Thermodynamics of the BMN matrix model at strong coupling

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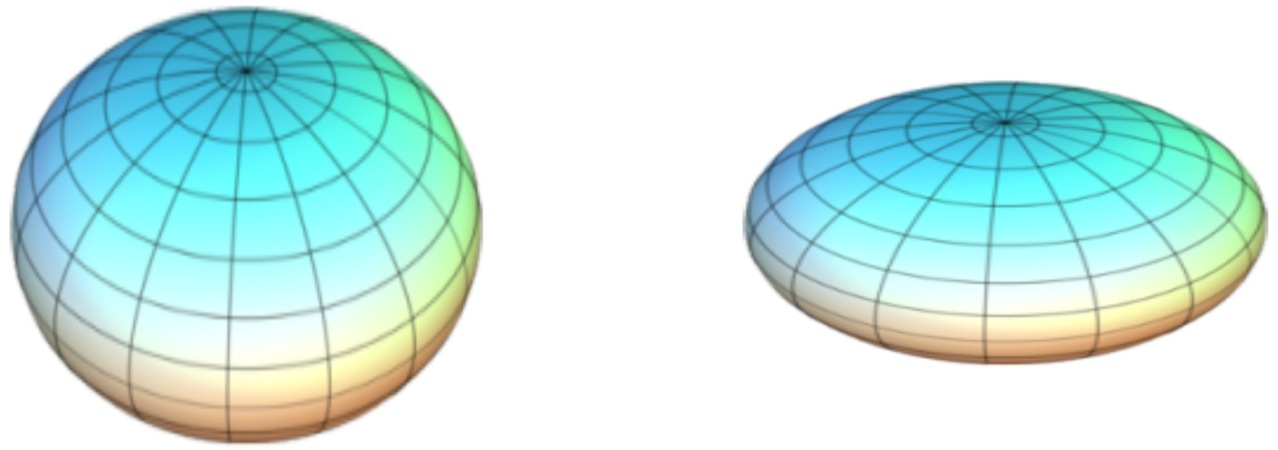
Work with L. Greenspan, J. Penedones and J. Santos



HoloGrav 2014, Reykjavik - August 2014

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- Gauge/gravity duality as definition of quantum gravity in AdS

Dual CFT is renormalizable and unitary. Problem: how to decode the hologram?

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Idea: Study thermodynamics of black holes dual to Matrix Quantum Mechanics that can be simulated on a computer using Monte-Carlo methods.

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$X^i \equiv SU(N)$ bosonic matrices $(i = 1, \ldots, 9)$

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- Consider theory on Euclidean time circle

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$$\lambda = g_{YM}^2 N = \frac{g_s N}{(2\pi)^2 l_s^3} \equiv \text{mass}^3$$

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- Can put theory on a computer using Monte Carlo simulations

[Catterall, Wiseman ´07,´08,´09; Anagnostopoulos et al ´07; Hanada et al ´08,´13]

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- 11D SUGRA solution (near horizon geometry of non-extremal D0-brane)

$$ds^2 = \frac{dr^2}{f(r)} + r^2 d\Omega_8^2 + \left(\frac{R}{r}\right)^7 dz^2 + f(r)dt\left(2dz - \left(\frac{r_0}{R}\right)^7 dt\right)$$

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^7 , \qquad \left(\frac{R}{\ell_s}\right)^7 = 60\pi^3 g_s N\, , \qquad \left(\frac{r_0}{\ell_s}\right)^5 = \frac{120\pi^2}{49}\left(2\pi g_s N\right)^{\frac{5}{3}} \tau^2$$

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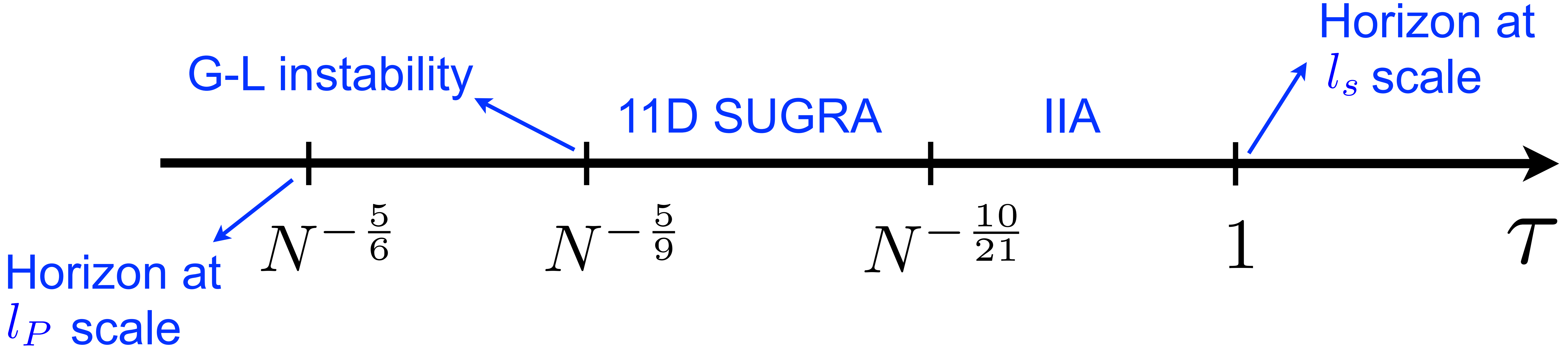
- Classical gravity domain (at horizon)

$$\tau = T/\lambda^{1/3}$$



G-L instability

11D SUGRA

IIA

Horizon at l_s scale

Horizon at l_P scale

$N^{-\frac{5}{6}}$ $N^{-\frac{5}{9}}$ $N^{-\frac{10}{21}}$ 1 τ

$$l_s^2 \mathcal{R}(r_0) \ll 1 \;\Rightarrow\; \tau \ll 1$$

$$g_s e^{\phi(r_0)} \ll 1 \;\Rightarrow\; \tau \gg N^{-\frac{10}{21}}$$

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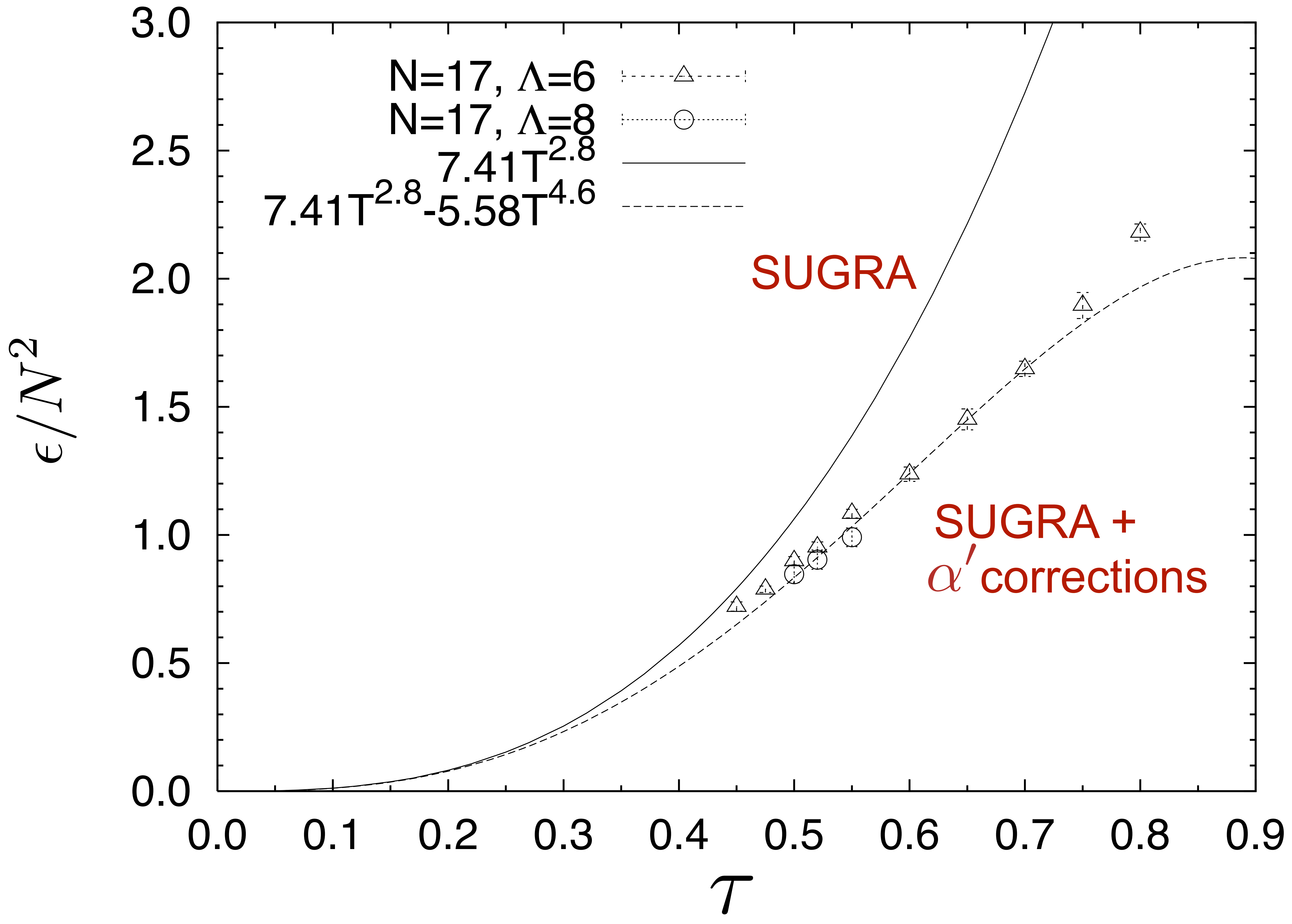

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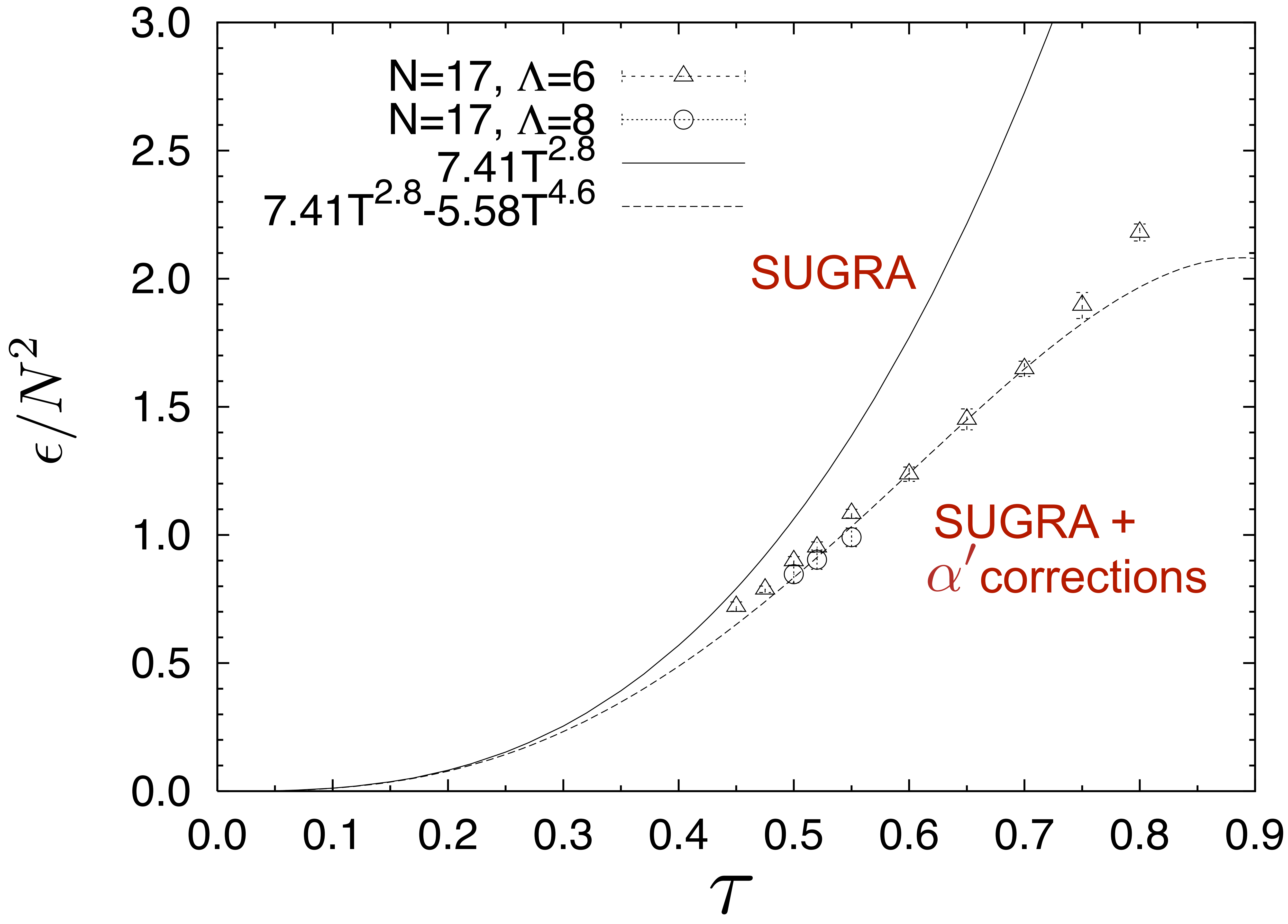

