Double-copy structures of pure and matter-coupled supergravities

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Nordita workshop:
Supersymmetric
Field Theories

Based on: arXiv:1407.4772, HJ, A. Ochirov,
Gravity double-copy structure

Graviton plane wave:

\[ \varepsilon^{\mu}(p)\varepsilon^{\nu}(p) e^{ip\cdot x} \]

On-shell 3-graviton vertex:

\[ i\kappa \left( \eta_{\mu_1\mu_2}(k_1 - k_2)_{\mu_3} + \text{cyclic} \right) \left( \eta_{\nu_1\nu_2}(k_1 - k_2)_{\nu_3} + \text{cyclic} \right) \]

Gravity scattering amplitude:

\[ M^\text{tree}_4(1, 2, 3, 4) = -i \frac{st}{u} A^\text{tree}_4(1, 2, 3, 4) \tilde{A}^\text{tree}_4(1, 2, 3, 4) \]
String theory tree-level identity:

\[ A_n \sim \int \frac{dx_1 \cdots dx_n}{\mathcal{V}_{abc}} \prod_{1 \leq i < j \leq n} |x_i - x_j|^{k_i \cdot k_j} \exp \left[ \sum_{i < j} \left( \frac{\epsilon_i \cdot \epsilon_j}{(x_i - x_j)^2} + \frac{k_i \cdot \epsilon_j - k_j \cdot \epsilon_i}{(x_i - x_j)^2} \right) \right] \]

KLT relations emerge after nontrivial world-sheet integral identities

Field theory limit \( \Rightarrow \) gravity theory \( \sim \) (gauge theory) \( \times \) (gauge theory)

\[
M_4^{\text{tree}}(1, 2, 3, 4) = -is_{12}A_4^{\text{tree}}(1, 2, 3, 4) \tilde{A}_4^{\text{tree}}(1, 2, 4, 3)
\]
\[
M_5^{\text{tree}}(1, 2, 3, 4, 5) = is_{12}s_{34}A_5^{\text{tree}}(1, 2, 3, 4, 5) \tilde{A}_5^{\text{tree}}(2, 1, 4, 3, 5)
+is_{13}s_{24}A_5^{\text{tree}}(1, 3, 2, 4, 5) \tilde{A}_5^{\text{tree}}(3, 1, 4, 2, 5)
\]

Gravity states are products of gauge theory states:
\[
|2\rangle = |1\rangle \otimes |1\rangle
\]
\[
|3/2\rangle = |1\rangle \otimes |1/2\rangle
\]
etc...
Yang-Mills theories are controlled by a hidden kinematic Lie algebra

- Amplitude represented by cubic graphs:

\[ A_m^{(L)} = \sum_{i \in \Gamma_3} \int \frac{d^{LD} \ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i c_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_l}^2} \]

Color & kinematic numerators satisfy same relations:

\[ f_{fbc} f_{fde} = f_{fbc} f_{fde} - f_{fbc} f_{fde} \]

Algebra enforces (BCJ) relations on partial amplitudes \( \rightarrow (n-3)! \) basis

(proven via string theory by Bjerrum-Bohr, Damgaard, Vanhove; Stieberger)
Gravity is a double copy of YM

- Gravity amplitudes obtained by replacing color with kinematics

\[ A_m^{(L)} = \sum_{i \in \Gamma_3} \int \frac{d^{nL} \ell}{(2\pi)^{nL}} \frac{1}{S_i} \frac{n_i c_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_l}^2} \]

\[ M_m^{(L)} = \sum_{i \in \Gamma_3} \int \frac{d^{nL} \ell}{(2\pi)^{nL}} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_l}^2} \]

- The two numerators can belong to different theories:

\( N = 4 \) × \( N = 4 \) \( \rightarrow \) \( N = 8 \) sugra

\( N = 4 \) × \( N = 2 \) \( \rightarrow \) \( N = 6 \) sugra

\( N = 4 \) × \( N = 0 \) \( \rightarrow \) \( N = 4 \) sugra

\( N = 0 \) × \( N = 0 \) \( \rightarrow \) Einstein gravity + axion+ dillaton

BCJ

similar to Kawai-Lewellen-Tye but works at loop level
General or accident?

