The superconformal bootstrap program

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CERN

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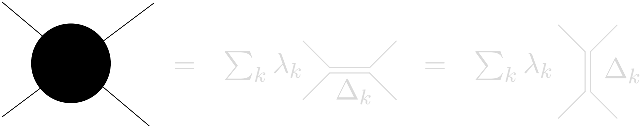
Based on work with C. Beem, M. Lemos, P. Liendo, W. Peelaers, L. Rastelli

Bootstrap

Bootstrap: use internal consistency conditions to fix the observables in a QFT

- Unitarity
- Global symmetries (Poincaré, conformal, supersymmetry, flavor)
- Crossing symmetry



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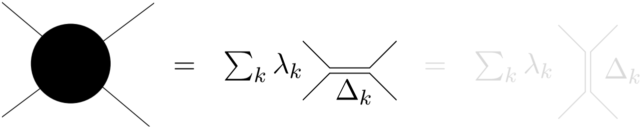

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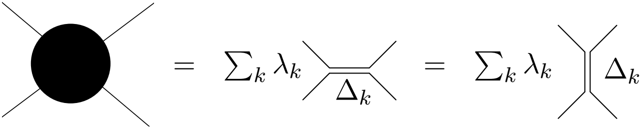

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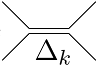
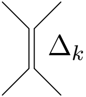
$\sum_k \lambda_k$ 

$=$ $\sum_k \lambda_k$ 

[Ferrara, Gatto, Grillo, Parisi (1972); Polyakov (1974)]

The bootstrap - is it possible?

Some success stories:

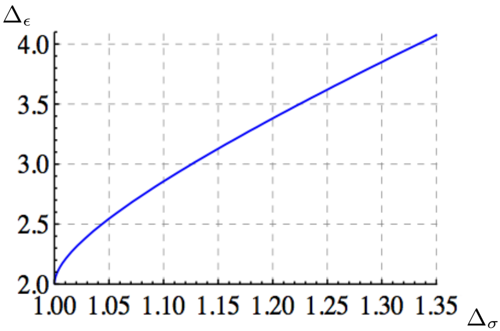
- Minimal models in two dimensions
  [Belavin, Polyakov, Zamolodchikov (1984)]
- Rational CFTs in two dimensions
  [Moore, Seiberg (1989), . . .]

Use crossing symmetry for CFTs in more than two dimensions?

Conformal field theory revisited

$$\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4) \rangle$$

$$\sigma(x_1)\sigma(x_2) \sim \frac{1}{(x_1 - x_2)^{2\Delta_\sigma}} + \frac{\epsilon(x_2)}{(x_1 - x_2)^{2\Delta_\sigma - \Delta_\epsilon}} + \ldots$$



[Rattazzi, Rychkov, Tonni, Vichi (2008)]

Back to the bootstrap

Conformal field theory revisited

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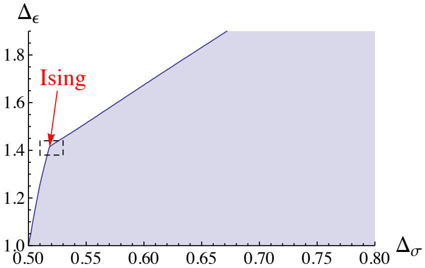


[El-Showk, Paulos, Poland, Simmons-Duffin, Rychkov, Vichi (2012)]

The conformal bootstrap

What can we say about the space of conformal field theories?

The conformal bootstrap

What can we say about the space of *super*conformal field theories?

The conformal bootstrap

What can we say about the space of $\mathcal{N}$-extended superconformal field theories?

Spaces of superconformal field theories

The space of all $\mathcal{N}$-extended superconformal field theories in four dimensions

$\mathcal{N} = 4$

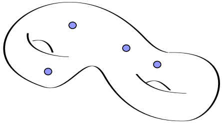
- Lagrangian theories are classified by

$$(G, \tau)/SL(2, \mathbb{Z})$$

  where G is a simple Lie group and $\tau \in H$
- No exotic theories?

$\mathcal{N} = 2$

- Quiver classification of Lagrangian theories [Bhardwaj, Tachikawa (2013)]
- Class S theories obtained from six dimensions [Gaiotto (2008)]
- Many non-Lagrangian theories
- Do we have a complete classification?



Can the bootstrap program help us?