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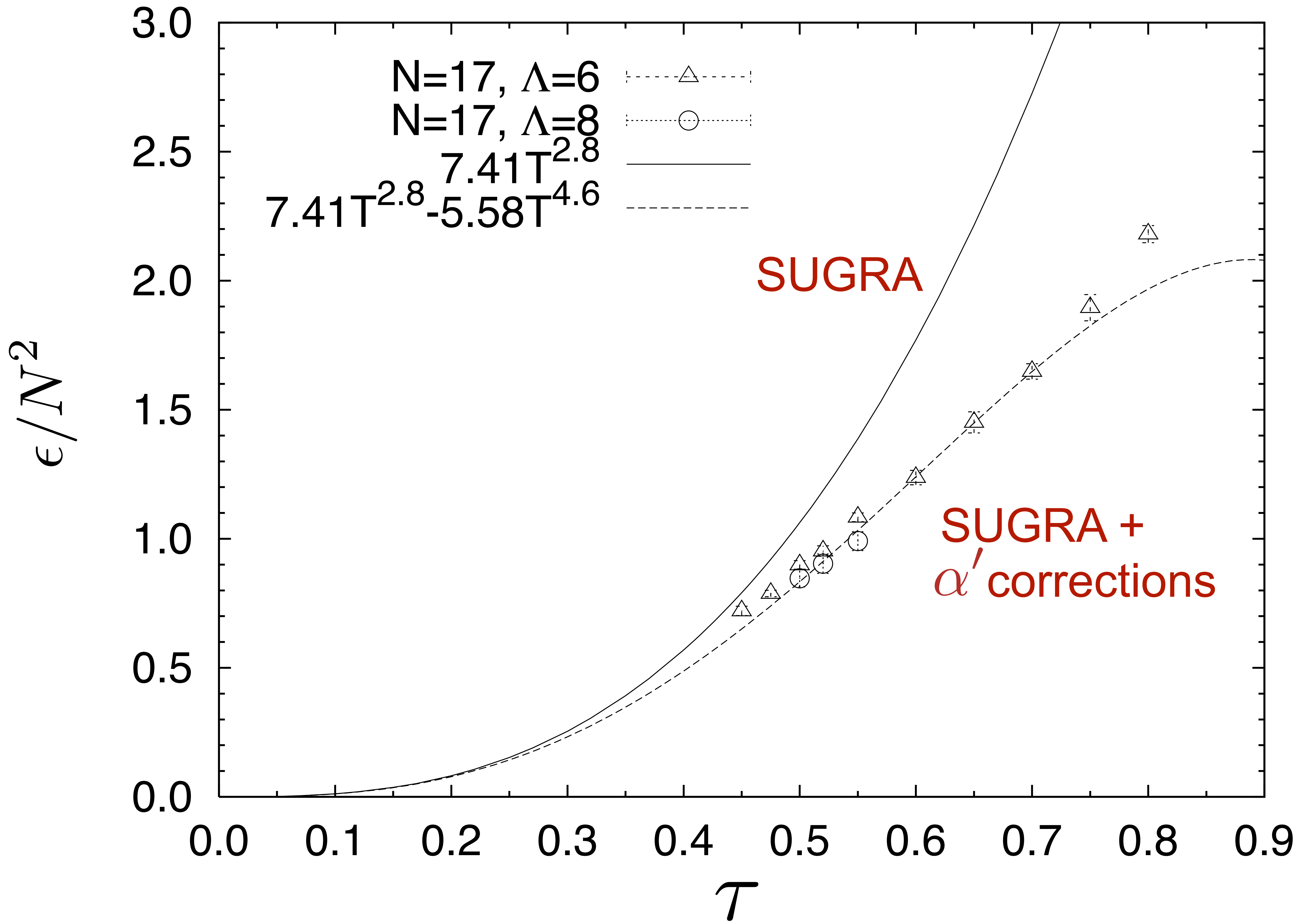

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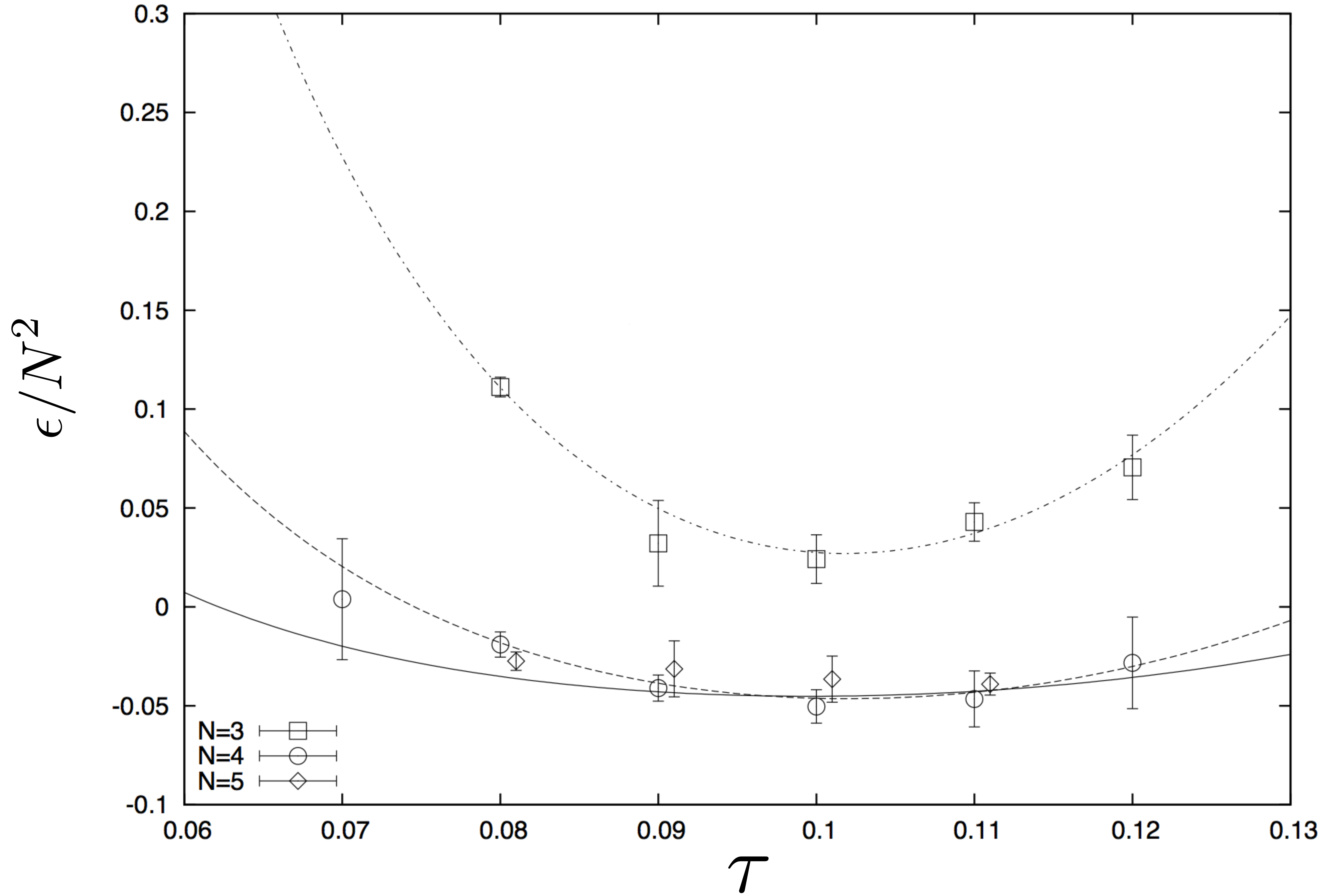

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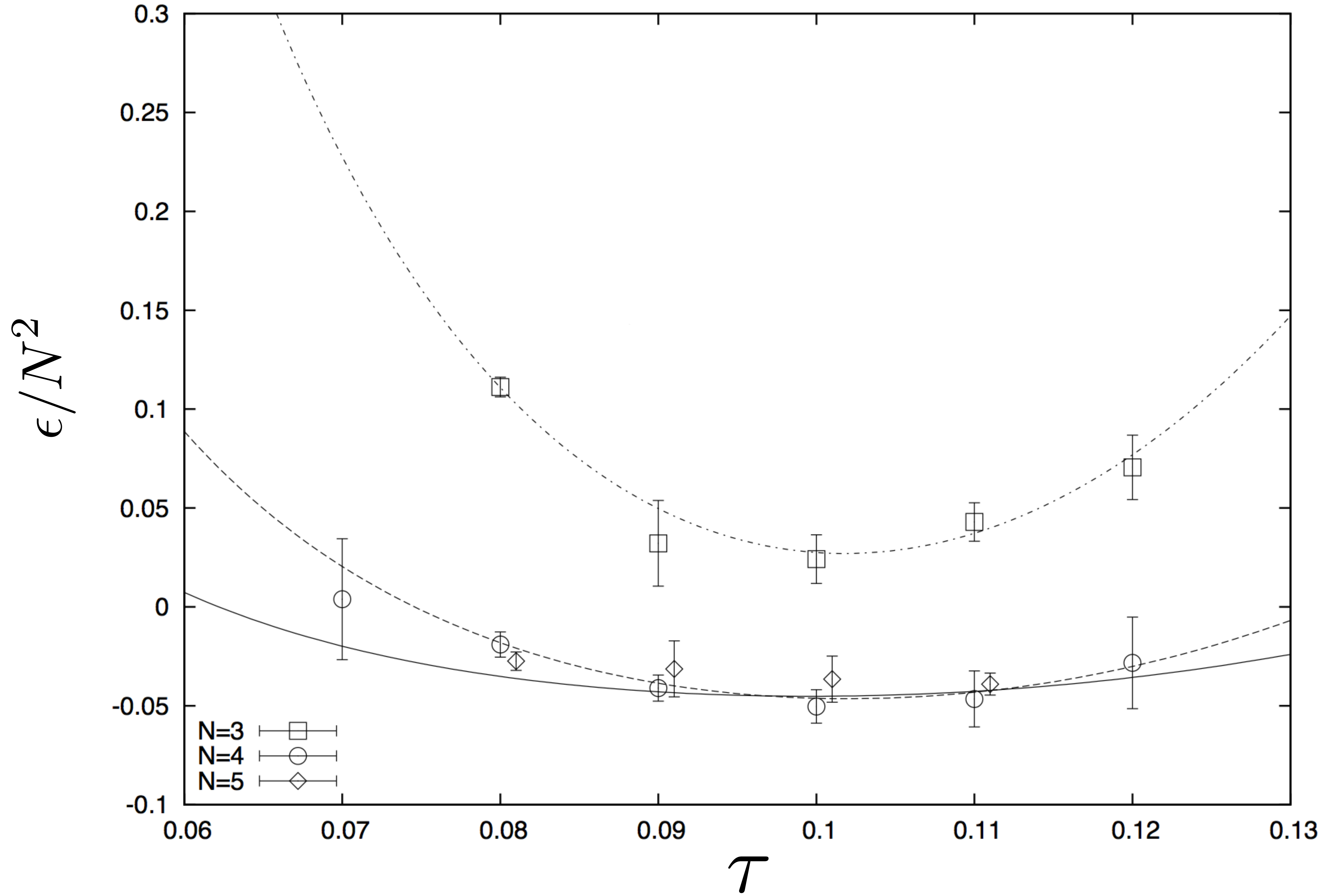

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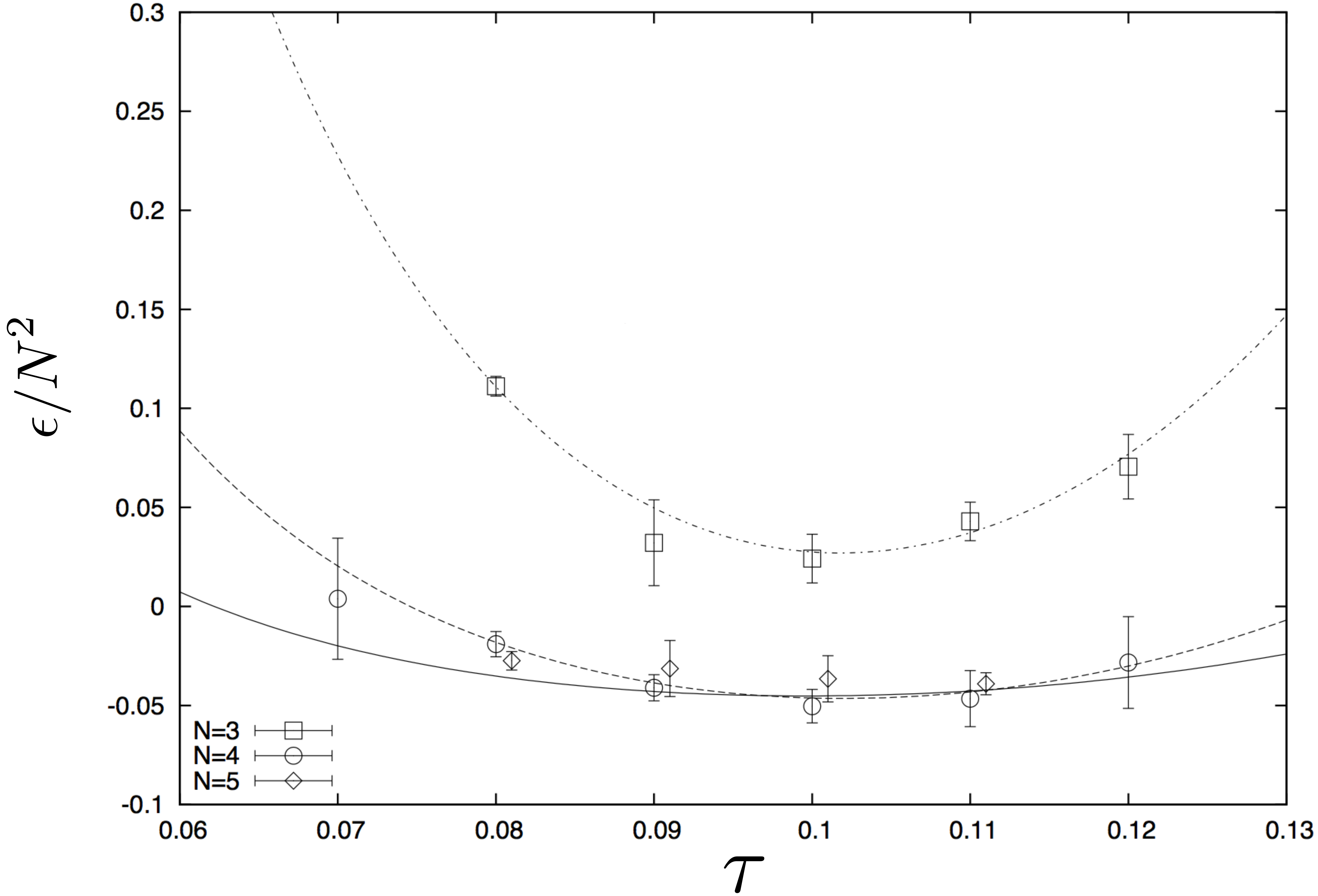

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$$\frac{F(T,r)}{N^2} \sim \mathcal{F}_{finite}(T) + \frac{9}{N}\ln r$$

Instability corresponds to Hawking radiation of D0-branes. At large N this is suppressed and black hole is stable (positive specific heat).