Is the double copy structure a generic feature of gravity? Or accident of maximal $N=8$ supergravity (and its truncations)?

Some known limitations (even for tree-level KLT):

- $N<4$ supergravities contaminated by extra matter (dilaton, axion,...)
  - How are pure $N<4$ supergravities obtained?

- Huge number of $N=<4$ supergravities that are not truncations of $N=8$ SG.
  - Can extra matter be introduced in $N=<4$ supergravities?
    - Abelian
    - Non-abelian (gauged supergravities)
    - Tensor matter in $D=6$

- There are gauge-theory amplitudes that cannot be used in the KLT formula
  - e.g. fundamental matter amplitudes — do they have a purpose?
Two generalizations of color-kinematics duality

1) Color-kinematics duality for fundamental rep. matter

→ “square” matter and vector states separately
   e.g. gravitons ~ (vector)^2
   matter/vector ~ (matter)^2

→ Pure N<4 supergravities
→ Generic abelian N=<4 matter (vectors, tensors, fermions...)

2) Color-kinematics duality for a certain bosonic YM + scalar theory

→ Supergravity coupled to abelian and non-abelian vectors
   i.e. Maxwell-Einstein and Yang-Mills-Einstein supergravity

These address/resolve the mentioned problems!
Motivation II: (super)gravity UV behavior

Old results on UV properties:
- susy forbids 1,2 loop div. $R^2, R^3$ [Ferrara, Zumino, Deser, Kay, Stelle, Howe, Lindström, Green, Schwarz, Brink, Marcus, Sagnotti]
- Pure gravity 1-loop finite, 2-loop divergent [Goroff & Sagnotti, van de Ven]
- With matter: 1-loop divergent ‘t Hooft & Veltman; (van Nieuwenhuizen; Fischler..)

New results on $N \geq 4$ UV properties:
- $N=8$ SG and $N=4$ SG 3-loop finite! [Bern, Carrasco, Dixon, HJ, Kosower, Roiban; Bern, Davies, Dennen, Huang]
- $N=8$ SG: no divergence before 7 loops
- First $N=4$ SG divergence at 4 loops (unclear interpretation, U(1) anomaly?) [Bern, Davies, Dennen, Smirnov, Smirnov; Carrasco, Kallosh, Roiban, Tseytlin]

However, no new results in pure $N < 4$ SG
- Need double-copy construction for pure $N < 4$ supergravity

Einstein

$N = 1, 2, 3$ SG

Dissect Goroff & Sagnotti; van de Ven
Motivation

Color-kinematics duality for fundamental rep.
- Gravity-matter amplitudes
- Application to pure gravities in $D=4$

Color-kinematics duality for bosonic YM+scalar theory
- Gauged supergravity (generic Jordan family of SGs)

Explicit loop-level checks

Conclusion
Color-kinematics duality for fundamental rep.
Since 2008, large body of evidence for color-kinematics duality

However, 4pt tree-level YM with matter seems broken?

Naïve application of color-kinematics duality

\[ A(\bar{q}, q, \bar{Q}, Q) = \frac{n_s c_s}{s} \]

\[ n_t = n_u = 0 \]

\[ n_s + n_t + n_u = 0 \]

→ YM amplitude forced to vanish.
→ Color-kinematics duality not generic?

or are we thinking about it incorrectly?
The four-quark tree amplitude:

\[
\begin{align*}
2^+, j & \quad 3^-, k \\
1^-, i & \quad 4^+, \bar{l}
\end{align*}
\]

\[
= -i T_{ij}^a T_{kl}^a \frac{\langle 13 \rangle [24]}{s} = \frac{c_1 n_1}{D_1}
\]

\[
\begin{align*}
2^+, j & \quad 3^-, k \\
1^-, i & \quad 4^+, \bar{l}
\end{align*}
\]

\[
= -i T_{il}^a T_{kj}^a \frac{\langle 13 \rangle [24]}{t} = \frac{c_2 n_2}{D_2}
\]

Note: 1) diagrams are gauge invariant – each is a partial amplitude
2) not meaningful to impose constraints on numerators
3) any numerator relation have to be manifest
4) this amplitude must have manifest color-kinematics duality
Gravity amplitudes from double copy