The superconformal bootstrap program

We are going to explore the consequences of crossing symmetry for superconformal field theories.

- Can we bootstrap specific superconformal theories?
- What can we learn about the space of all superconformal theories?

The superconformal bootstrap

Is there a *protected, solvable* subsector of the crossing symmetry constraints for superconformal field theories?

Yes, for

$d = 4$ theories with $\mathcal{N} = 2$ susy

$d = 6$ theories with $(2, 0)$ susy

$d = 2$ theories with $(0, 4)$ susy

More precisely, we find that *twisted correlation functions* of certain *protected operators* become those of a two-dimensional *chiral algebra* and can be completely solved.

For example,

$$T(z)T(w) \sim \frac{c/2}{(z-w)^2} + \frac{2T}{(z-w)^2} + \frac{\partial T}{z-w}$$

completely determines all the correlation functions of $T(z)$.

[Beem, Lemos, Liendo, Peelaers, Rastelli, BvR (2013)]

The superconformal bootstrap program

Consequently, our program splits into two parts:

Minibootstrap		Maxibootstrap
• Protected	$\longrightarrow$	• Not protected
• Meromorphic		• Numerical
• Virasoro, Kac-Moody, W, ...		• Linear programming

Outline

1. Introduction
2. The minibootstrap in $d = 4$
3. The maxibootstrap in $d = 4$
4. Conclusions

Outline

Definition

Take an $\mathcal{N} = 2$ superconformal field theory. Recall that the $\mathcal{N} = 2$ superconformal algebra is $\mathfrak{su}(2, 2|2)$ with maximal bosonic subgroup

$$\mathfrak{su}(2, 2) \times \mathfrak{su}(2)_R \times \mathfrak{u}(1)_r$$

so irreps are labeled with Δ, (j_1, j_2) and (R, r).

Consider now an n-point correlation function

$$\langle \mathcal{O}^{I_1}(x_1) \ldots \mathcal{O}^{I_n}(x_n) \rangle$$

and restrict it in the following way:

1. Take all operators to be 'Schur' operators satisfying $\Delta = 2R + j_1 + j_2$.
2. Take all n points to lie in a two-plane $\mathbf{R}^2 \subset \mathbf{R}^4$.
3. Contract the $\mathfrak{su}(2)_R$ indices with *position-dependent* vectors $v_I(\bar{z})$. For example, for the fundamental representation $v(\bar{z}) = (1, \bar{z})$.

Claim: the resulting correlation function is *meromorphic* in all the positions.

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Consider now an n-point correlation function

$$\frac{\partial}{\partial\bar{z}_k}\left(v_{I_1}(\bar{z}_1)\ldots v_{I_n}(\bar{z}_n)\langle\mathcal{O}^{I_1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{I_n}(z_n,\bar{z}_n)\rangle\right)=0$$

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3. Contract the $\mathfrak{su}(2)_R$ indices with *position-dependent* vectors $v_I(\bar{z})$. For example, for the fundamental representation $v(\bar{z})=(1,\bar{z})$.

Claim: the resulting correlation function is *meromorphic* in all the positions.

Example: free hypermultiplet

In a free hypermultiplet the scalars $Q^I = (Q, \tilde{Q}^*)$ and $\tilde{Q}^J = (\tilde{Q}, -Q^*)$ form two $\mathfrak{su}(2)_R$ doublets and satisfy $\Delta = 2R + j_1 + j_2$. Their OPE is

$$Q^I(z,\bar{z})\tilde{Q}^J(0) \sim \frac{-\epsilon^{IJ}}{z\bar{z}}$$

so

$$v_I(\bar{z})Q^I(z,\bar{z})v_J(0)\tilde{Q}^J(0) \sim \frac{-v_I(\bar{z})v_J(0)\epsilon^{IJ}}{z\bar{z}} = \frac{1}{z}$$

Defining $q(z) = v_I Q^I$ and $\tilde{q}(z) = v_I \tilde{Q}^I$ we find the two-dimensional OPE

$$q(z)\tilde{q}(0) \sim \frac{1}{z}$$

corresponding to a (non-unitary) pair of symplectic bosons of dimension 1/2.