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- **Today’s talk is about BMN matrix model** [Berenstein, Maldacena, Nastase ´02]

Mass deformation resolves IR divergence - canonical ensemble well defined.

Much richer thermodynamics with a 1st order phase transition
(at large N there are two dimensionless parameters).

BMN matrix model

$$S = S_{D0} - \frac{N}{2\lambda}\int dt\,\mathrm{Tr}\left[\frac{\mu^2}{3^2}(X^i)^2 + \frac{\mu^2}{6^2}(X^a)^2 + \frac{\mu}{4}\Psi^\alpha\left(\gamma^{123}\right)_{\alpha\beta}\Psi^\beta + i\frac{2\mu}{3}\epsilon_{ijk}X^iX^jX^k\right]$$

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$a = 4, \ldots, 9 \quad i = 1, 2, 3$

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Canonical ensemble is well defined and may still be simulated on a computer.

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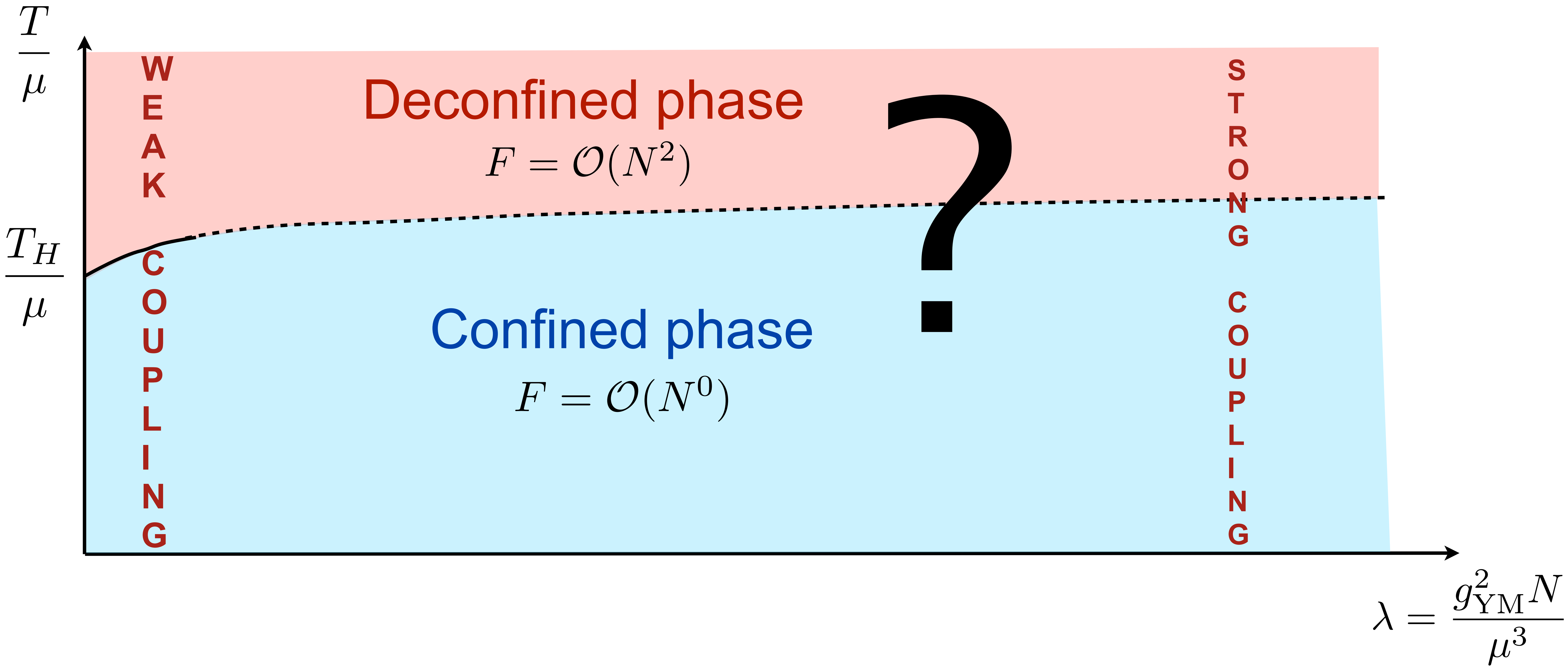

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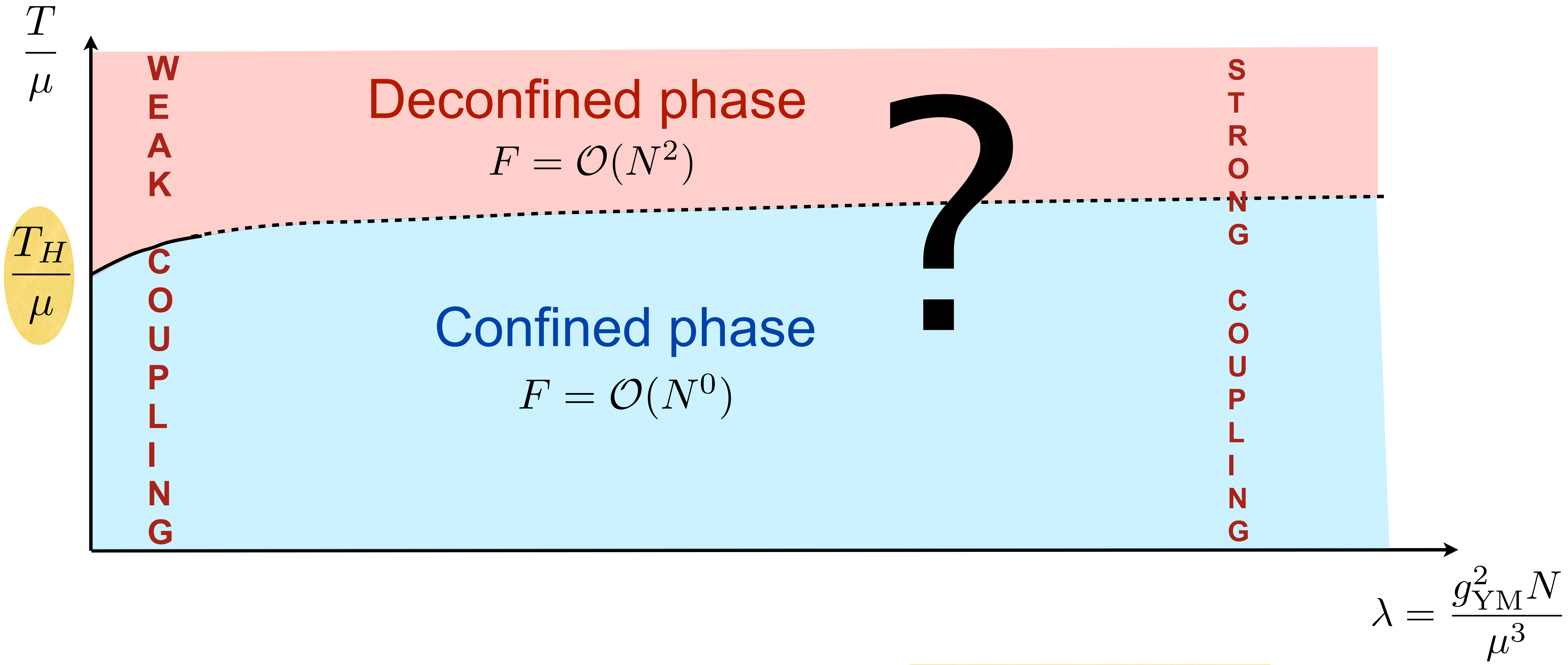

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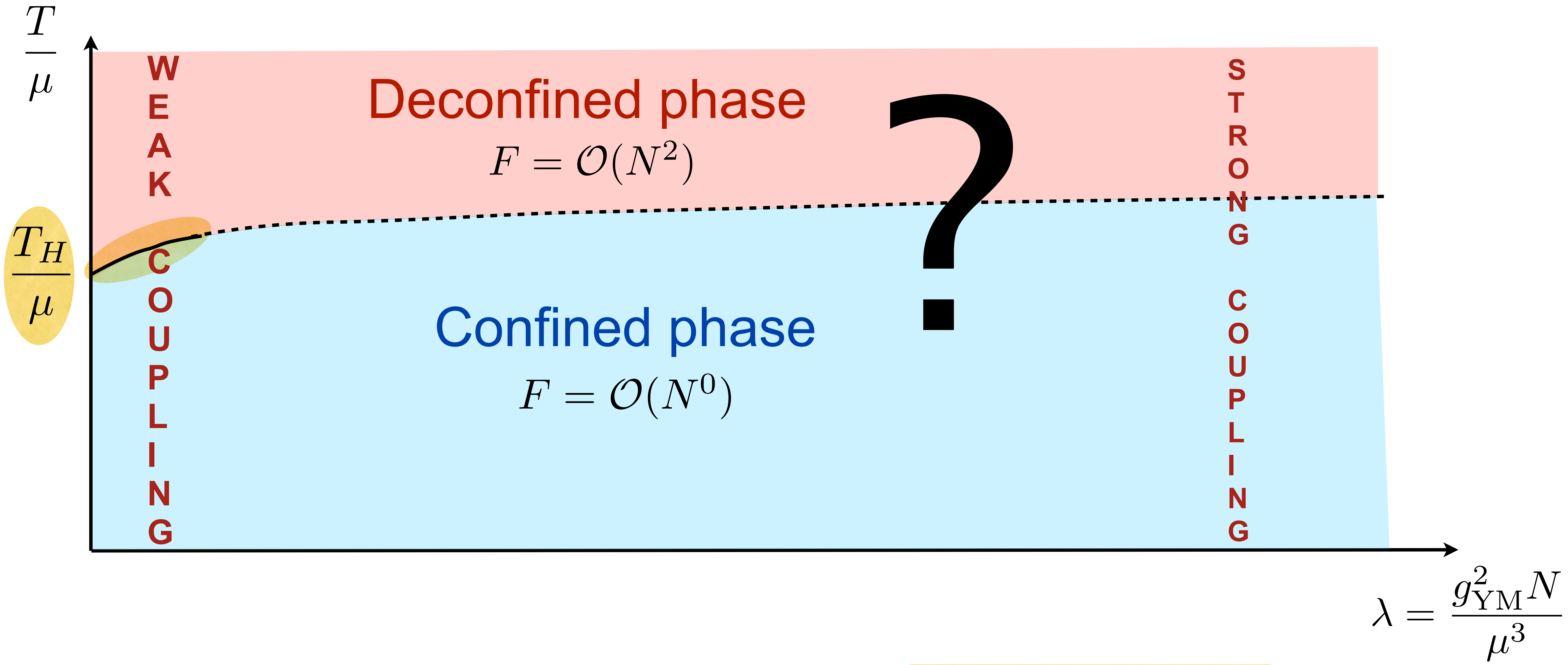

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First-order phase transition at
$$\frac{T_c}{\mu} = \frac{1}{12 \log 3}\left[1 + \frac{2^6 5}{3^4}\lambda - c\,\lambda^2 + O(\lambda^3)\right] \approx 0.076 + \mathcal{O}(\lambda)$$