Four-photon amplitude in GR (distinguishable matter):

\[ A(1^-, 2^+, 3^-, 4^+) = \frac{n_s^2}{s} = \frac{\langle 13 \rangle^2 [24]^2}{s} \]

Four-scalar amplitude in GR (distinguishable matter):

\[ A(1^{++}, 2^{+-}, 3^{-+}, 4^{+-}) = \frac{n_s \bar{n}_s}{s} = \frac{u^2}{s} \]

Indistinguishable matter:

\[ A(1^{++}, 2^{+-}, 3^{-+}, 4^{+-}) = \frac{n_s \bar{n}_s}{s} + \frac{n_s \bar{n}_s}{t} = \frac{u^2}{s} + \frac{u^2}{t} \]
More complicated 5pt example

Look at 3 Feynman diagrams out of 10 in total:

\[
\begin{align*}
3^-, k & 4^+, \bar{l} \\
5, a & = \frac{i}{\sqrt{2}} \frac{1}{s_{15}s_{34}} T^a_{im} T^b_{mj} T^b_{k\bar{l}} \langle 1|\varepsilon_5|1+5|3 \rangle [24] = \frac{c_1 n_1}{D_1} \\
2^+, \bar{j} & 1^-, i \\
3^-, k & 4^+, \bar{l} \\
5, a & = -\frac{i}{\sqrt{2}} \frac{1}{s_{25}s_{34}} T^b_{im} T^a_{mj} T^b_{k\bar{l}} \langle 13 \rangle [2|\varepsilon_5|2+5|4] = \frac{c_2 n_2}{D_2} \\
2^+, \bar{j} & 1^-, i \\
3^-, k & 4^+, \bar{l} \\
5, a & = \frac{i}{\sqrt{2}} \frac{1}{s_{12}s_{34}} \tilde{f}^{abc} T^b_{ij} T^c_{k\bar{l}} \left( \langle 1|\varepsilon_5|2 \rangle \langle 3|5|4 \rangle - \langle 1|5|2 \rangle \langle 3|\varepsilon_5|4 \rangle \right) - 2 \langle 13 \rangle [24]((k_1+k_2)\cdot\varepsilon_5) = \frac{c_5 n_5}{D_5} \\
2^+, \bar{j} & 1^-, i
\end{align*}
\]

Not gauge invariant, but accidentally satisfy color-kinematics duality

\[
C_1 - C_2 = -C_5 \iff n_1 - n_2 = -n_5
\]
Double copy = gravity amplitudes

Indistinguishable matter:

\[ A(1^-, 2^+, 3^-, 4^+, 5_{h}^{++}) = \sum_{i=1}^{10} \frac{n_i^2}{D_i} \]

\[ A(1_{\phi}^{--}, 2_{\phi}^{+-}, 3_{\phi}^{--}, 4_{\phi}^{+-}, 5_{h}^{++}) = \sum_{i=1}^{10} \frac{n_i \bar{n}_i}{D_i} \]

distinguishable matter

\[ A(1^-, 2^+, 3^-, 4^+, 5_{h}^{++}) = \sum_{i=1}^{5} \frac{n_i^2}{D_i} \]

\[ A(1_{\phi}^{--}, 2_{\phi}^{+-}, 3_{\phi}^{--}, 4_{\phi}^{+-}, 5_{h}^{++}) = \sum_{i=1}^{5} \frac{n_i \bar{n}_i}{D_i} \quad (\varepsilon_5 = \varepsilon_5^+) \]

4 and 5pts are well behaved for accidental reasons.
What kinematic algebra should be imposed on numerators in general?
‘Adjoint’ Color-Kinematics Duality

pure Yang-Mills theories are controlled by an ‘adjoint’ kinematic algebra

- Amplitude in cubic graph expansion:

\[ \mathcal{A}^{(L)}_m = \sum_{i \in \Gamma_3} \int \frac{dL^D \ell}{(2\pi)^L} \frac{1}{S_i} \frac{n_i c_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_l}^2} \]

Color & kinematic numerators satisfy same relations:

- Jacobi identity

\[ f^{adc} f^{ceb} = f^{eac} f^{cbd} - f^{abc} f^{cde} \]

- Antisymmetry

\[ f^{bac} = - f^{abc} \]

Bern, Carrasco, HJ
The ‘adjoint’ & ‘fundamental’ algebra

Jacobi Id.

