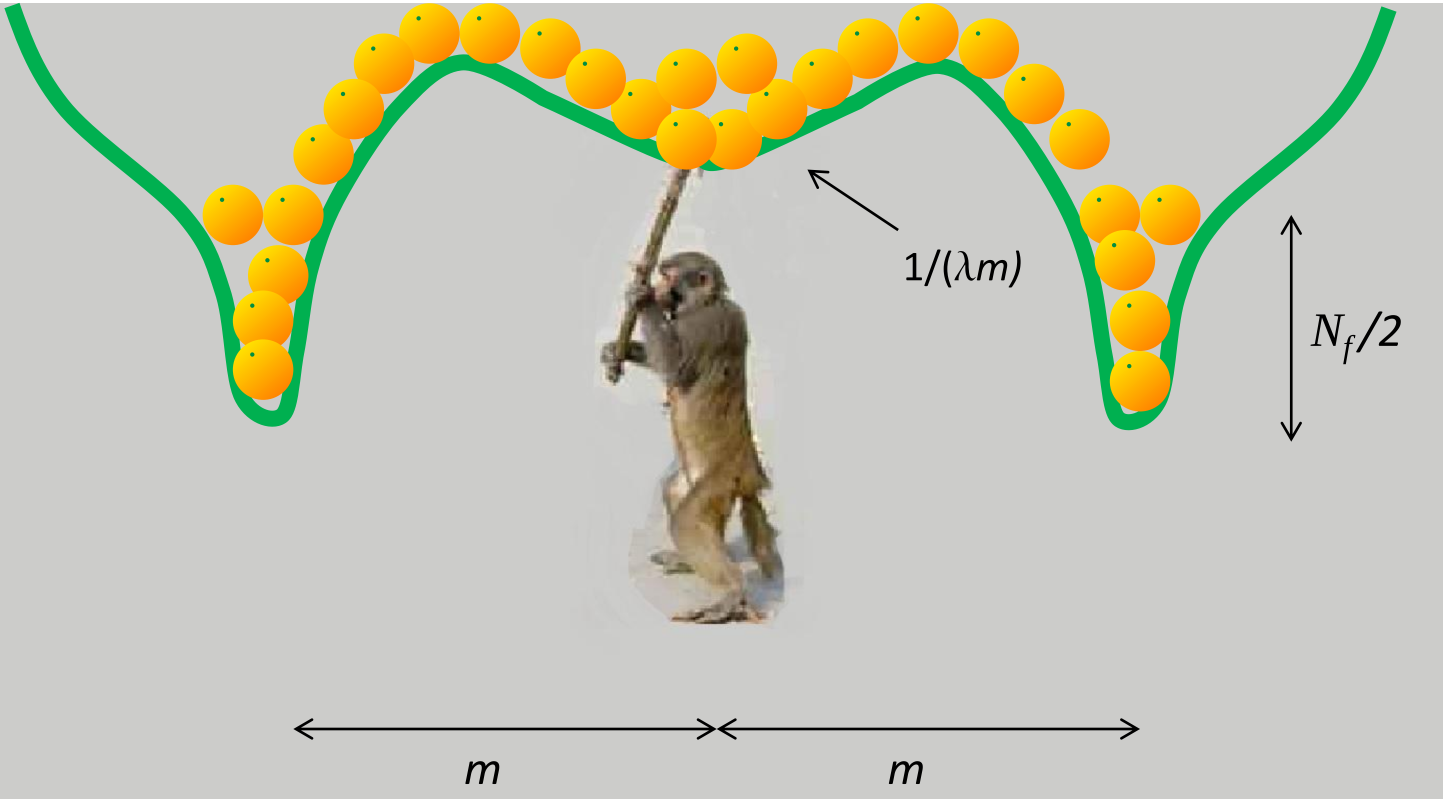
$$\rho(\mu) = \frac{1}{2m\lambda} + \frac{\zeta}{2}\big(\delta(\mu+m)+\delta(\mu-m)\big) \quad , \qquad \mu \in [-m\lambda(1-\zeta), m\lambda(1-\zeta)]$$

We assumed: $0 < \zeta < 1 \quad (0 < N_f < N)$

When $\zeta \geq 1 \quad (N_f \geq N)$ only phase I, II remain

Analogy: N oranges on a canvas with one well at the center and two wells at +m and –m, where only $N_f/2$ oranges can fit.