Definition

Claim:

$$\frac{\partial}{\partial \bar{z}_k} \langle v_{I_1}(\bar{z}_1) \mathcal{O}^{I_1}(z_1, \bar{z}_1) \ldots \ldots v_{I_n}(\bar{z}_n) \mathcal{O}^{I_n}(z_n, \bar{z}_n) \rangle = 0$$

Proof:

- There exists a particular nilpotent supercharge $\mathbb{Q}$ such that

  $$\{\mathbb{Q}, \mathbb{Q}^\dagger\} = \mathcal{H} - 2R - \mathcal{M}_+{}^+ - \mathcal{M}_{\dot{+}}{}^{\dot{+}}$$

  so necessarily $\Delta - 2R - j_1 - j_2 \geq 0$ and a Schur operator satisfies

  $$[\mathbb{Q}, \mathcal{O}^{1\ldots1}_{+\ldots+\dot{+}\ldots\dot{+}}(0)\} = 0\,.$$

  We can pick $\mathbb{Q} = \mathcal{Q}^1_- - \tilde{\mathcal{S}}^{\dot{2}-}$.
- Holomorphic translations are $\mathbb{Q}$ closed

  $$[\mathbb{Q}, P_z] = 0$$

- In the antiholomorphic direction we find that

  $$\partial_{\bar{z}} \left(v_I(\bar{z}) \mathcal{O}^I(z, \bar{z}) \right) = v_I(\bar{z}) [P_{\bar{z}} + R^-, \mathcal{O}^I(\bar{z})]$$

  and such *twisted* antiholomorphic translations are $\mathbb{Q}$ exact

  $$P_{\bar{z}} + R^- = \{\mathbb{Q}, \ldots\}$$

Meromorphicity then follows from the usual cohomological argument.

Properties

Dictionary

Hypermultiplet $\rightarrow$ Symplectic bosons

Flavor symmetry $\rightarrow$ affine Kac-Moody symmetry

$$j^A(z)j^B(0) \sim \frac{k_{2d}}{z^2} + \frac{f^{AB}{}_C}{z} j^C(0)$$
$$k_{2d} = -k_{4d}/2$$

Stress tensor $\rightarrow$ Virasoro stress tensor

$$T(z)T(0) \sim \frac{c_{2d}}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}$$
$$c_{2d} = -12c_{4d}$$

. . .

The minibootstrap

To summarize, $\mathcal{N} = 2$ SCFTs in $d = 4$ always have infinite chiral symmetry in a protected sector. In particular we have Virasoro symmetry, but there is often much more.

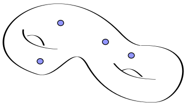
Results for $\mathcal{N} = 2$ theories in $d = 4$:

- Flavor symmetry enhanced to Kac-Moody (or QDS thereof)
- New unitarity bounds
- New three-point couplings
- Holographic interpretation?
- ...

[Beem, Lemos, Liendo, Peelaers, Rastelli, BvR (2014)]

Results for $(2, 0)$ theories in $d = 6$:

- W_g
- All half-BPS three-point couplings
- Microscopic understanding of AGT?
- Connections to geometric Langlands?
- ...



[Beem, Rastelli, BvR (2014)]

Outline

The $\mathcal{N} = 4$ maxibootstrap

In theories with $\mathcal{N} = 4$ superconformal symmetry, the primary $\mathcal{O}^i_{\mathbf{20'}}$ is a *universal* operator. So let's bootstrap its four-point function,

$$\langle \, \mathcal{O}^{i_1}_{\mathbf{20'}}(x_1)\mathcal{O}^{i_2}_{\mathbf{20'}}(x_2)\mathcal{O}^{i_3}_{\mathbf{20'}}(x_3)\mathcal{O}^{i_4}_{\mathbf{20'}}(x_4) \, \rangle = \frac{A^{i_1 i_2 i_3 i_4}(x_{ij})}{x_{12}^4 x_{34}^4}$$

A priori there are 6 different functions but they are fixed in terms of a *single* unconstrained function $A(x_{ij})$ and two meromorphic functions $f_1(z_i)$ and $f_2(z_i)$.

- The meromorphic functions are fixed by the chiral algebra in terms of $a = \text{dim}\; G/4$ so they are *input* for the numerical bootstrap.
- The unconstrained function $A(x_{ij})$ contains information on the *unprotected* operators only and is analyzed numerically.