Fundamental algebra

Are there additional algebraic relations?
Recall the 4pt example

The four-quark tree amplitude has an obvious numerator identity

\[
\begin{align*}
2^+, j & \quad 3^-, k \\
1^-, i & \quad 4^+, \bar{l}
\end{align*}
\]

\[
\begin{align*}
&= -i T_{ij}^a T_{kl}^a \langle 13 \rangle [24] \\
&\quad s = \frac{c_1 n_1}{D_1}
\end{align*}
\]

\[
\begin{align*}
2^+, j & \quad 3^-, k \\
1^-, i & \quad 4^+, \bar{l}
\end{align*}
\]

\[
\begin{align*}
&= -i T_{il}^a T_{kj}^a \langle 13 \rangle [24] \\
&\quad t = \frac{c_2 n_2}{D_2}
\end{align*}
\]

\[\rightarrow \quad \mathcal{N}_s = \mathcal{N}_t\]

Is the identity part of the kinematic algebra?
Optional kinematical identity

Two-term Id.

Possible to enforce for single scalar or fermion in $D=3,4,6,10$ (Chiodaroli, Jin, Roiban)

Color identity? Not for fundamental matter, but holds for certain complex representations of $U(N)$

$$T_{ij}^a T_{kl}^a = T_{il}^a T_{kj}^a, \quad U(1) : T_{ij}^a = 1$$
Amplitude representation for non-pure SYM

super-Yang-Mills amplitude with one fundamental matter multiplet:

\[ A_m^{(L)} = \sum_i \int \frac{d^{LD} \ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i c_i}{D_i} \]

sum is over all cubic vector–matter graphs with vertices

Color factors \( c_i \) are built out of \( f^{abc}, T^a_{ij} \)

If \( n_i \) satisfy the kinematic algebra, and external legs are vector multiplets, then consistent sugra ampl’s are generated by replacing the color factors:

\[ c_i \rightarrow (N_V)^{|i|} n_i \quad \text{or} \quad c_i \rightarrow (N_X)^{|i|} \bar{n}_i \]

\(|i|\) counts number of closed matter loops

\( \bar{n}_i \) conjugation denotes reversal of all matter arrows

\( N_X + 1 = \) number of complex matter multiplets in theory

\( N_V = \) number of abelian vector multiplets in theory
Factorizable non-pure gravities

archetype: \((\mathcal{N}=0) \times (\mathcal{N}=0) \rightarrow \text{Einstein gravity + axion+ dilaton}\)

\begin{align*}
(\mathcal{N}=1) \times (\mathcal{N}=0) & \rightarrow \mathcal{N}=1 \text{ sugra } + \mathcal{N}=2 \text{ matter} \\
(\mathcal{N}=1) \times (\mathcal{N}=1) & \rightarrow \mathcal{N}=2 \text{ sugra } + \text{two } \mathcal{N}=2 \text{ matter} \\
(\mathcal{N}=2) \times (\mathcal{N}=0) & \rightarrow \mathcal{N}=2 \text{ sugra } + \mathcal{N}=2 \text{ vector} \\
(\mathcal{N}=2) \times (\mathcal{N}=1) & \rightarrow \mathcal{N}=3 \text{ sugra } + \mathcal{N}=4 \text{ vector} \\
(\mathcal{N}=2) \times (\mathcal{N}=2) & \rightarrow \mathcal{N}=4 \text{ sugra } + \text{two } \mathcal{N}=4 \text{ vectors}
\end{align*}

**Definition:** \(\mathcal{N}=2\) matter: \((\lambda, 2\phi, \bar{\lambda})\)

In terms of on-shell superspace multiplets this can be summarized as:

\[
\mathcal{H}_{\mathcal{N}+\mathcal{M}} \equiv \mathcal{V}_{\mathcal{N}} \otimes \mathcal{V'}_{\mathcal{M}} = \mathcal{H}_{\mathcal{N}+\mathcal{M}} \oplus X_{\mathcal{N}+\mathcal{M}} \oplus \overline{X}_{\mathcal{N}+\mathcal{M}}
\]