$$N\,\rho(\mu) = \frac{N}{2m\lambda} + \frac{N_f}{2}\big(\delta(\mu+m) + \delta(\mu-m)\big) \quad , \qquad N\mu \in [-m\lambda(N-N_f), m\lambda(N-N_f)]$$



General solution

By standard matrix model methods, one can find the general solution to the integral equation for finite three-sphere radius *R*

$$\rho(\mu) = \frac{1}{\pi t}\arctan\left(\sqrt{\frac{\cosh^2(A/2)}{\cosh^2(\mu/2)}-1}\right) + \frac{\zeta}{\pi}\frac{\cosh\mu/2\cosh m/2}{\cosh\mu+\cosh m}\sqrt{\frac{\cosh A-\cosh\mu}{\cosh A+\cosh m}}$$

where

$$\log(\cosh^2(A/2)) = t(1-\zeta) + t\zeta\frac{\sqrt{2}\cosh(m/2)}{\sqrt{\cosh A+\cosh m}}$$

Special cases:

a) Massless flavors:

$$\rho(\mu) = \frac{1}{\pi t}\arctan\left(\sqrt{\frac{\cosh^2(A/2)}{\cosh^2(\mu/2)}-1}\right) + \frac{\zeta}{2\pi}\sqrt{\mathrm{sech}^2(\mu/2)-\mathrm{sech}^2(A/2)}$$

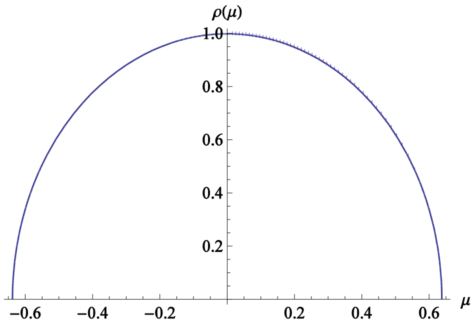
$$\log(X) = -t(1-\zeta+\zeta X) \quad , \qquad X \equiv \mathrm{sech}(A/2)$$

b) Pure N = 2 CS theory ($\zeta = 0$)

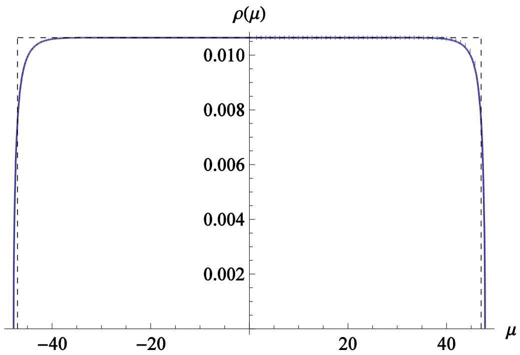
$$\rho(\mu) = \frac{1}{\pi t}\arctan\left(\sqrt{\frac{\cosh^2(A/2)}{\cosh^2(\mu/2)}-1}\right) \quad , \qquad \cosh(A/2) = \exp(t)$$

Reproduces known results of the literature [Marino]

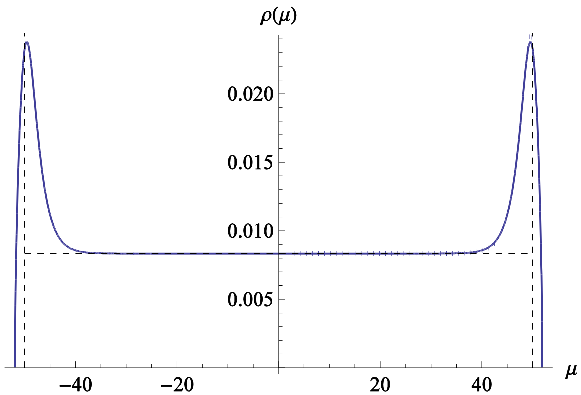
Eigenvalue density



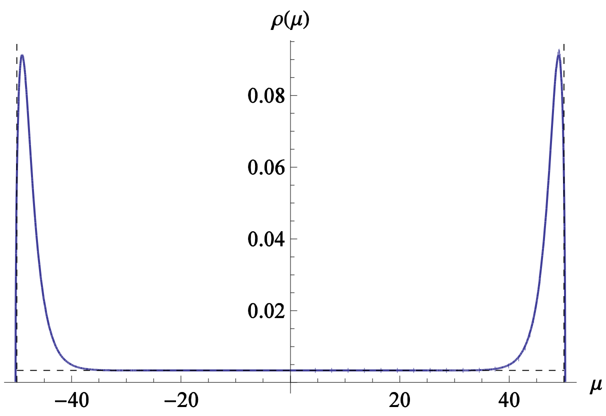
$t \ll 1$: $\rho(\mu)$ approaches the Wigner distribution (t=0.1, m=50, ζ = 0.25)