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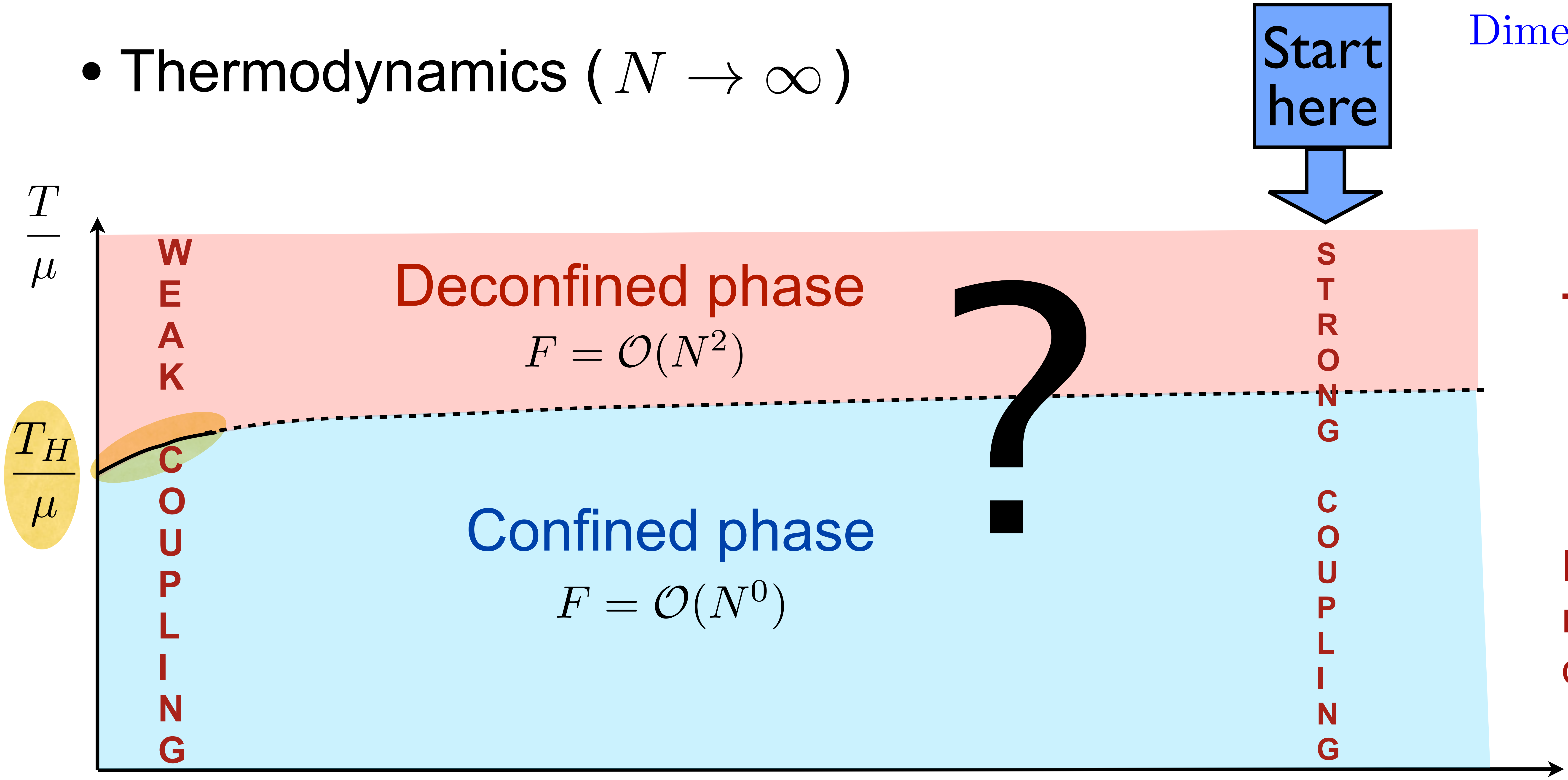

Today: strongly coupled limit

$$\mu \to 0\,, \qquad \frac{T}{\mu} \text{ fixed and large}$$

Dual geometry is SO(9) invariant non-extremal D0-brane with deformation turned on

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Need back-reaction to decrease temperature and study phase transition at strong coupling. In particular,

$$SO(9) \to SO(6) \times SO(3)$$

- Ansatz for 11D SUGRA

$$ds^2 = -A\,\frac{(1-y^7)}{y^7}d\eta^2 + T_4\,y^7\left[d\zeta + \Omega\,\frac{(1-y^7)d\eta}{y^7}\right]^2$$

$$+\frac{1}{y^2}\left[B\,\frac{(dy+F dx)^2}{(1-y^7)y^2} + T_1\,\frac{4dx^2}{2-x^2} + T_2\,x^2(2-x^2)d\Omega_2^2 + T_3\,(1-x^2)^2 d\Omega_5^2\right]$$

$$C = (M\,d\eta + L\,d\zeta)\wedge d^2\Omega_2$$

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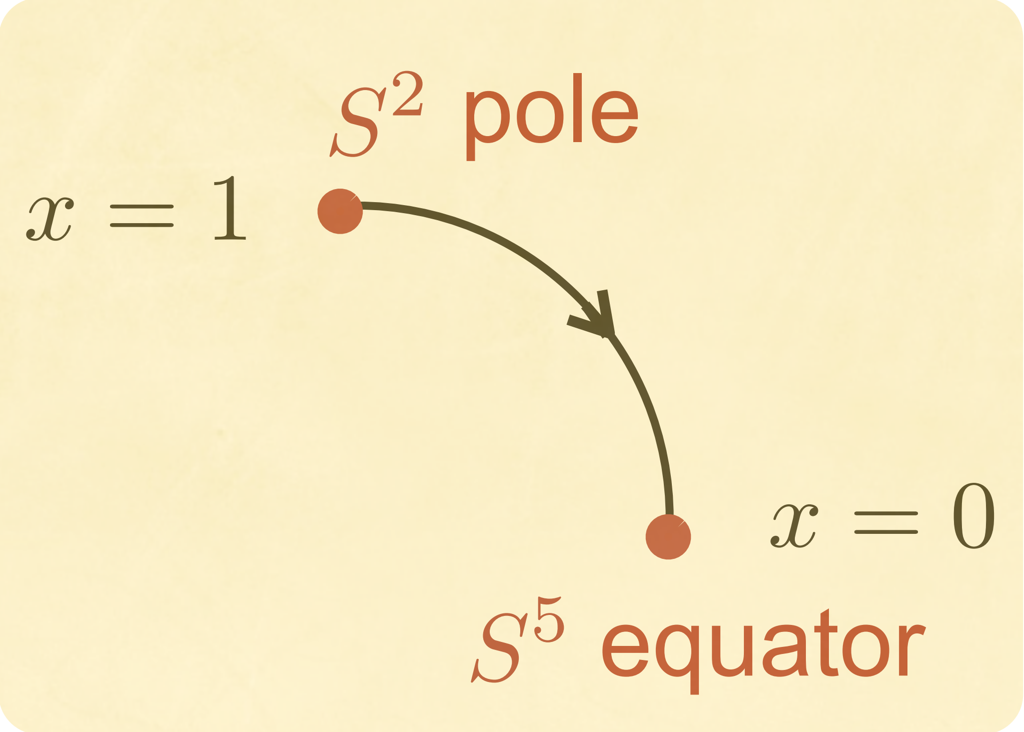

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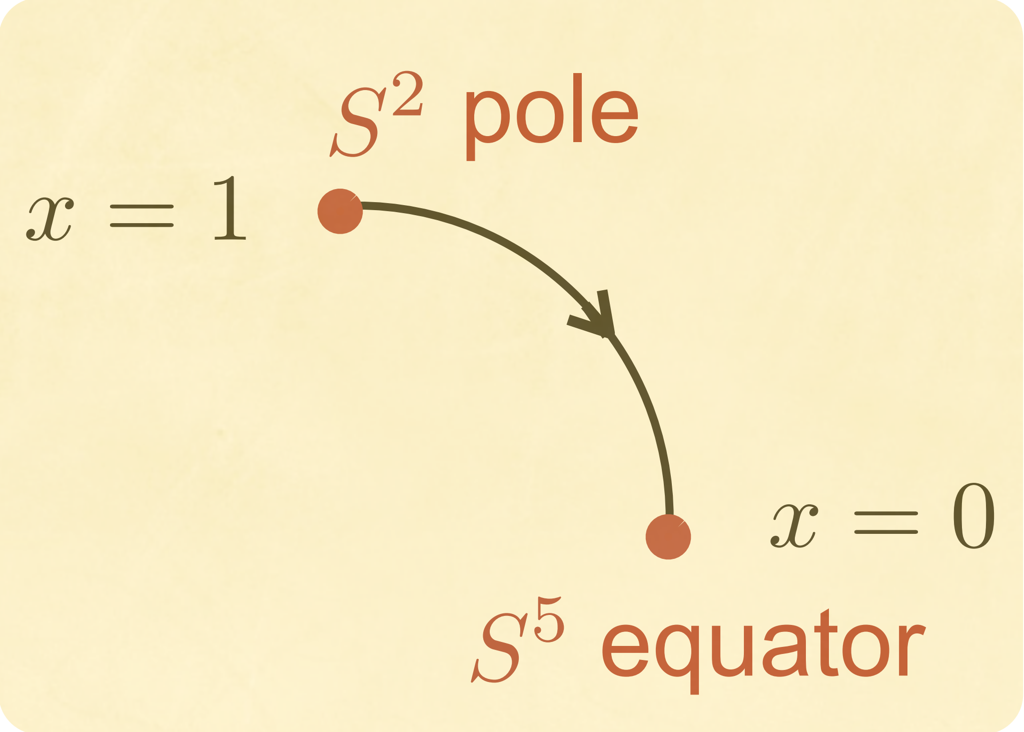

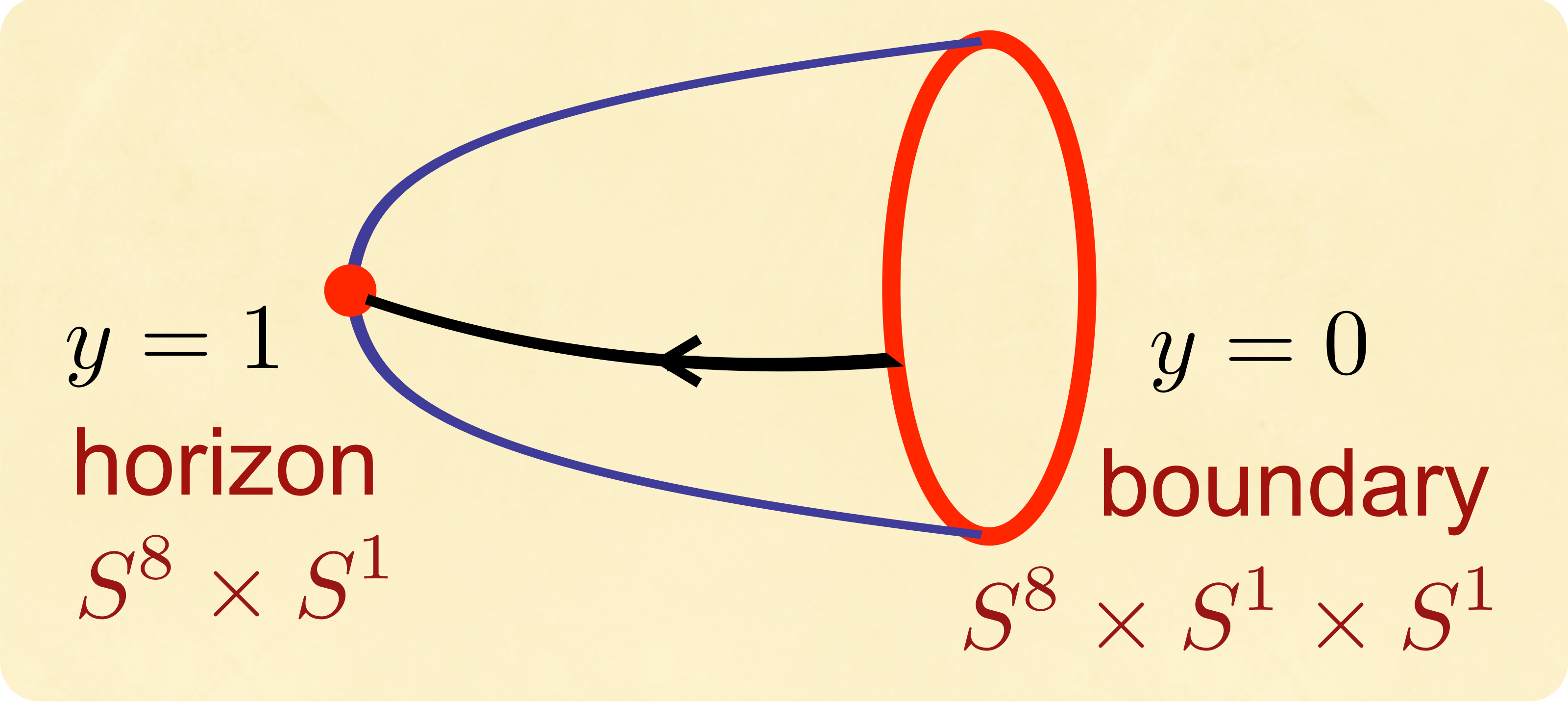

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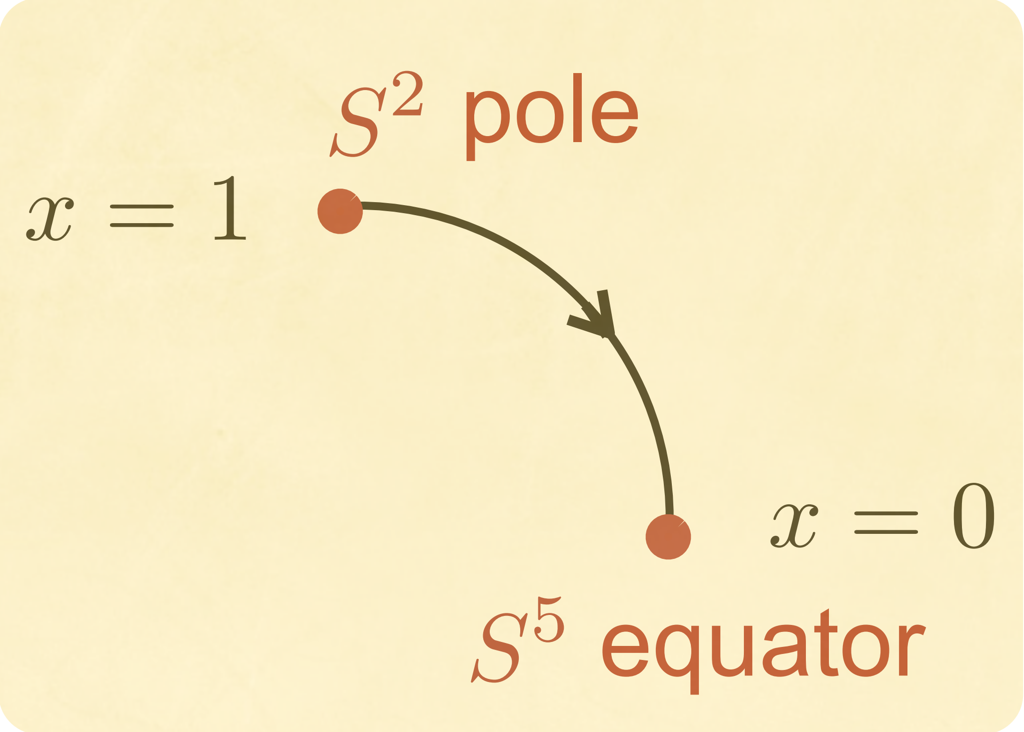