- factorizable graviton multiplet: \(\mathcal{H}_{\mathcal{N}+\mathcal{M}} \equiv \mathcal{V}_{\mathcal{N}} \otimes \mathcal{V'}_{\mathcal{M}}\)
- gravity matter: \(X_{\mathcal{N}+\mathcal{M}} \equiv \Phi_{\mathcal{N}} \otimes \Phi'_{\mathcal{M}}\)
- gravity antimatter: \(\overline{X}_{\mathcal{N}+\mathcal{M}} \equiv \overline{\Phi}_{\mathcal{N}} \otimes \overline{\Phi'}_{\mathcal{M}}\)
Obtaining Pure Supergravities

recall:
\[ N_\phi + 1 = \text{number of complex scalars in theory} \]
\[ N_X + 1 = \text{number of complex matter multiplets in theory} \]

if we want no extra matter in the (super)gravity theory
we are forced to pick \( N_\phi = -1 \) and \( N_X = -1 \)

What does it mean?

The double-copied matter has the wrong-sign statistics;
that is, those matter fields are ghosts!

This is a welcome feature of the construction, not a bug
– Removes the unwanted states in the vector double copy
– Preserves the double-copy factorization of states
– Preserves Lorentz invariance
Pure $\mathcal{N}=0,1,2,3$ supergravity amplitudes

(super-)Yang-Mills amplitude with one fundamental matter multiplet

$$A_{m}^{(L)} = \sum_{i} \int \frac{d^{LD} \ell}{(2\pi)^{LD}} \frac{1}{S_{i}} \frac{n_{i} c_{i}}{D_{i}}$$

sum is over all cubic vector–matter graphs with vertices

Color factors $c_{i}$ are built out of $f^{abc}, T^{a}_{i\bar{j}}$

If $n_{i}, n'_{i}$ satisfy the kinematic algebra $\rightarrow$ pure (super-)gravity

$$M_{m}^{(L)} = \sum_{i} \int \frac{d^{LD} \ell}{(2\pi)^{LD}} (-1)^{\mid i \mid} \frac{1}{S_{i}} \frac{n_{i} \bar{n}'_{i}}{D_{i}}$$

$\mid i \mid$ counts number of closed matter loops (i.e. ghost loops)

$\bar{n}_{i}$ conjugation denotes reversal of all matter arrows
Example: one-loop 4pt

Collect all diagrams with the same denominators

\[ i = \{ \text{Box, triangle, bubble} \} \]

\[
\mathcal{M}_{4}^{(1)} = \sum_{S_{4}} \sum_{i=\{B,t,b\}} \int \frac{d^{D}\ell}{(2\pi)^{D}} \frac{1}{S_{i}} \frac{n_{i}^{V} n_{i}^{V'} - \bar{n}_{i}^{m} n_{i}^{m'} - n_{i}^{m} \bar{n}_{i}^{m'}}{D_{i}}
\]

If left and right states are the same \( \rightarrow \) effective gravity numerator is

\[
\text{GR} = \left( \begin{array}{c}
\text{YM} \\
\text{matter} \end{array} \right)^{2} - 2 \left( \begin{array}{c}
\text{matter} \\
\text{matter} \end{array} \right)
\]

...and similarly for triangle and bubble.
Bosonic YM+scalar theory → YM-Einstein SUGRA
Consider the bosonic YM + scalar theory

\[ L_{N=0} = - \frac{1}{4} F_{\mu \nu}^{\hat{a}} F_{\mu \nu}^{\hat{a}} + \frac{1}{2} (D_{\mu} \phi^{\hat{a}})(D_{\mu} \phi^{\hat{b}}) \delta_{\hat{a} \hat{b}} + \frac{g^2}{4} \left( i f_{\hat{a} \hat{b} \hat{c}} \phi^{\hat{b}} \phi^{\hat{c}} \right) \left( i f_{\hat{b} \hat{b'} \hat{c'}} \phi^{\hat{b'}} \phi^{\hat{c'}} \right) \delta_{\hat{b} \hat{b'}} \delta_{\hat{c} \hat{c'}} \]

\[ + \frac{g g'}{3!} \left( i f_{\hat{a} \hat{b} \hat{c}} \right) F_{abc} \phi^{\hat{a} a} \phi^{\hat{b} b} \phi^{\hat{c} c} \]

It has two coupling constants, and two copies of Lie algebras: color & flavor

Non-trivially satisfies color-kinematics duality!
(changed at tree level up to 6pts, and at 1-loop 4pts.)