The $\mathcal{N} = 4$ maxibootstrap

$$\langle\, \mathcal{O}^{i_1}_{\mathbf{20'}}(x_1)\mathcal{O}^{i_2}_{\mathbf{20'}}(x_2)\mathcal{O}^{i_3}_{\mathbf{20'}}(x_3)\mathcal{O}^{i_4}_{\mathbf{20'}}(x_4)\,\rangle = \frac{A^{i_1 i_2 i_3 i_4}(x_{ij})}{x_{12}^4 x_{34}^4}$$

Note: the *only* long multiplets that can appear have $\mathbf{R} = 0$ and even spin.

Examples are:

Konishi, Tr$(\Phi^I \Phi_I)$:

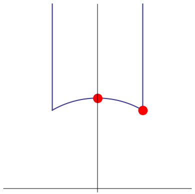
$$\Delta_{\text{kon}} = 2 + \frac{3Ng}{\pi} - \frac{3N^2 g^2}{\pi^2} + \frac{21 N^3 g^3}{4\pi^3} + \left(-39 + 9\,\zeta(3) - 45\,\zeta(5)\left(\tfrac{1}{2} + \tfrac{6}{N^2}\right)\right)\frac{N^4 g^4}{4\pi^4} + \cdots$$

[Velizhanin, . . .]

Double-trace, Tr$(\Phi^I \Phi^J)$Tr$(\Phi_I \Phi_J)$:

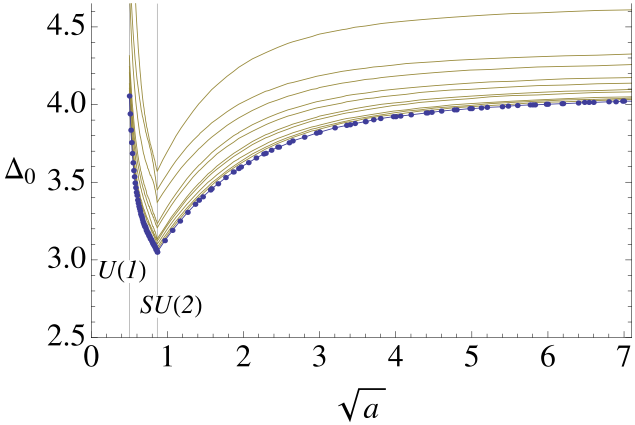
$$\Delta_{\text{dt}} = 4 - \frac{16}{N^2} + \ldots$$

[D'Hoker et al (1999)]



$\tau = \frac{\theta}{2\pi} + \frac{i}{g}$

Results for the first unprotected scalar with $\mathbf{R} = 0$



[Beem, Rastelli, BvR (2013)]

The $\mathcal{N} = 4$ maxibootstrap

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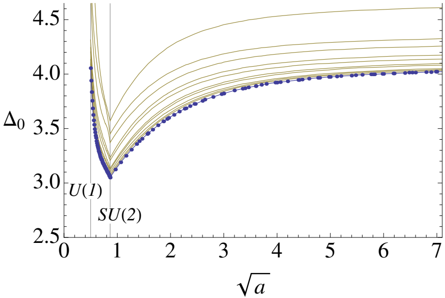

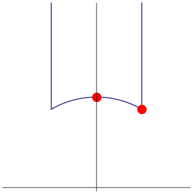

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Conjecture: for $N = 2$, with $(g, \theta) = (1, 0)$ or $(2/\sqrt{3}, 1/2)$, we have

$$\boxed{\Delta_{\text{kon}} \simeq 3.05}$$

The $\mathcal{N} = 2$ maxibootstrap

In theories with $\mathcal{N} = 2$ superconformal symmetry, a flavor symmetry multiplet contains a dimension two scalar $\mu^{A,IJ}$ in the triplet of $\mathfrak{su}(2)_R$ known as the moment map.

Its four-point function is decomposed into a set of meromorphic functions $f^{ABCD}(z_i)$ and unconstrained functions $\mathcal{G}^{ABCD}(x_{ij})$. As before, the meromorphic functions are fixed from the chiral algebra and we analyze the two-variable functions numerically.