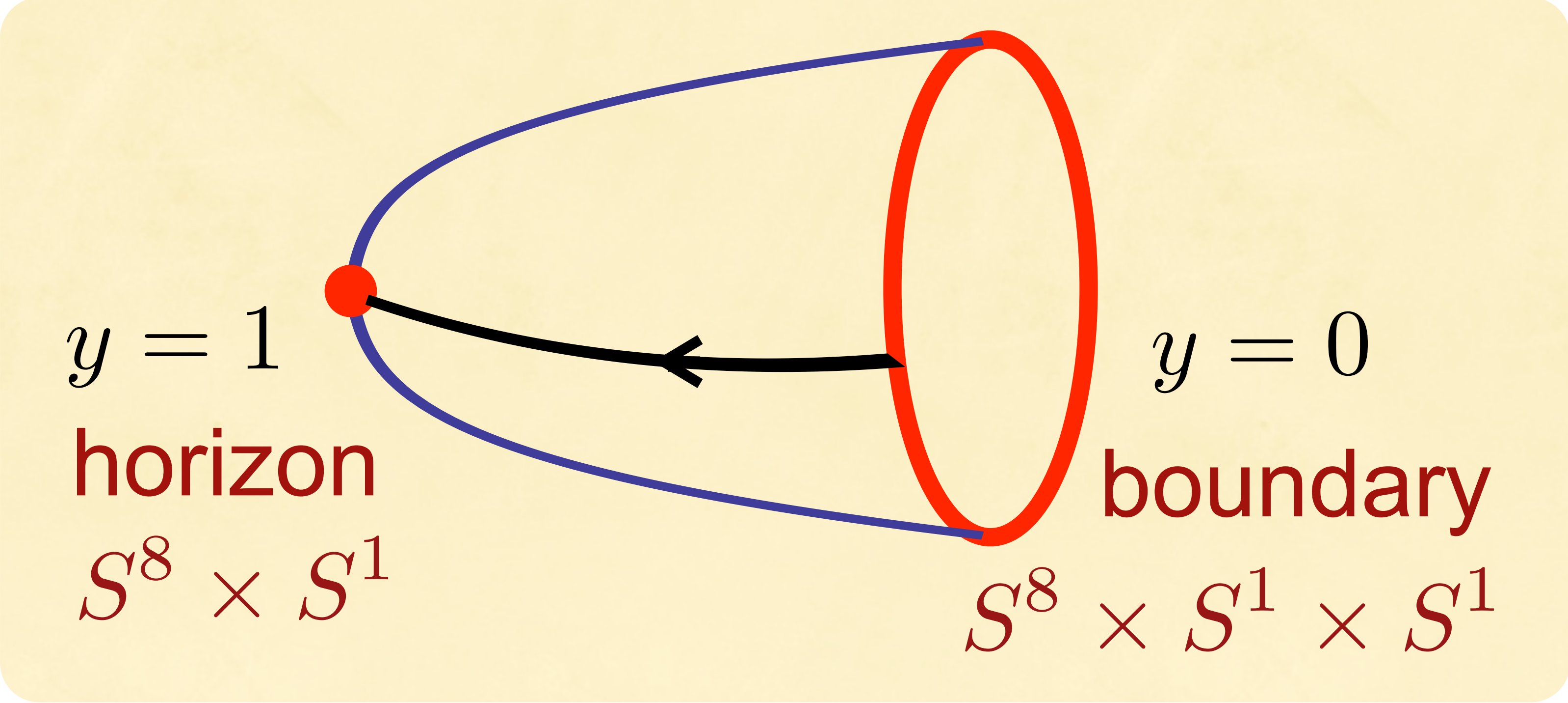

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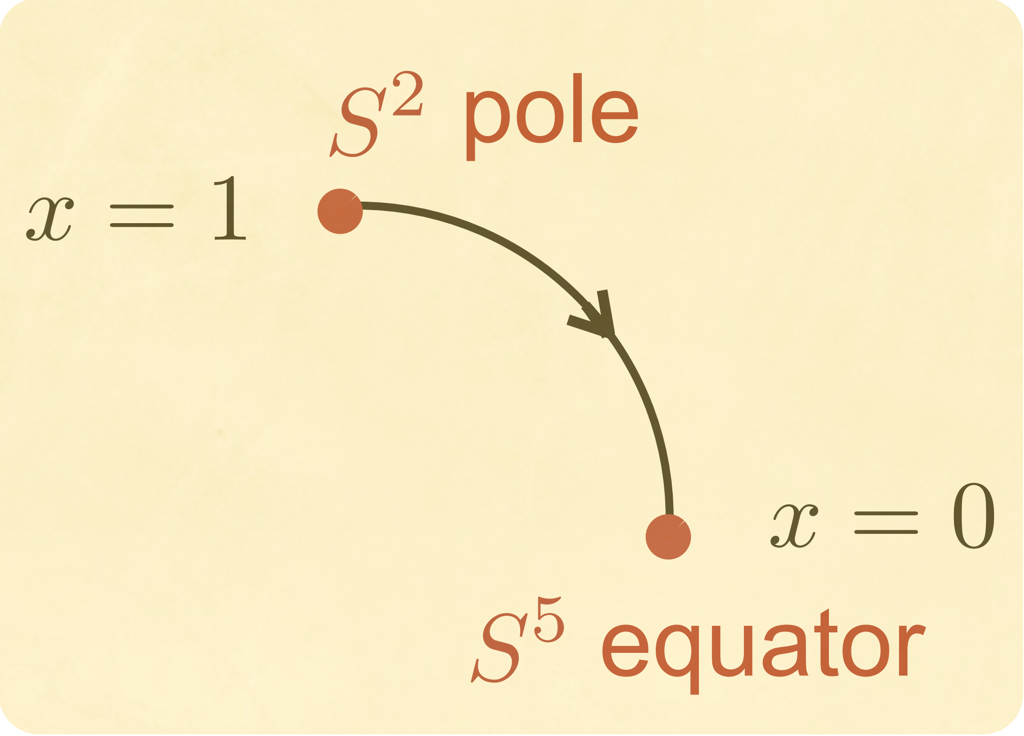

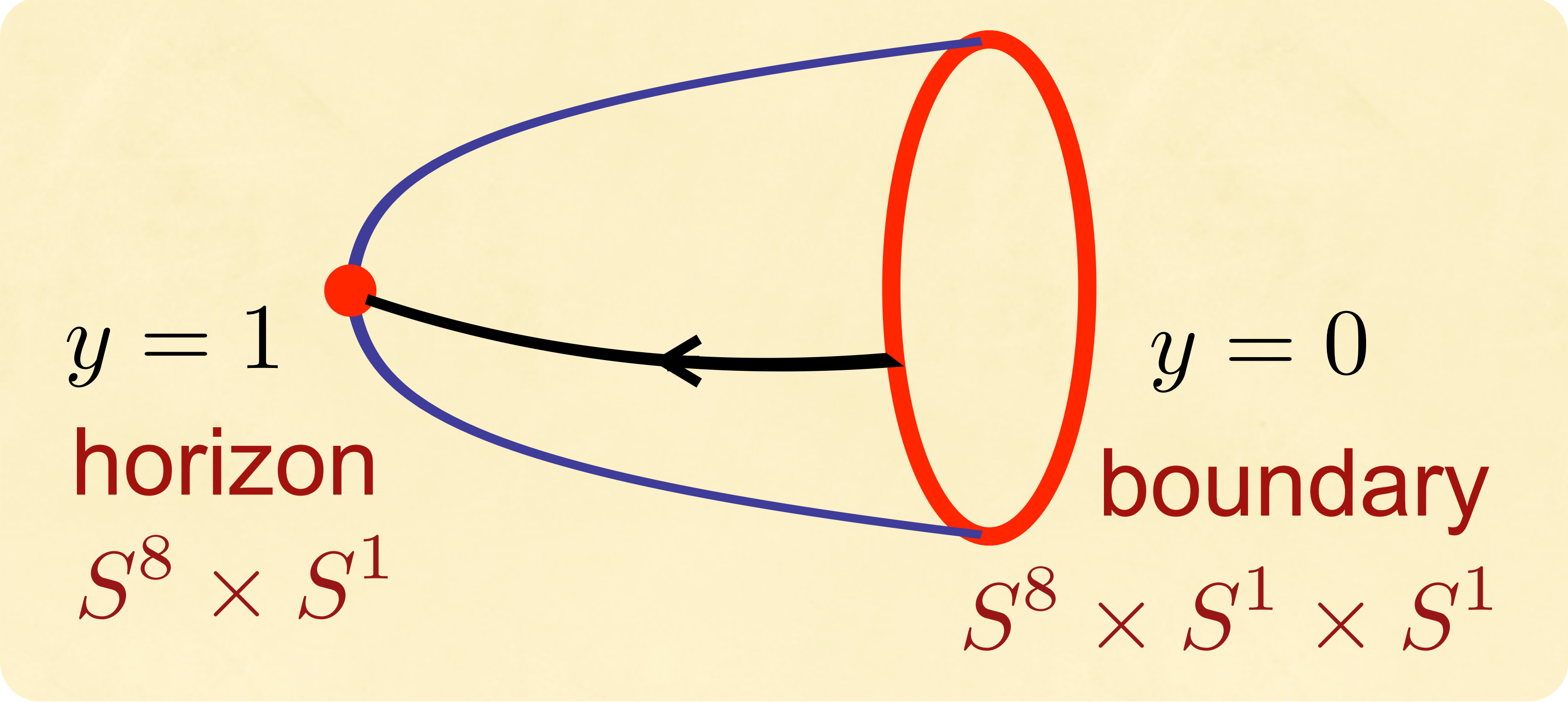

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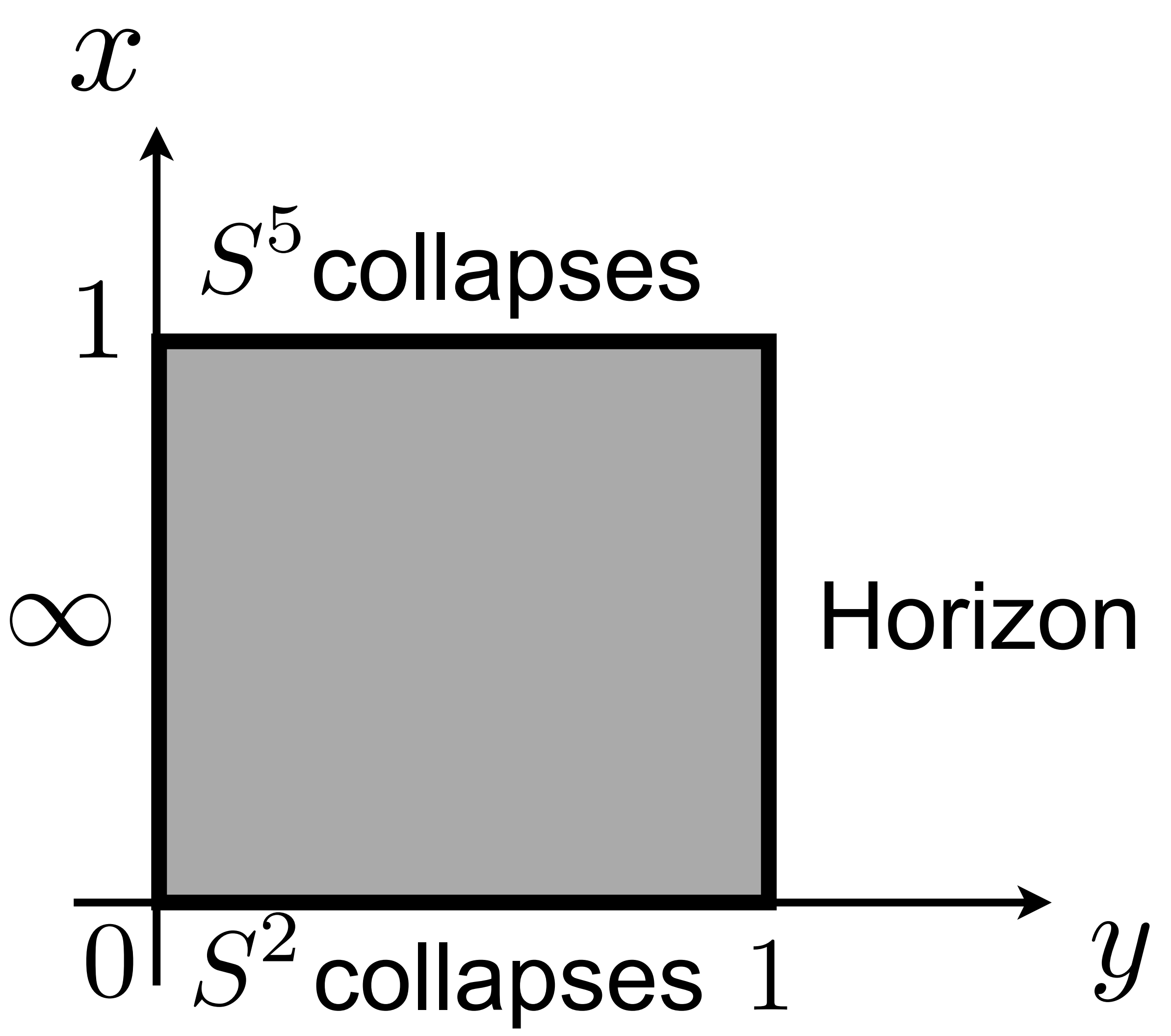
Tailored to numerical implementation
(domain of unknown is the unit square; everything dimensionless)