Multiple gauged supergravities obtained after double copying:
e.g. certain $N=0,1,2,4$ Yang-Mills-Einstein supergravities
Consider the tensor product of the spectrum

\[(\mathcal{N} = 0 \text{ YM + scalars}) \times (\mathcal{N} = 2 \text{ SYM})\]

\[\{A_+, \phi^a, A_-\} \otimes \{A_+, \lambda_+, \varphi, \bar{\varphi}, \lambda_-, A_-\}\]

This gives the spectrum of the generic Jordan family of

\(N = 2\) Maxwell-Einstein and Yang-Mills-Einstein supergravities

Gunaydin, Sierra, Townsend;
de Wit, Lauwers, Philippe, Su, Van Proeyen;
de Wit, Lauwers, Van Proeyen

Indeed, using the explicit Feynman rules of this \(N=2\) gravity theory we have confirmed that the double-copy construction reproduces amplitudes in this theory in \(D=4\) and \(D=5\) dimensions.
One-loop 4pt amplitudes
Algebra for one-loop 4pt calculations

diagrams:

kinematic algebra:

\[ n_{\text{tri}}(1, 2, 3, 4, \ell) = n_{\text{box}}([1, 2], 3, 4, \ell), \]
\[ n_{\text{bub}}(1, 2, 3, 4, \ell) = n_{\text{box}}([1, 2], [3, 4], \ell), \]
\[ n_{\text{snail}}(1, 2, 3, 4, \ell) = n_{\text{box}}([[1, 2], 3], 4, \ell), \]
\[ n_{\text{tadpole}}(1, 2, 3, 4, \ell) = n_{\text{box}}([[1, 2], [3, 4]], \ell), \]
\[ n_{\text{xtadpole}}(1, 2, 3, 4, \ell) = n_{\text{box}}([[1, 2], [3], 4], \ell). \]
Amplitude in YM + scalar theory

Consider the 4pt amplitude with external scalars in the YM+ scalar theory

Three types of contributions:

\[ n_{\text{box}}^{(a)}(1, 2, 3, 4) = -ig'^4 F^{bac} F^{ca2d} F^{drae} F^{e4b} \]

\[ n_{\text{box}}^{(b)}(1, 2, 3, 4, \ell) = -\frac{i}{12} g'^2 \left\{ (N_V + 2)(F^{a1 a4 b} F^{ba a2} (\ell_2^2 + \ell_4^2)) + F^{a1 a2 b} F^{ba a4} (\ell_1^2 + \ell_3^2) \right\} 
+ 24(s F^{a1 a4 b} F^{ba a2} + t F^{a1 a2 b} F^{ba a4}) + \delta^{a3 a4} \text{Tr}_{12}(6\ell_3^2 - \ell_2^2 - \ell_4^2) 
+ \delta^{a2 a3} \text{Tr}_{14}(6\ell_2^2 - \ell_1^2 - \ell_3^2) + \delta^{a1 a4} \text{Tr}_{23}(6\ell_4^2 - \ell_1^2 - \ell_3^2) 
+ \delta^{a1 a2} \text{Tr}_{34}(6\ell_1^2 - \ell_2^2 - \ell_4^2) + (\ell_1^2 + \ell_2^2 + \ell_3^2 + \ell_4^2)(\delta^{a2 a4} \text{Tr}_{13} + \delta^{a1 a3} \text{Tr}_{24}) \} \]
Consider the 4pt amplitude with external vectors in the YM-Einstein theory