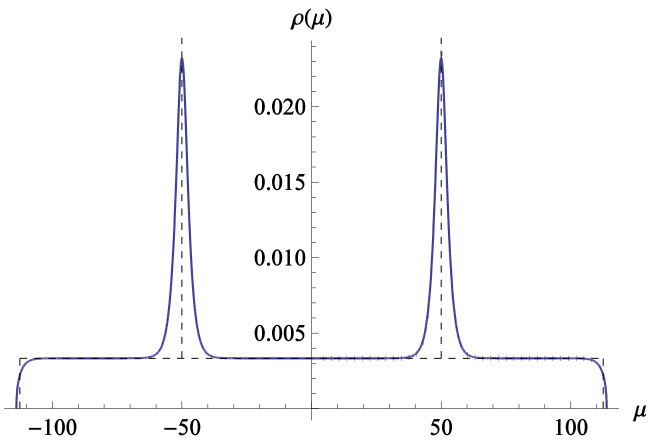
Phase I : $\rho(\mu)$ for m = 50, ζ = 0.25, t = 47.



Phase II : $\rho(\mu)$ for m = 50, ζ = 0.25, t = 60.



Phase II : Case $N_f > N$. $\rho(\mu)$ for m = 50, ζ = 2, t = 150.



Phase III : $\rho(\mu)$ for m = 50, ζ =0.25, t = 150.

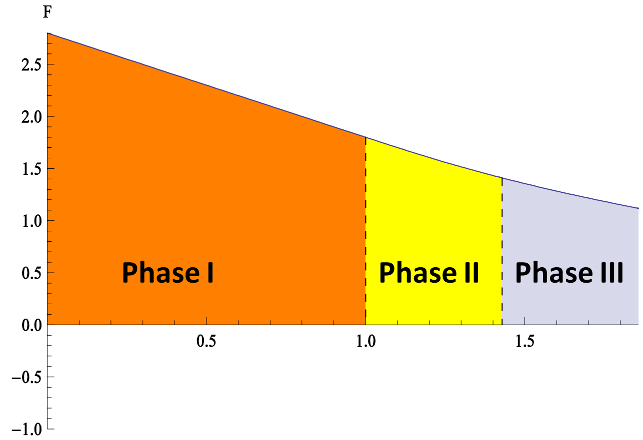
Free energy:

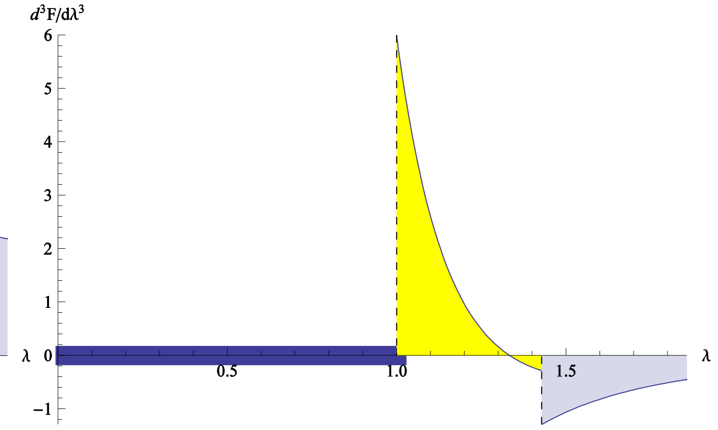
$$F = -\frac{1}{N^2}\log Z$$

$$F_I = \frac{mR}{6}(6\zeta - \lambda)$$

$$F_{II} = \frac{mR}{6\lambda^2}(3(2\zeta - 1)\lambda^2 + 3\lambda - 1)$$

$$F_{III} = \frac{mR}{6\lambda}((\zeta - 1)^3\lambda^2 + 3\zeta^2\lambda + 3\zeta)$$





This implies a discontinuity in the **third** derivative at both critical points, $\lambda = 1$ and $\lambda = (1 - \zeta)^{-1}$:

$$\partial^3(F_I - F_{II})\Big|_{\lambda=1} = -mR \quad , \qquad \partial^3(F_{II} - F_{III})\Big|_{\lambda=1/(1-\zeta)} = mR(1-\zeta)^5$$