Input parameters:

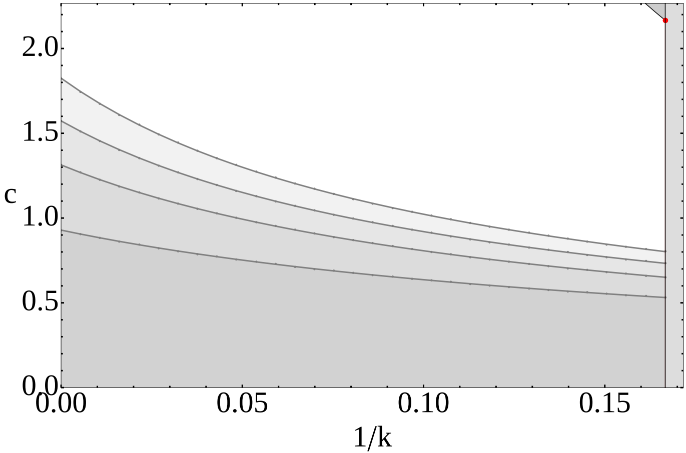
- global symmetry algebra G
- flavor central charge k
- central charge c

Output:

- Can the theory exist?
- Bounds on e.g. scalar operators
- ...

The $\mathcal{N} = 2$ maxibootstrap

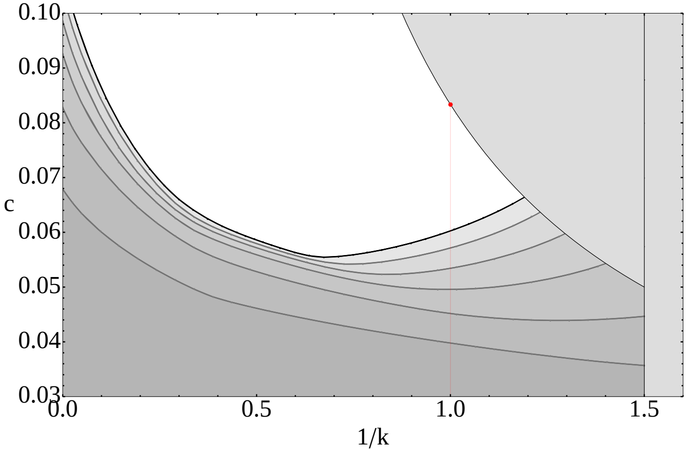
Global symmetry group: E_6



[Beem, Lemos, Liendo, Rastelli, BvR (to appear)]

The $\mathcal{N} = 2$ maxibootstrap

Global symmetry group: $\mathfrak{su}(2)$



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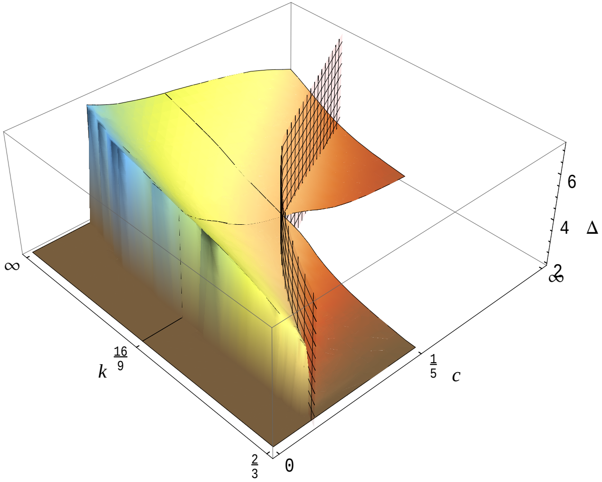


[Beem, Lemos, Liendo, Rastelli, BvR (to appear)]

Other results

Other work on the superconformal maxibootstrap:

- $\mathcal{N} = 1$ in $d = 4$
  [Poland, Simmons-Duffin, Vichi (2010-2011);
  Berkooz, Yacoby, Zait (2014)]
- $\mathcal{N} = 4$ in $d = 4$ [Alday, Bissi (2013-2014)]
- $\mathcal{N} = 8$ in $d = 3$ [Chester, Lee, Pufu, Yacoby (2014)]
- $(2, 0)$ in $d = 6$ [Beem, Lemos, Rastelli, BvR (to appear)]
- ...

Lots of analytic work, for example on the computation of superconformal blocks, the lightcone limit, etc.
[Fitzpatrick, Kaplan, Khandker, Komargodski, Li, Poland, Simmons-Duffin, Zhiboedov, ... (2010-2014)]

Conclusions

We are only beginning to understand the consequences of crossing symmetry for superconformal theories. The results so far have been very promising.