The three types of contributions are obtained as double copies

\[
\tilde{n}_{\text{box}}^{N=2,\text{mat.}}(1, 2, 3, 4, \ell) = \frac{(\kappa_{12} + \kappa_{34})(s - \ell_s)^2}{2s^2} + \frac{(\kappa_{23} + \kappa_{14})\ell_t^2}{2t^2} + \frac{(\kappa_{13} + \kappa_{24})st + (s + \ell_u)^2}{2u^2}
\]

\[
+ \mu^2 \left( \frac{\kappa_{12} + \kappa_{34}}{s} + \frac{\kappa_{23} + \kappa_{14}}{t} + \frac{\kappa_{13} + \kappa_{24}}{u} \right)
\]

\[
- 2i\epsilon(1, 2, 3, \ell)\frac{\kappa_{13} - \kappa_{24}}{u^2},
\]
Ampl’s for fundamental $\mathcal{N}=1$ SYM and pure $\mathcal{N}=2$ SG

\[ n_{\text{box}}^{\mathcal{N}=1, \text{odd}} = -\left(\kappa_{12} - \kappa_{34}\right)\frac{(s + \tau_{35} + \tau_{45})^3}{2s^3} - \left(\kappa_{14} - \kappa_{23}\right)\frac{(\tau_{25} + \tau_{35})^3}{2t^3} + (\kappa_{13} - \kappa_{24}) \left[ s\left( \frac{1}{2u} - \frac{3(\tau_{15} + \tau_{35})}{2u^2} + \frac{3(\tau_{15} + \tau_{35})^2}{2u^3} \right) \right. \\
\left. + \frac{(\tau_{15} + \tau_{35})^3}{2u^3} \right] - 2i(\kappa_{13} + \kappa_{24}) \frac{2\tau_{15} + 2\tau_{35} - u}{u^3} \epsilon(1, 2, 3, 5) \]

parity odd: \text{HJ, Ochirov}

parity even: Carrasco, Chiodaroli, Gunaydin, Roiban; Nohle; Ochirov, Tourkine

E.g. pure $\mathcal{N}=2$ gravity numerator:

\[ n_{\text{box}}^{\mathcal{N}=2 \ \text{SG}} = \left(n_{\text{box}}^{\mathcal{N}=1 \ \text{SYM}}\right)^2 - 2\left(n_{\text{box}}^{\mathcal{N}=1, \text{even}} + n_{\text{box}}^{\mathcal{N}=1, \text{odd}}\right)\left(n_{\text{box}}^{\mathcal{N}=1, \text{even}} - n_{\text{box}}^{\mathcal{N}=1, \text{odd}}\right) \]

\[ = \left(n_{\text{box}}^{\mathcal{N}=1 \ \text{SYM}}\right)^2 - 2\left(n_{\text{box}}^{\mathcal{N}=1, \text{even}}\right)^2 + 2\left(n_{\text{box}}^{\mathcal{N}=1, \text{odd}}\right)^2 \]

\[ \tau_{i5} = 2k_i \cdot \ell \]

After integration: agreement with Dunbar & Norridge (‘94)
Two-loop check

We have checked that the ghost prescription removes all dilaton and axion contributions in the physical unitarity cuts in 2-loop 4pt Einstein gravity

The double s-channel cut is the most nontrivial. It mixes two diagrams of different statistics:

The cancellation between the ghosts, dilaton and axion is highly intricate in this case

→ works well for \((\lambda^+ \otimes \lambda^-) \oplus (\lambda^- \otimes \lambda^+) \rightarrow \phi, a\)

→ glitch/feature: \((\phi^+ \otimes \phi^-) \oplus (\phi^- \otimes \phi^+) \rightarrow \phi, \phi'\)
In the past color-kinematics duality could not be used for pure N<4 (super)gravity theories – impeded studies of gravity UV behavior.

Similarly, how to properly “double-copy construct” supergravity coupled to abelian or non-abelian vectors was an open problem.

The problems are solved by the introduction of fundamental-matter color-kinematics duality, and a bosonic YM+scalar theory, respectively.

Double copies give amplitudes in a wide range of matter-coupled (super)gravity theories.

A ghost prescription is proposed to give pure gravities. Ghosts obtained from double copies of matter-antimatter pairs.

Checks: At tree-level, one and two loops.