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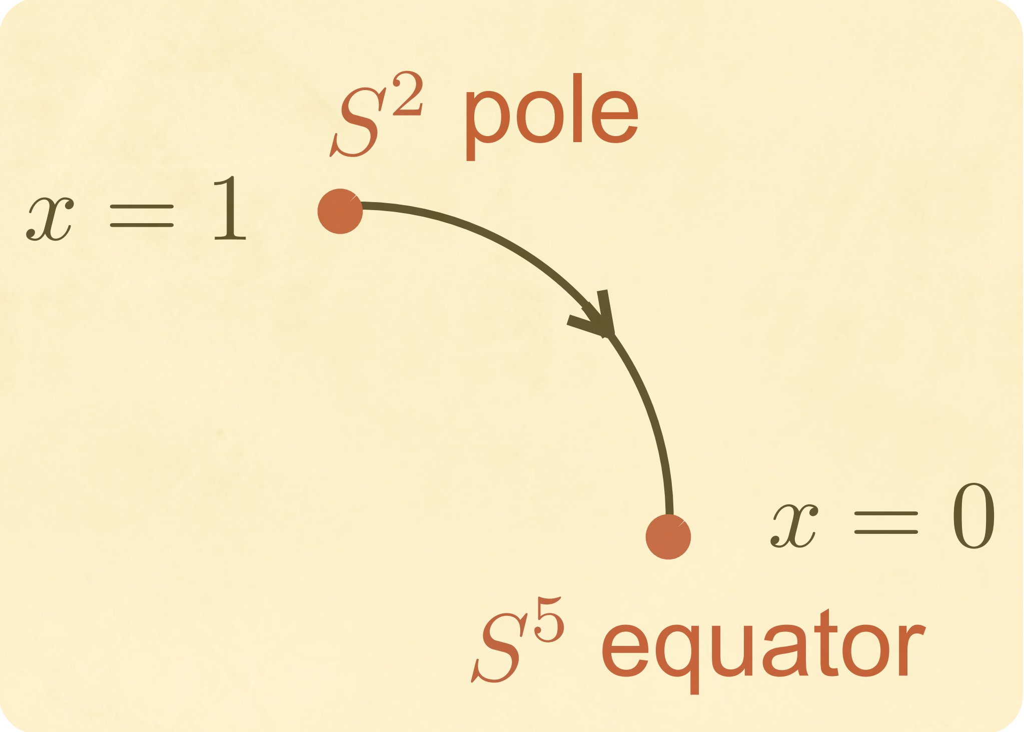

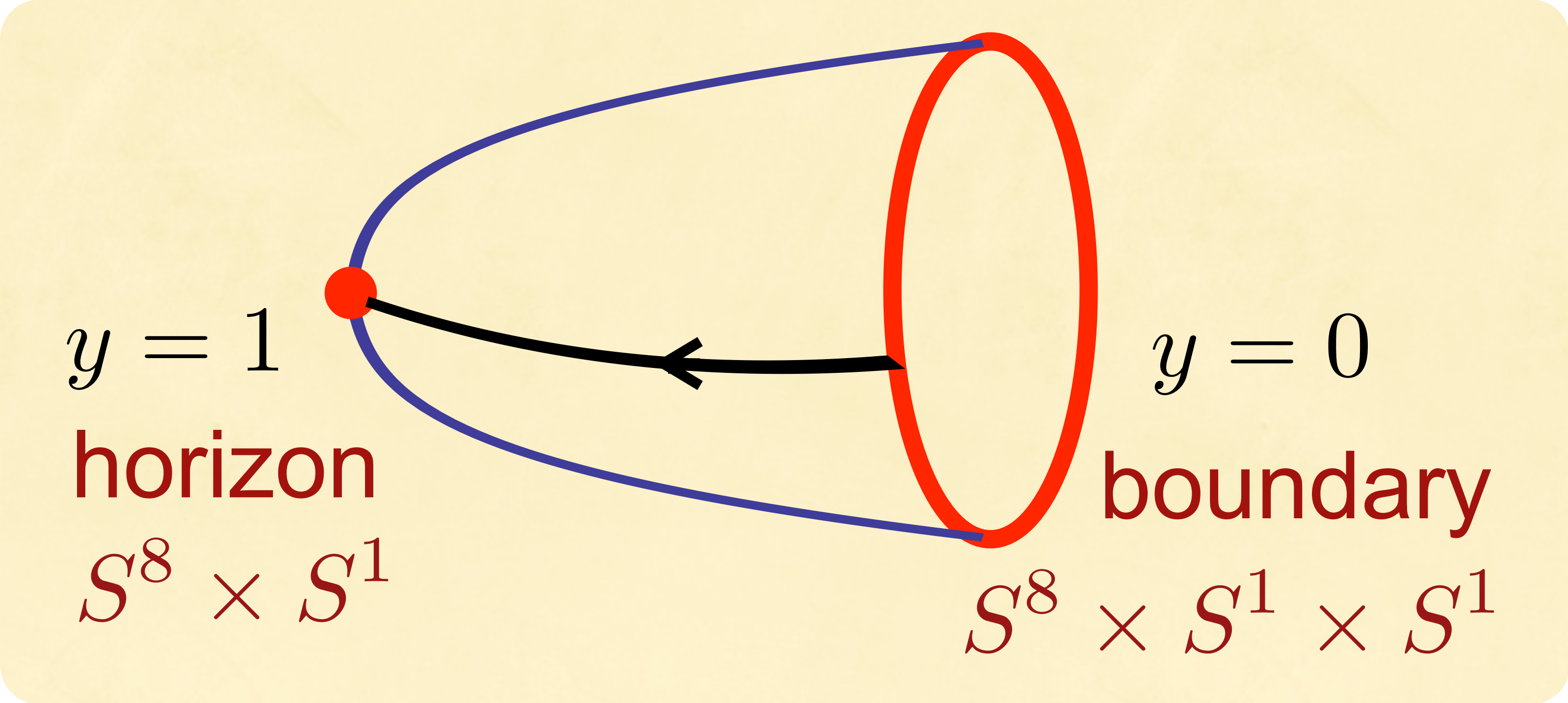

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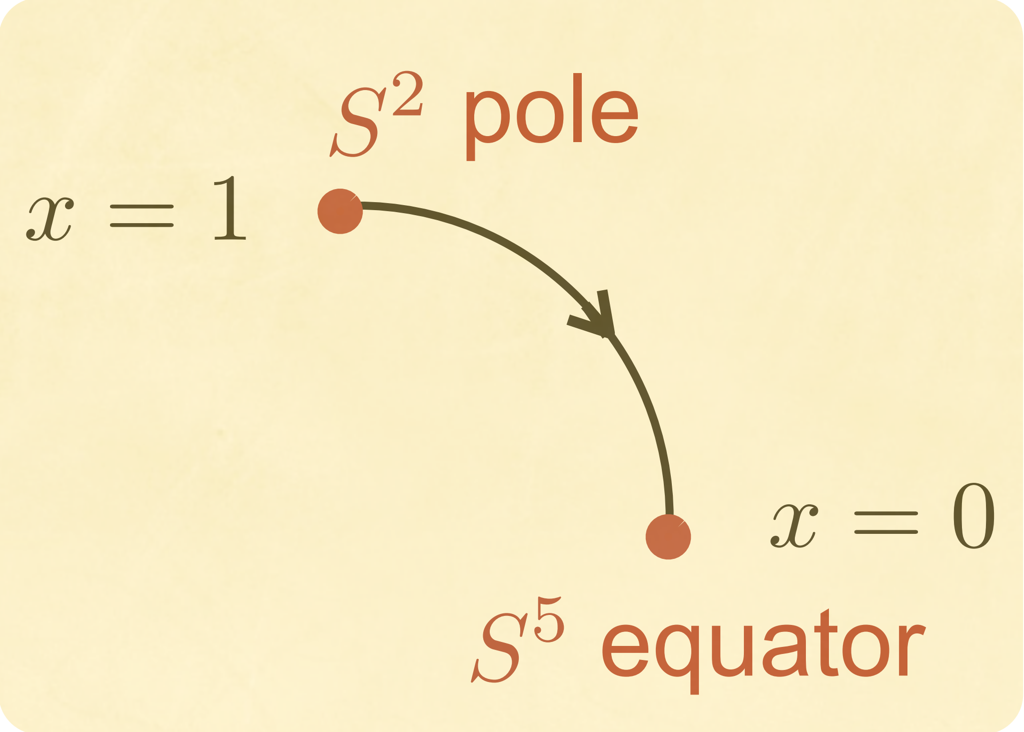

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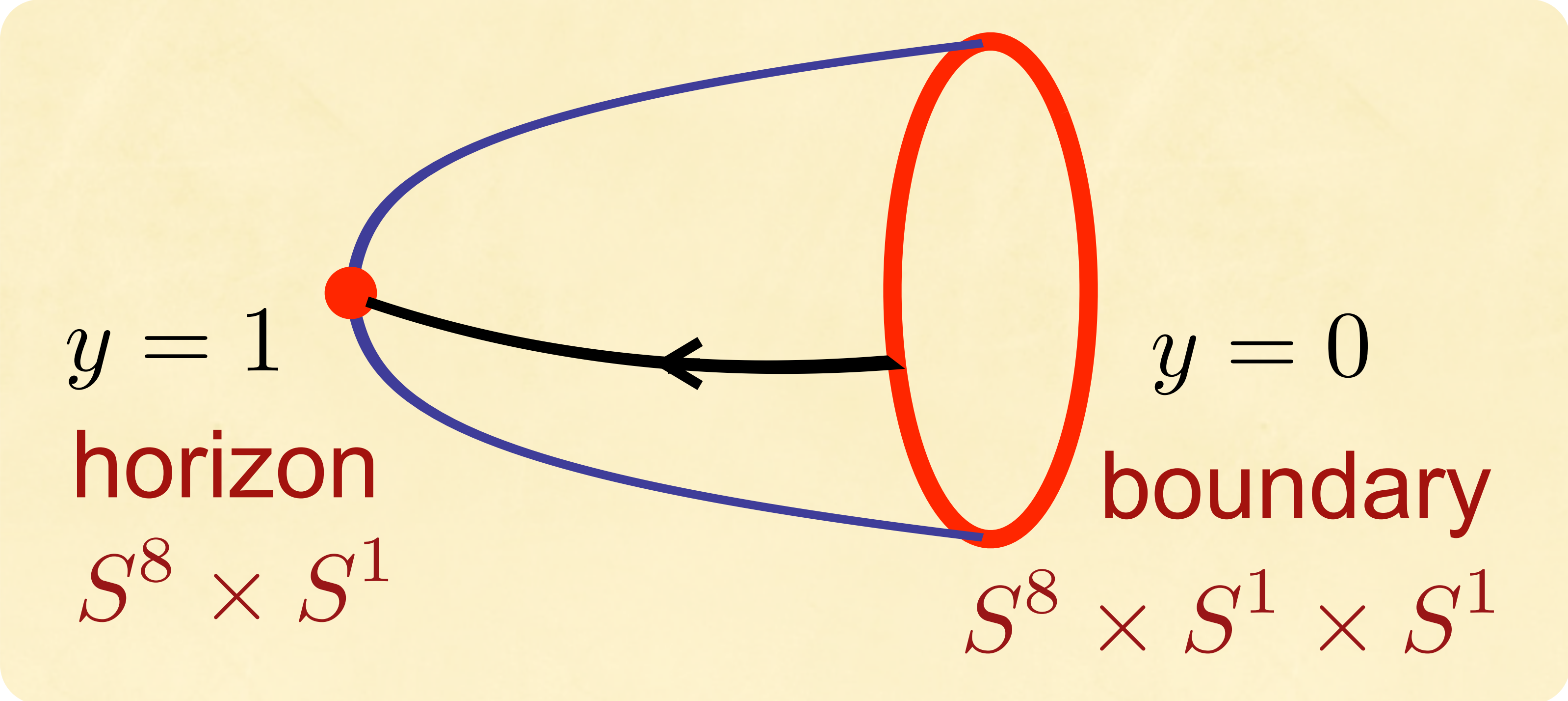

Non-extremal D0-brane solution corresponds to

$$A = B = T_1 = T_2 = T_3 = T_4 = \Omega = 1\,, \qquad F = M = L = 0\,, \qquad \beta = \frac{4\pi}{7}\ \text{(Euclidean time circle)}$$