Highlights so far:

- Infinite chiral symmetry in four, six and two dimensions
- Quantitative results for strongly coupled non-planar theories

Grand questions:

- Spaces of SCFTs?
- Microscopic derivation of AGT?

Conclusion

The superconformal bootstrap program works!

Extra slides

Definition

In fact, by restricting ourselves to $\mathbf{R}^2 \subset \mathbf{R}^4$ we preserve

$$\mathfrak{sl}(2)_L \times \mathfrak{sl}(2|2)_R \subset \mathfrak{su}(2,2|2)$$

The entire $\mathfrak{sl}(2)_L$ is closed

$$[\mathbb{Q}, L_{-1}] = 0 \qquad [\mathbb{Q}, L_0] = 0 \qquad [\mathbb{Q}, L_1] = 0$$

and the entire *twisted* $\mathfrak{sl}(2)_R$ is exact

$$\bar{L}_{-1} + R^- = \{\mathbb{Q}, \ldots\} \qquad \bar{L}_0 - R = \{\mathbb{Q}, \ldots\} \qquad \bar{L}_1 - R^+ = \{\mathbb{Q}, \ldots\}$$

→ We have a *superconformal* twist. Notice that

$$L_0 = \frac{1}{2}\left(\mathcal{H} + \mathcal{M}_+{}^+ + \mathcal{M}_{\dot{+}}{}^{\dot{+}}\right) \qquad \to \qquad h = \frac{1}{2}(\Delta + j_1 + j_2)$$

$$\bar{L}_0 - R = \frac{1}{2}\left(\mathcal{H} - 2R - \mathcal{M}_+{}^+ - \mathcal{M}_{\dot{+}}{}^{\dot{+}}\right) = \frac{1}{2}\{\mathbb{Q}, \mathbb{Q}^\dagger\} = 0$$

→ The twist works for any superconformal algebra with an $\mathfrak{sl}(2|2)$ subalgebra, so chiral algebras also exist for $(2,0)$ SUSY in $d = 6$ and $(0,4)$ SUSY in $d = 2$.

Universal properties

- Consider a flavor symmetry multiplet containing (among other operators)

$$\mu^A_{IJ}(x) \qquad J^A_\mu(x)$$

  Here $\mu^A_{IJ}(x)$ is the moment map with $\Delta = 2$ and $R = 1$. In the chiral algebra it becomes a dimension one current $j^A(z)$. The four-dimensional OPE determines

$$j^A(z)j^B(0) \sim \frac{k_{2d}}{z^2} + \frac{f^{AB}{}_C}{z} j^C(0)$$

  so we find an *affine Kac-Moody algebra* with $k_{2d} = -k_{4d}/2$.

- Similarly, the $\mathfrak{su}(2)_R$ symmetry current $J^{IJ}_\mu(x)$ becomes a stress tensor $T(z)$ with

$$T(z)T(0) \sim \frac{c_{2d}}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}$$

  so we find a *Virasoro algebra* with $c_{2d} = -12c_{4d}$.

- The elements of the *Higgs branch* chiral ring correspond to Virasoro primaries, with null states indicating relations.
- The elements of the *Coulomb branch* chiral ring are *not* Schur.

Gauging prescription

In a free vectormultiplet the Schur operators are the gauginos λ^A_α and $\tilde{\lambda}^A_{\dot{\alpha}}$. Defining

$$b^A(z) \sim v(\bar{z}) \cdot \lambda^A_+(z,\bar{z}) \qquad\qquad \partial c^A(z) \sim v(\bar{z}) \cdot \tilde{\lambda}^A_{\dot{+}}(z,\bar{z})$$

we find a (small) (b,c) ghost system with dimensions $(1,0)$ and OPE

$$b^A(z)c^B(0) \sim \frac{\delta^{AB}}{z}$$

Gauging of a flavor symmetry now corresponds to a restriction to the cohomology of

$$Q_{\text{BRST}} = \frac{1}{2\pi i} \oint dz \left(c_A J^A - \frac{1}{2} f^{AB}{}_C c_A c_B b^C \right)$$

in the chiral algebra.

This operator is nilpotent precisely if the four-dimensional beta function vanishes!