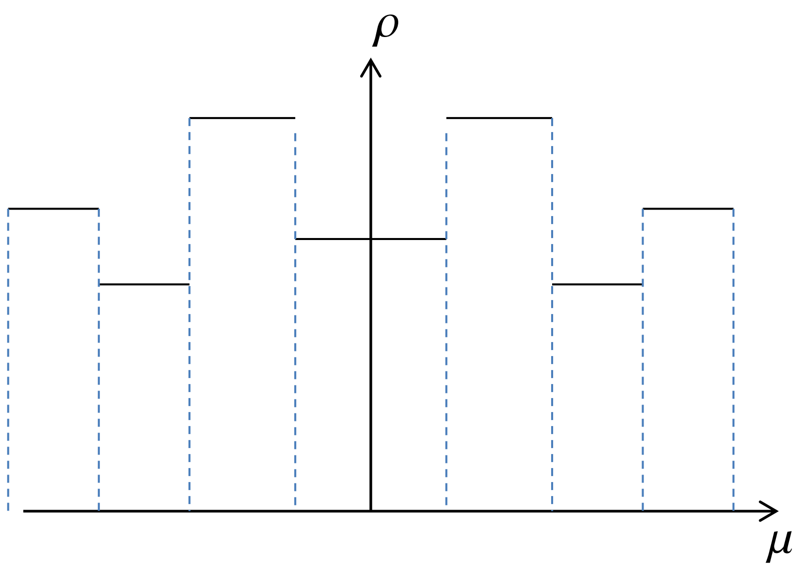
Therefore, both phase transitions are third order.

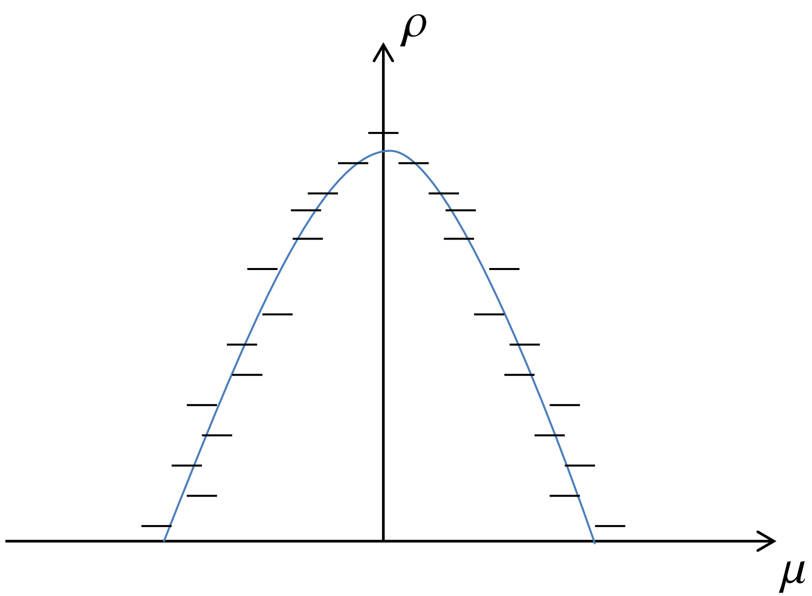
Case of adjoint matter (e.g. ABJM)

[Anderson, Zarembo 1406.3366]

Infinitely many phase transitions like in $\mathcal{N}$=2* theory.
They can be understood in terms of resonances and n*th*-order secondary resonances.

U(N)xU(N) with equal couplings





Strong coupling limit g >> m

$$\rightarrow \quad \rho(\mu) = n(A^2 - \mu^2)$$

Derivation from AdS/CFT?

3. $\mathcal{N}$ = 2 U(N) Chern-Simons-matter with finite N

[J.R., G. Silva and M. Tierz, arxiv:1407.4794]

Consider

$$Z_{N_f}^{U(N)} = \int d^N\mu \quad e^{-\frac{1}{2g}\sum_i \mu_i^2} \frac{\prod_{i<j}(4\sinh^2(\mu_i-\mu_j))}{\prod_i [4\cosh(\mu_i - m)\cosh(\mu_i+m)]^{N_f}} \quad , \qquad g = \frac{2\pi i}{k}$$

The previous saddle-point method only computes Z in the special corner of the parameter space

$$(g, m, N, N_f)$$

involving the planar N=infinity limit

Finite *N*: One can use the method of orthogonal polynomials.
Then computing Z reduces to the problem of computing the integral:

$$I = \int_{-\infty}^{\infty} \frac{e^{at^2+bt}}{e^{ct}+d}\, dt$$

Since 19th century, this integral appeared in many interesting contexts. In particular, studies of Riemann zeta functions by Siegel and Mock theta functions by Ramanujan.
In 1933, it was studied in great detail in a classical paper by Mordell, who gave the result for all possible values of parameters a, b, c, d.