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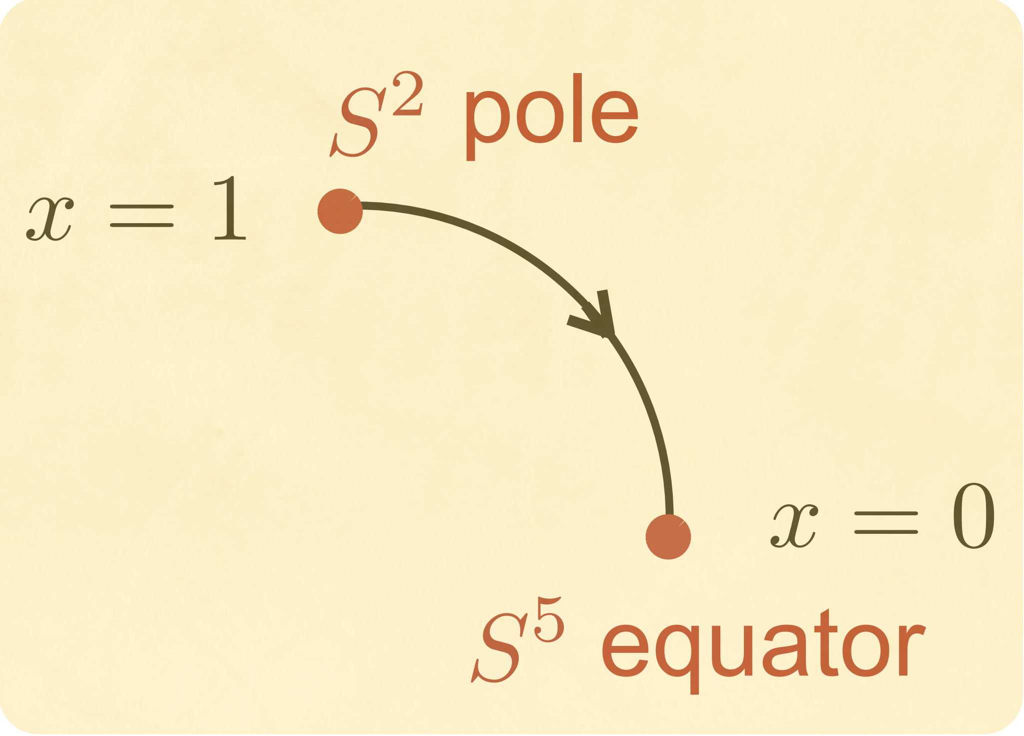

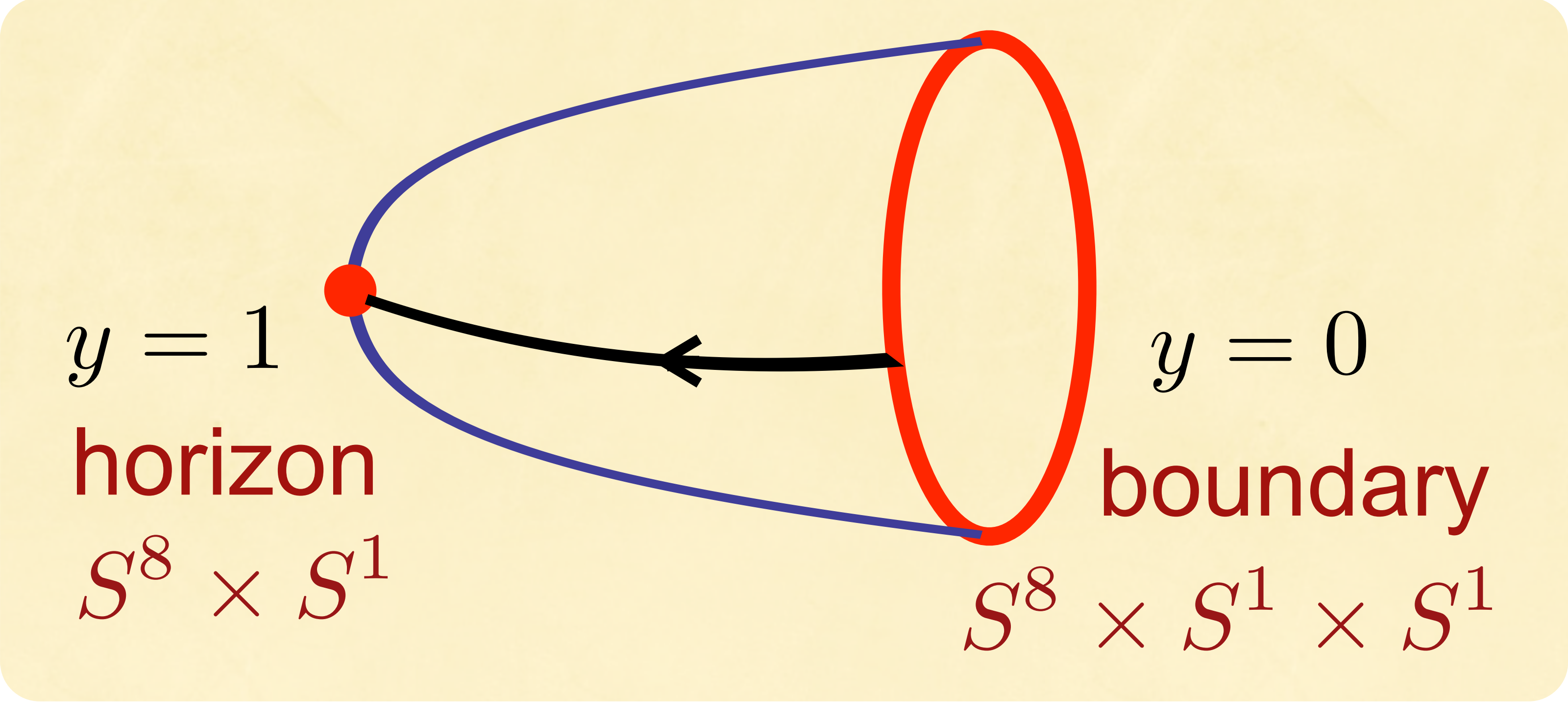

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and need to use scaling symmetry of 11D SUGRA action

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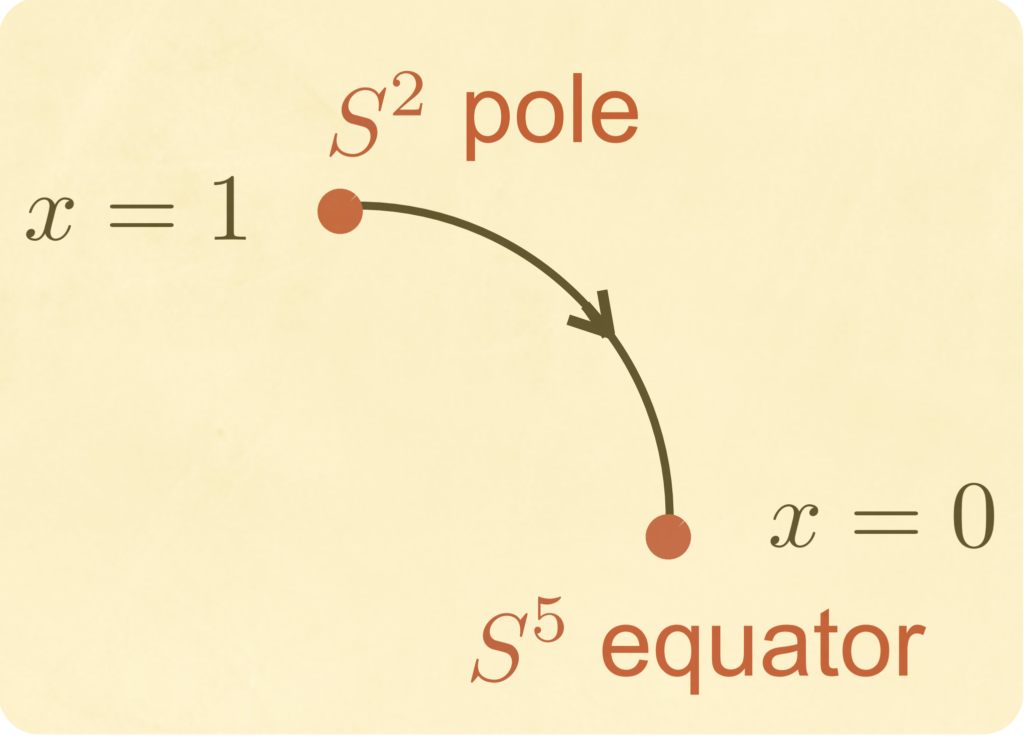

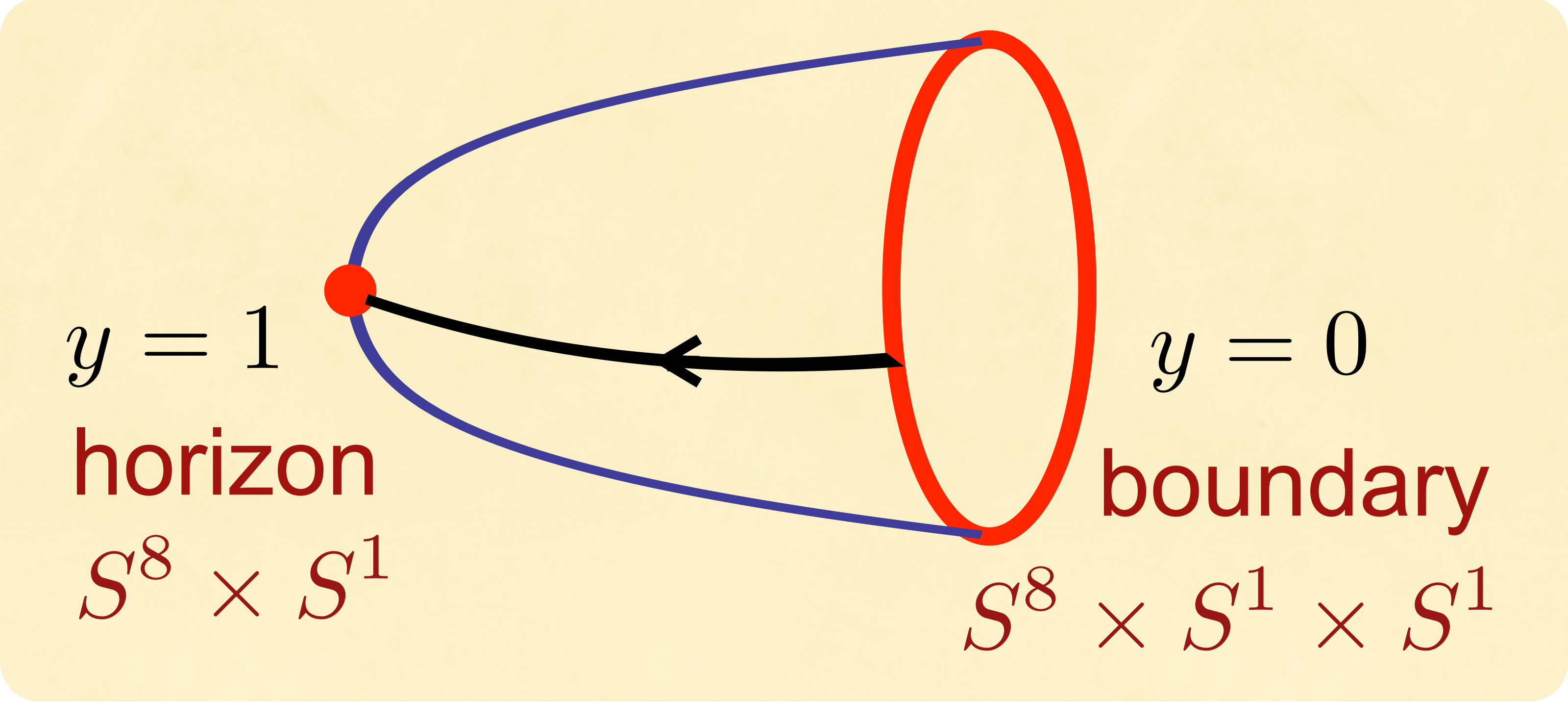

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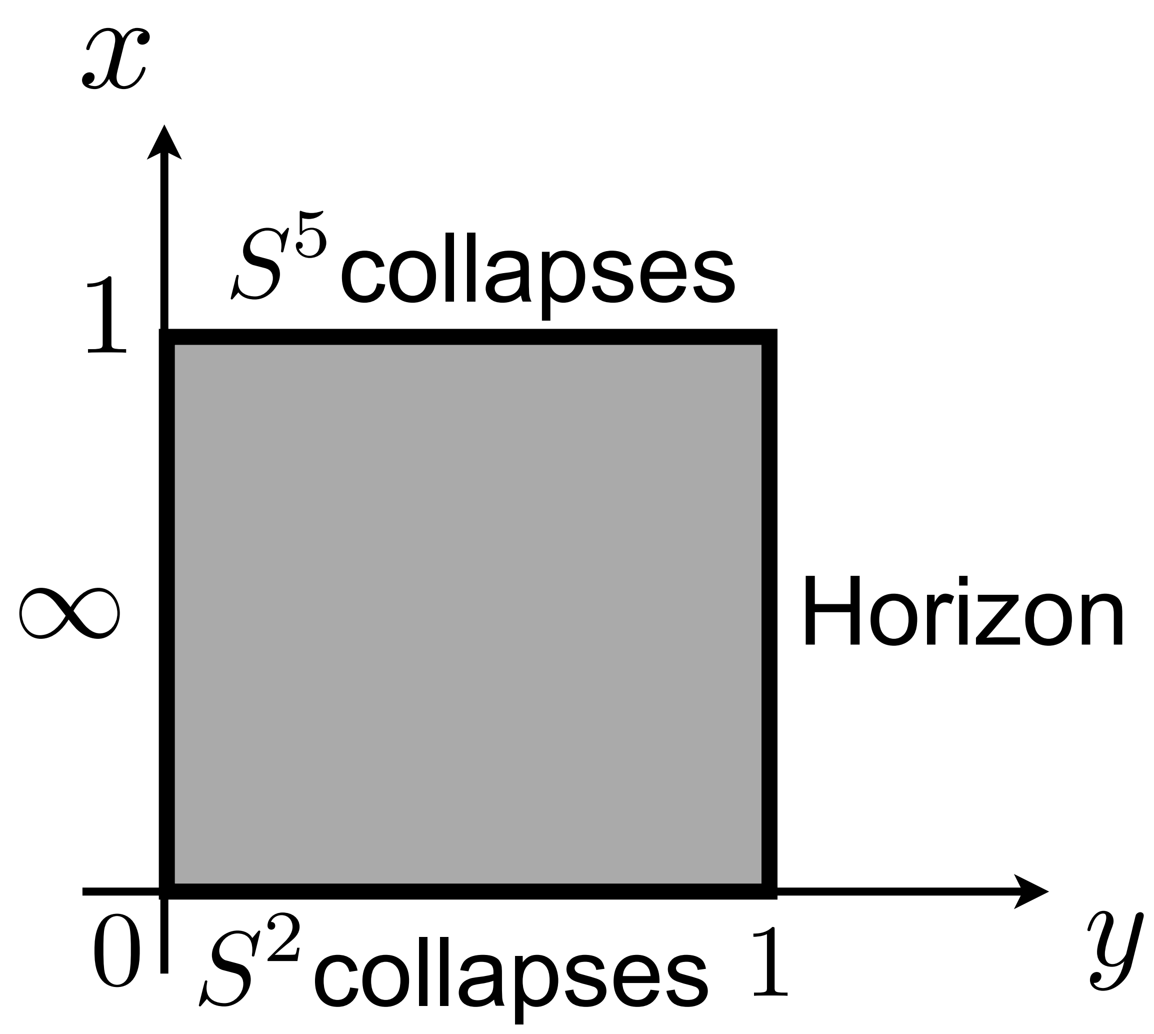
This scaling symmetry will be important later...