The big picture

For specific theories we have precise claims for the chiral algebra:

- $\mathfrak{su}(2)$ with four fundamental flavors:

  $\mathfrak{so}(8)_{-2}$ AKM algebra

- Minahan-Nemeschansky E_6 theory:

  $(\mathfrak{e}_6)_{-3}$ AKM algebra

- $\mathfrak{su}(N_c)$ with $N_f = 2N_c$ fundamental flavors:

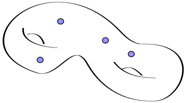
  $\mathfrak{u}(1) \times \mathfrak{su}(N_f)_{-N_c}$ AKM algebra + baryons

- $\mathcal{N} = 4$ SYM theories:

  small $\mathcal{N} = 4$ algebra + primaries from half BPS chiral ring

Furthermore, for class S theories:

- Chiral algebra of T_N?
- Gauging: as before
- Maximal puncture: AKM algebra at $k = -h^\vee$
- Closing puncture: quantum Drinfeld-Sokolov reduction



Conformal multiplets

The conformal group in four dimensions is $SU(2,2) \sim SO(4,2)$ with generators

$$P^\mu \qquad M_{\mu\nu} \qquad D \qquad K_\mu$$

Consider the set of local operators in a CFT

$$\{\mathcal{O}_i^{\Delta,j_1,j_2}(x)\}$$

where $[D,\mathcal{O}] = \Delta\mathcal{O}$ and (j_1,j_2) are the Lorentz quantum numbers.

They can be organized in conformal multiplets consisting of

primary:	$[K_\mu, \mathcal{O}_i^{\Delta,j_1,j_2}(0)] = 0$
descendants:	$\partial_{\mu_1} \ldots \partial_{\mu_n} \mathcal{O}_i^{\Delta,j_1,j_2}(0)$

Sometimes representations are *short*, e.g.

$$\partial_\mu J^\mu = 0 \qquad \Box\phi = 0$$

and then the dimensions are fixed

$$[D, J^\mu] = 3 \qquad [D,\phi] = 1$$

Superconformal multiplets

The $\mathcal{N}=4$ superconformal group in four dimensions is $PSU(2,2|4)$ with generators

$$P^\mu \qquad M_{\mu\nu} \qquad D \qquad K_\mu \qquad Q^I_\alpha \qquad \tilde{Q}_{\dot\alpha I} \qquad S^\alpha_I \qquad \tilde{S}^{\dot\alpha I} \qquad R^J_I$$

The local operators

$$\{\mathcal{O}_i^{\Delta,j_1,j_2,\mathbf{R}}(x)\}$$

can be organized in superconformal multiplets consisting of

superconformal primary: $[S^\alpha_I, \mathcal{O}_i^{\Delta,j_1,j_2,\mathbf{R}}(0)] = 0 \quad [\tilde{S}^{\dot\alpha I}, \mathcal{O}_i^{\Delta,j_1,j_2,\mathbf{R}}(0)] = 0$

superconformal descendants: $Q\ldots Q\tilde{Q}\ldots\tilde{Q}\,\mathcal{O}_i^{\Delta,j_1,j_2,\mathbf{R}}$

Generic superconformal multiplets contain 2^8 conformal multiplets.

Sometimes representations are *short* or *semishort*, e.g.

$$Q^3_\alpha\mathcal{O} = Q^4_\alpha\mathcal{O} = 0 \qquad\qquad \tilde{Q}_{\dot\alpha 3}\mathcal{O} = \tilde{Q}_{\dot\alpha 4}\mathcal{O} = 0$$

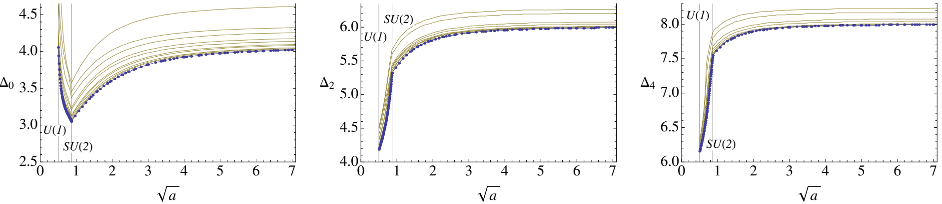
(but $Q^1_\alpha\mathcal{O}$, $Q^2_\alpha\mathcal{O}$, $\tilde{Q}_{\dot\alpha 1}\mathcal{O}$, $\tilde{Q}_{\dot\alpha 2}\mathcal{O} \neq 0$).