U(N) partition function from orthogonal polynomials

Using the method of orthogonal polynomials, the partition function can be written as follows:

$$Z_{N_f}^{U(N)} = N!\, e^{-\frac{gN}{2}(N^2 - N_f^2)} \det(f_i, f_j)$$

$$(f_i, f_j) = e^{g(N-N_f)(i+j+1-\frac{1}{2}(N-N_f))} \int_{-\infty}^{\infty} dx \, \frac{e^{x(i+j+1+N_f-N) \; - \; \frac{1}{2g}x^2}}{(1+e^{x+m})^{N_f}(1+e^{x-m})^{N_f}}$$

Case $N_f = 1$

$$(f_i, f_j) = e^{\frac{g}{2}(N^2-1) + \ell g(N-1)} \, \frac{I(\ell,m) - I(\ell,-m)}{2\sinh m}$$

$$\ell = i + j + 1 - N, \qquad \ell = 1-N, \ldots, N-1$$

$$I(\ell,m) = e^m \int_{-\infty}^{\infty} dx \, \frac{e^{(\ell+1)x - \frac{1}{2g}x^2}}{1+e^{x+m}}$$

The integral $I(\ell,m)$ is a particular case of the Mordell integral. It can in general be evaluated in terms of expressions involving infinite sums. However**, in two cases**, it reduces to a finite sum of terms.

Luckily, one of these cases is when $k = 2\pi i/g$ is an integer, which in CS theory is required by gauge invariance.

Using Mordell's formula, we find

$$I(\ell,m)=\frac{2\pi\, e^{-i\pi(\ell+\frac{k}{4})}e^{-m(\ell+\frac{k}{2})+\frac{i\,km^2}{4\pi}}}{e^{-km}-1}\left(-\sqrt{\frac{i}{k}}\sum_{n=0}^{k-1}e^{\frac{i\pi}{k}(n-\ell-\frac{k}{2}+\frac{i\,km}{2\pi})^2}+i\right)$$

Examples:

$$Z_k^{U(1)}=\frac{2\pi\, e^{-m+\frac{i\,k(m-i\pi)^2}{4\pi}}}{(1-e^{-2m})(e^{km}-1)}\left(\sqrt{\frac{i}{k}}\sum_{n=0}^{k-1}(e^{\frac{i\pi}{k}(n-\frac{k}{2}-\frac{i\,km}{2\pi})^2}+e^{\frac{i\pi}{k}(n-\frac{k}{2}+\frac{i\,km}{2\pi})^2})-2i\right)$$

The formula contains perturbative as well as non-perturbative terms

Perturbative: e.g. $\exp(gn^2/2)$

Non-perturbative: e.g. $\exp(2\pi im/g)$

Massless case

$$Z_k^{U(1)}(m=0)=\frac{1}{2}e^{-\frac{i\pi k}{4}}+\frac{\pi}{k^{\frac{3}{2}}}e^{\frac{i\pi}{4}}\left(\sum_{n=0}^{k-1}(-1)^n e^{\frac{i\pi}{k}n^2}(n-\tfrac{k}{2})^2\right)$$

Finite dimensional Hilbert space?

Higher rank

$$Z_{k=2}^{U(3)}=\frac{24\sqrt{2}\,\pi^3 e^{\frac{i\pi}{4}}e^{m+\frac{i\,m^2}{\pi}}}{(e^{2m}-1)^2}\left(e^{2m}+2ie^{m}+1-2\sqrt{2}e^{\frac{i\pi}{4}}e^{m-\frac{i\,m^2}{2\pi}}\right)$$

Large *g* coupling and phase transitions

The partition function is proportional to the determinant of J_{ij} defined by the basic integral

$$J_{ij} = \int_{-\infty}^{\infty} dx \, \frac{e^{g(x\ell - \frac{1}{2}x^2)}}{\left(4\cosh\frac{1}{2}(gx+m)\cosh\frac{1}{2}(gx-m)\right)^{N_f}} \quad , \qquad \ell = i + j + 1 - N$$