Opens a new window into the study of gravity UV properties.
Notation for one-loop calculations

ansatz for 4pt MHV amplitude with internal matter, in any SYM theory: HJ, Ochirov

\[ n_{\text{box}}(1, 2, 3, 4, \ell) = \sum_{1 \leq i < j \leq 4} \kappa_{ij} \left( \sum_{k} a_{ij;k} M_{k}^{(N)} + \varepsilon(1, 2, 3, \ell) \sum_{k} \bar{a}_{ij;k} M_{k}^{(N-2)} \right) \]

power-counting factor: \( N = 4 - \mathcal{N} \leftrightarrow \text{SUSY} \)

momentum monomials: \( M^{(N)} = \left\{ \prod_{i=1}^{N} m_{i} \middle| m_{i} \in \{ s, t, \ell \cdot k_{j}, \ell^2, \mu^2 \} \right\} \)

state dependence: \( \kappa_{ij} = \frac{[1 2] [3 4]}{\langle 1 2 \rangle \langle 3 4 \rangle} \delta^{(2\mathcal{N})}(Q) \langle i \ j \rangle^{4-\mathcal{N}} \theta_{i} \theta_{j} \)

( vector multiplet: \( V_{\mathcal{N}} = V_{\mathcal{N}} + \bar{V}_{\mathcal{N}} \theta \) )
Other Spacetime Dimensions

The prescription for adding matter-antimatter ghosts is similar in other dimensions, however, starting in $D=6$ an unwanted 4-form show up.

<table>
<thead>
<tr>
<th>Dim</th>
<th>$A^\mu \otimes A^\nu \rightarrow \phi \oplus h^{\mu\nu} \oplus B^{\mu\nu}$</th>
<th>tensoring matter</th>
<th>(matter)$^2 \rightarrow \phi \oplus B^{\mu\nu} \oplus D^{\mu\nu\rho\sigma}$</th>
<th>resulting states</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D = 3$</td>
<td>$1 \otimes 1 \rightarrow 1 \oplus 0 \oplus 0$</td>
<td>$\lambda \otimes \bar{\lambda}$</td>
<td>$1 \otimes 1 \rightarrow 1 \oplus 0 \oplus 0$</td>
<td>topological</td>
</tr>
<tr>
<td>$D = 4$</td>
<td>$2 \otimes 2 \rightarrow 1 \oplus 2 \oplus 1$</td>
<td>$\left(\lambda^+ \otimes \lambda^-\right) \oplus \left(\lambda^- \otimes \lambda^+\right)$</td>
<td>$1 \otimes 1 \oplus (1 \otimes 1) \rightarrow 1 \oplus 1 \oplus 0$</td>
<td>$h_{\mu\nu}$</td>
</tr>
<tr>
<td>$D = 5$</td>
<td>$3 \otimes 3 \rightarrow 1 \oplus 5 \oplus 3$</td>
<td>$\lambda^\alpha \otimes \bar{\lambda}^\beta$</td>
<td>$2 \otimes 2 \rightarrow 1 \oplus 3 \oplus 0$</td>
<td>$h_{\mu\nu}$</td>
</tr>
<tr>
<td>$D = 6$</td>
<td>$4 \otimes 4 \rightarrow 1 \oplus 9 \oplus 6$</td>
<td>$\left(\lambda^\alpha \otimes \lambda^\beta\right) \oplus \left(\bar{\lambda}^\dot{\alpha} \otimes \bar{\lambda}^\dot{\beta}\right)$</td>
<td>$2 \otimes 2 \oplus (2 \otimes 2) \rightarrow 1 \oplus 6 \oplus 1$</td>
<td>$h_{\mu\nu}, D^{\mu\nu\rho\sigma}$ (ghost)</td>
</tr>
<tr>
<td>$D = 10$</td>
<td>$8 \otimes 8 \rightarrow 1 \oplus 35 \oplus 28$</td>
<td>$\lambda^A \otimes \lambda^B$</td>
<td>$8 \otimes 8 \rightarrow 1 \oplus 28 \oplus 35$</td>
<td>$h_{\mu\nu}, D^{\mu\nu\rho\sigma}$ (ghost)</td>
</tr>
</tbody>
</table>

possibly cured cured by:

- Projecting out states in the matter-antimatter double copy?
- Adding back a 4-form (axion in $D=6$)?
- Other way?