Then there are relations between the quantum numbers. For this case:

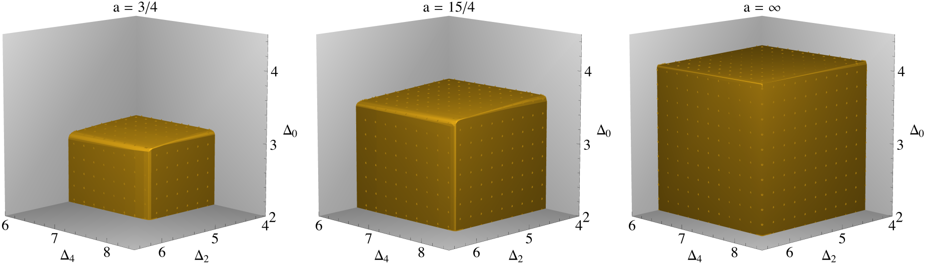
$$j_1 = j_2 = 0 \qquad R = [0,p,0] \qquad \Delta = p$$

(These are the chiral primaries, with $\mathcal{O} = \mathrm{Tr}\,(\Phi^{\{I_1}\ldots\Phi^{I_p\}})$ in $\mathcal{N}=4$ SYM.)

Results for the first three spins

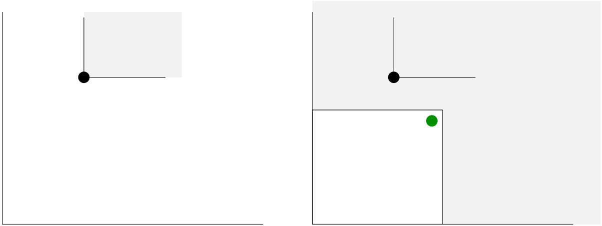


Combining spins



Why the cubes?

We are looking at *bounds*



$\rightarrow$ there is a special solution to crossing symmetry at the corner

We conjecture that it corresponds to a self-dual point of $\mathcal{N} = 4$ SYM.

This leads e.g to

$$\Delta \lesssim 2.90$$

for the Konishi operator $\mathrm{Tr}\,(\Phi^I \Phi_I)$ in $SU(2)$ $\mathcal{N} = 4$ SYM at $\tau = i$ or at $\tau = \exp(i\pi/3)$.

Not in disagreement with resumming the four-loop result...
[Beem, Rastelli, Sen, BvR (2013), Alday, Bissi (2013)]

Can we find the rest of the conformal manifold?

The conformal bootstrap

We are interested in correlation functions of local operators

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle$$

These are heavily constrained by conformal symmetry, for example

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(y) \rangle = \frac{\delta_{ij}}{(x-y)^{2\Delta}}$$

Conformal invariance further guarantees the existence of a convergent *operator product expansion* (or OPE) of the form

$$\mathcal{O}_i(x) \mathcal{O}_j(y) \sim \sum_k \lambda_{ij}^{\ k} \, C[x-y, \partial_y] \mathcal{O}_k(y)$$

We can use the OPE to decompose correlation functions as

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle$$
$$= \sum_k \lambda_{12}^{\ k} \lambda_{34}^{\ k} C[x_1 - x_2, \partial_2] C[x_3 - x_4, \partial_4] \langle \mathcal{O}_k(x_2) \mathcal{O}_k(x_4) \rangle$$

The conformal bootstrap

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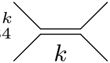

The conformal bootstrap

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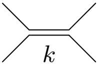
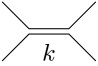


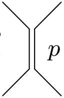
The conformal bootstrap

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle$$

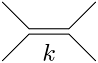
$$= \sum_k \lambda_{12}^{\;\;k}\lambda_{34}^{\;\;k}$$

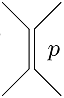


$$= \sum_p \lambda_{13}^{\;\;p}\lambda_{24}^{\;\;p}$$



The conformal bootstrap

$$\sum_k \lambda_{12}^{\ k} \lambda_{34}^{\ k}$$


$$= \sum_p \lambda_{13}^{\ p} \lambda_{24}^{\ p}$$


Crossing symmetry: an infinite set of constraints for Δ_k and $\lambda_{ij}^{\ k}$

Can we solve them? Could we determine the theory using

- global symmetries
- unitarity
- crossing symmetry

and nothing else? In other words, can we *bootstrap* the theory?

[Ferrara, Gatto, Grillo, Parisi (1972); Polyakov (1974)]