When g is large, the main contribution of the integral comes from the saddle-point at $x = \ell$
Define *m = g p,* and consider large *g* with fixed *p.* This is equivalent to the decompactification limit considered before, but now we take it at finite *N* (even low *N*, e.g. *N* = 2)

This implies

$$2\cosh\tfrac{1}{2}(gx \pm m) \to \exp\left(\tfrac{1}{2} g\,|x \pm p|\right)$$

$$J_{ij} \to \int_{-\infty}^{\infty} dx \, e^{-\frac{1}{2}g\left(x^2 - 2x\ell + N_f|x+p| + N_f|x-p|\right)}$$

$$= \int_{-\infty}^{-p} dx \, e^{-\frac{1}{2}g(x^2-2x(\ell+N_f))} + e^{-N_f g p}\int_{-p}^{p} dx \, e^{-\frac{1}{2}g(x^2-2x\ell)} + \int_{p}^{\infty} dx \, e^{-\frac{1}{2}g(x^2-2x(\ell-N_f))}$$

Which term is dominant depends on whether the saddle-point lies within the interval (-*p,p*) or outside. This leads to three different cases , which are in one-to-one correspondence to the three phases encountered in the planar limit).

I) $g(N\text{-}1) < m$. In this case we simply have

$$Z_{N_f}^{U(N)} = N!(2\pi g)^{N/2} e^{-N N_f m} e^{\frac{1}{6} g N (N^2-1)}$$
$$= e^{-N N_f m} Z_{CS}(\mathbf{S}^3)$$

Large N: the free energy becomes

$$F_{N_f}^{U(N)} \equiv -\frac{1}{N^2} \ln Z_{N_f}^{U(N)} \;\to\; m(\zeta - \tfrac{1}{6}\lambda) \quad , \qquad \zeta = \frac{N_f}{N} \quad , \quad \lambda = gN/m$$

II) $g(N\text{-}1\text{-}N_f) < m \le g(N\text{-}1)$. In this case Z is given by a finite product. At large N

$$F_{N_f}^{U(N)} \;\to\; \frac{m}{6\lambda^2}(3(2\zeta - 1)\lambda^2 + 3\lambda - 1)$$

III) $m \le g(N\text{-}1\text{-}N_f)$

$$F_{N_f}^{U(N)} \;\to\; \frac{m}{6\lambda}((\zeta - 1)^3 \lambda^2 + 3\zeta^2 \lambda + 3\zeta)$$

We thus reproduce the free energies found in the planar limit in phases 1,2 and 3 starting with exact finite N expressions.

Unitary matrix model formulation and large N

An interesting limit is the large N limit at fixed Chern-Simons level k.
A convenient approach is to formulate the unitary version of the matrix model, where the eigenvalues lie on S^1. In pure CS theory one can show that this approach gives equivalent results [Romo, Tierz, 1103.2421]

The unitary matrix model can be viewed as a deformation of the contour of integration to imaginary eigenvalues. The partition function now has trigonometric functions

$$\tilde{Z}_{N_f}^{U(N)} = \int d^N\mu \quad e^{-\frac{1}{2g}\sum_i \mu_i^2} \frac{\prod_{i<j}(4\sin^2\tfrac{1}{2}(\mu_i-\mu_j))}{\prod_i [4\cos\tfrac{1}{2}(\mu_i+im)\cos\tfrac{1}{2}(\mu_i-im)]^{N_f}}$$

This can be written as an integration on the compact interval

$$\begin{aligned}
\tilde{Z}_{N_f}^{U(N)} &= \int_{[0,2\pi]^N} d^N\mu \quad \prod_{i=1}^{N} \Big(\sum_{n_i=-\infty}^{\infty} e^{-\frac{1}{2g^2}(\mu_i+2\pi n_i)^2}\Big) \frac{\prod_{i<j}(4\sin^2\tfrac{1}{2}(\mu_i-\mu_j))}{\prod_i [4\cos\tfrac{1}{2}(\mu_i+im)\cos\tfrac{1}{2}(\mu_i-im)]^{N_f}} \\
&= \int_{[0,2\pi]^N} d^N\mu \quad \prod_{i=1}^{N} \Big(\sum_{n_i=-\infty}^{\infty} e^{-\frac{n_i^2 g}{2}+in_i\mu_i}\Big) \frac{\prod_{i<j}(4\sin^2\tfrac{1}{2}(\mu_i-\mu_j))}{\prod_i [4\cos\tfrac{1}{2}(\mu_i+im)\cos\tfrac{1}{2}(\mu_i-im)]^{N_f}} \\
&= \int_{[0,2\pi]^N} d^N\mu \quad \prod_{i=1}^{N} \frac{\theta_3(e^{i\mu_j}\,|\,q)}{[4\cos\tfrac{1}{2}(\mu_i+im)\cos\tfrac{1}{2}(\mu_i-im)]^{N_f}} \prod_{i<j}(4\sin^2\tfrac{1}{2}(\mu_i-\mu_j)) \quad , \qquad q=e^{-g}
\end{aligned}$$

where we used Poisson resummation formula. This form permits to compute the large N limit by writing Z as a Toeplitz determinant and use the Szegő theorem.

Toeplitz determinants and Szegő theorem

Let

Toeplitz matrix

$$\begin{pmatrix} a & b & c & d \\ f & a & b & c \\ g & f & a & b \\ h & g & f & a \end{pmatrix}$$

$$f(z) = \sum_{k \in Z} f_k \, z^k \quad , \qquad T_N(f) = (f_{i-j})_{i,j=1,\dots,N}$$

Heine-Szegő formula

$$Z_N[f] \equiv \det T_N(f) = \int_{(0,2\pi)^N} \frac{d^N \phi}{(2\pi)^N} \prod_{l<k} \left| e^{i\phi_l} - e^{i\phi_k} \right|^2 \prod_{j=1}^{N} f(e^{i\phi_j})$$

i.e. the Toeplitz determinant is the partition function of a U(N) unitary matrix model
Then the strong Szegő limit theorem states that

$$\frac{\det T_N(f)}{G(f)^N} \xrightarrow[N\to\infty]{} \exp\left(\sum_{k=1}^{\infty} k[\ln f]_k [\ln f]_{-k}\right)$$

where

$$\ln f(z) = \sum_{k=-\infty}^{\infty} [\ln f]_k \, z^k$$

$[\ln f]_k$ are the coefficients of the Fourier expansion of ln *f,* and *G(f)*=exp([log f]$_0$)

In our case this gives

$$\widetilde{Z}_{N_f}^{U(N)} = \left(\frac{g}{2\pi}\right)^{\frac{N}{2}} \frac{e^{-N N_f |m|}}{(1-e^{-2|m|})^{N_f^2}} \prod_{j=1}^{\infty} (1-q)^{N-j} (1-q^{j-\frac{1}{2}} e^{-|m|})^{2N_f} \quad , \qquad N >> 1$$

If we further take the limit of g to infinity at fixed m/g, then this reproduces the expression of Z for phase I.

The other phases II and III cannot be recovered because in the unitary model $|m \pm i\mu|$ can never be zero.

Conclusions

Massive supersymmetric gauge theories exhibit large N phase transitions at critical couplings.
Transitions occur when the eigenvalue distribution expands and the largest eigenvalue hits the mass.
Then extra massless states contribute to the planar free energy.

Examples:

Four dimensional $\mathcal{N}=2^*$ SYM and $\mathcal{N}=2$ SQCD.

$\mathcal{N}=2^*$ theory has a gravity dual. It predicts something special occurring at $\sqrt{\lambda} \approx n\pi, \; n >> 1$

$\mathcal{N}=2$ $U(N)$ Chern-Simons with $2N_f$ massive flavors:
The planar theory presents three phases when $0 < N_f < N$ and two phases when $N_f \geq N$.

Finite N: The matrix integral defining Z can be computed explicitly using Mordell integrals.
Large N with fixed k: Z can be computed using Szegő limit theorem.

Mass deformed ABJM Model
Rich structure of phase transitions at imaginary values of the couplings, analogous to $\mathcal{N}=2^*$ SYM [Anderson, Zarembo 1406.3366]. Gravity dual phenomenon in terms of M2 branes?

Phase transitions also in 5d $\mathcal{N}=1$ SYM+CS term and adjoint matter [Minahan, Nedelin 1408.2767]
- Phase transition at some critical relative value of the two couplings.
- Pure CS case with adjoint matter: evidence for infinite number of phase transitions.

PHASE DIAGRAM FOR $\mathcal{N} = 2^*$ THEORY ON S^4