- Boundary conditions



- Boundary conditions

At infinity ($y = 0$): $A, B, T_1, T_2, T_3, T_4, \Omega \to 1\,, \quad F \to 0$

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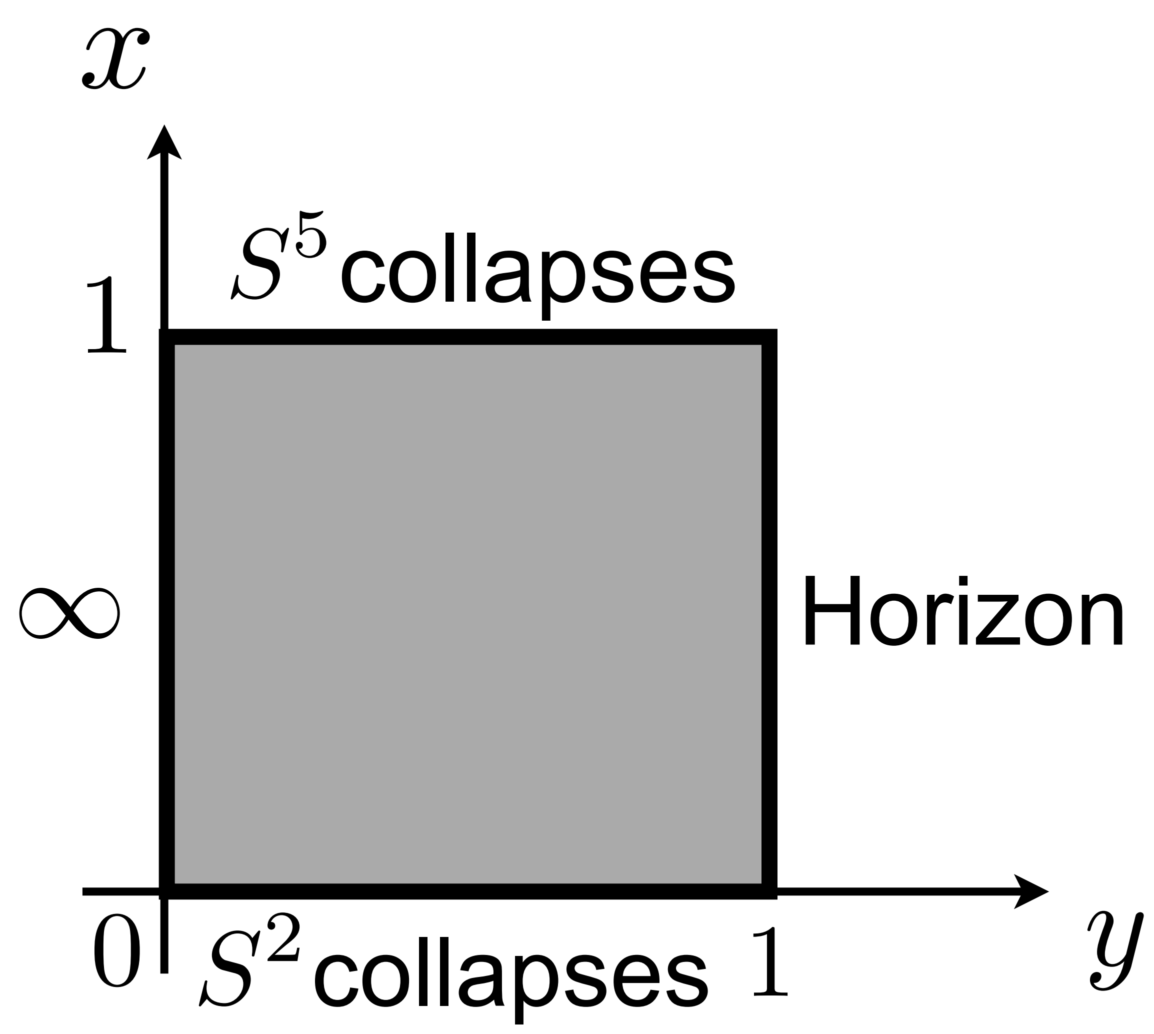

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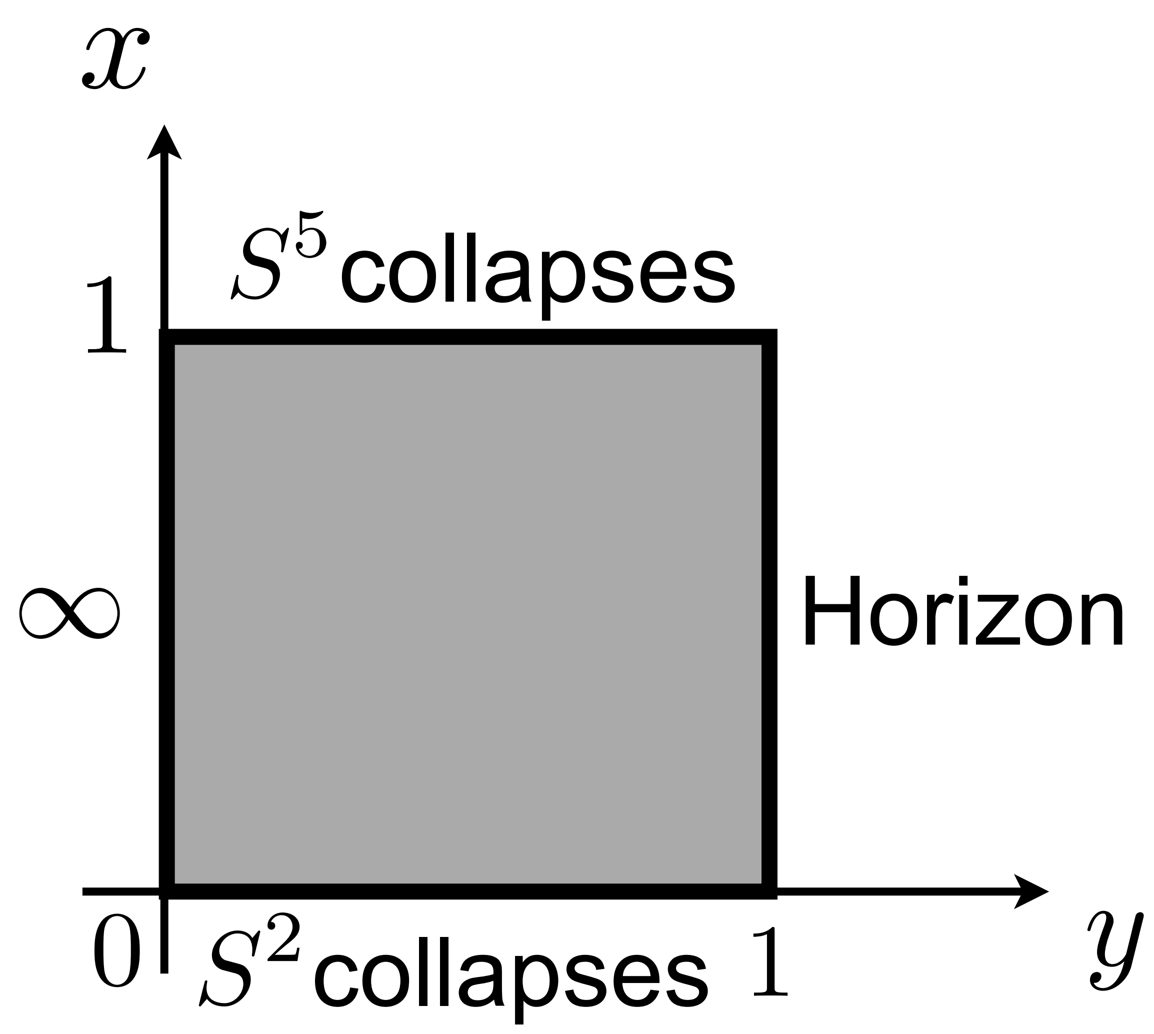

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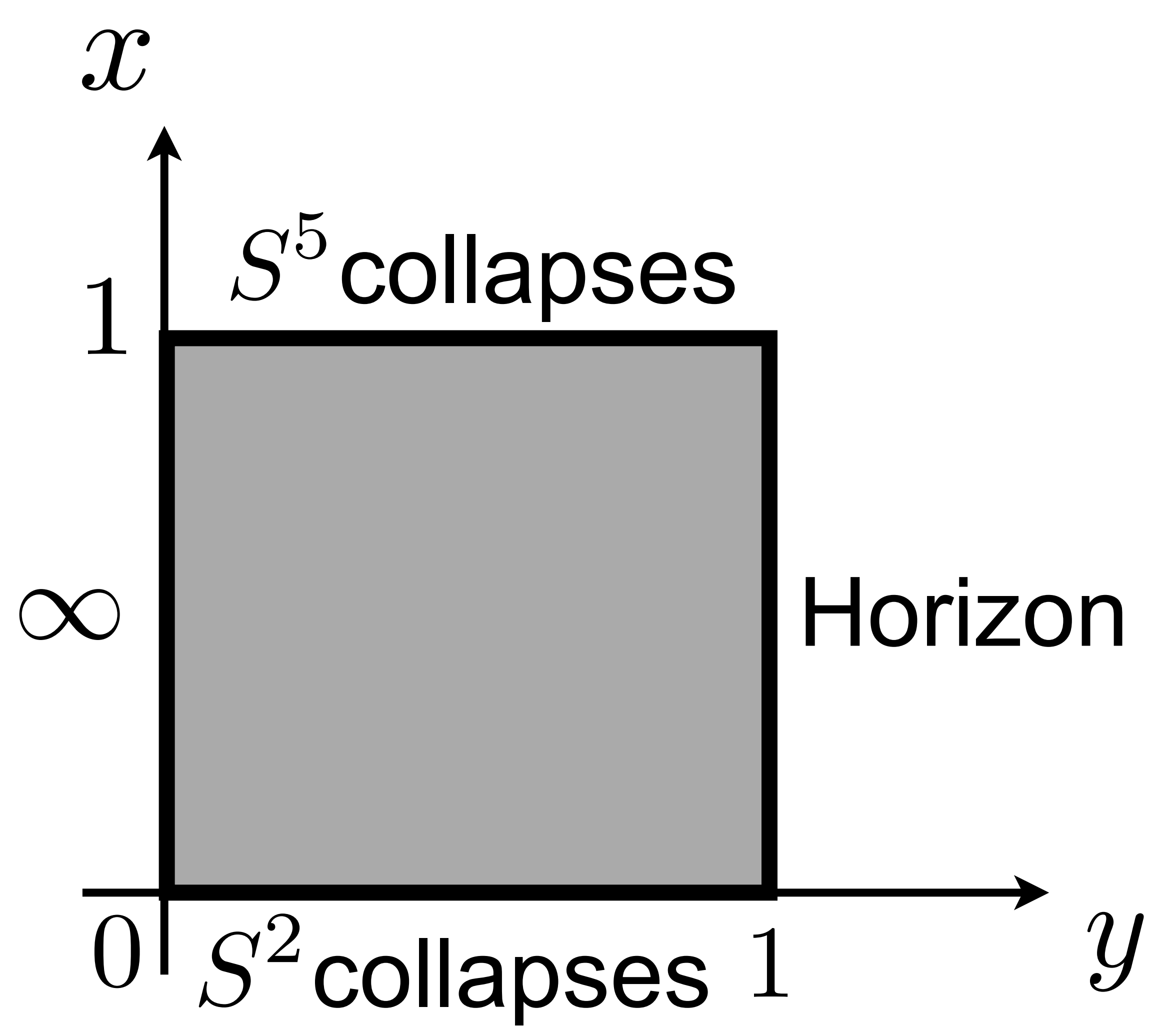

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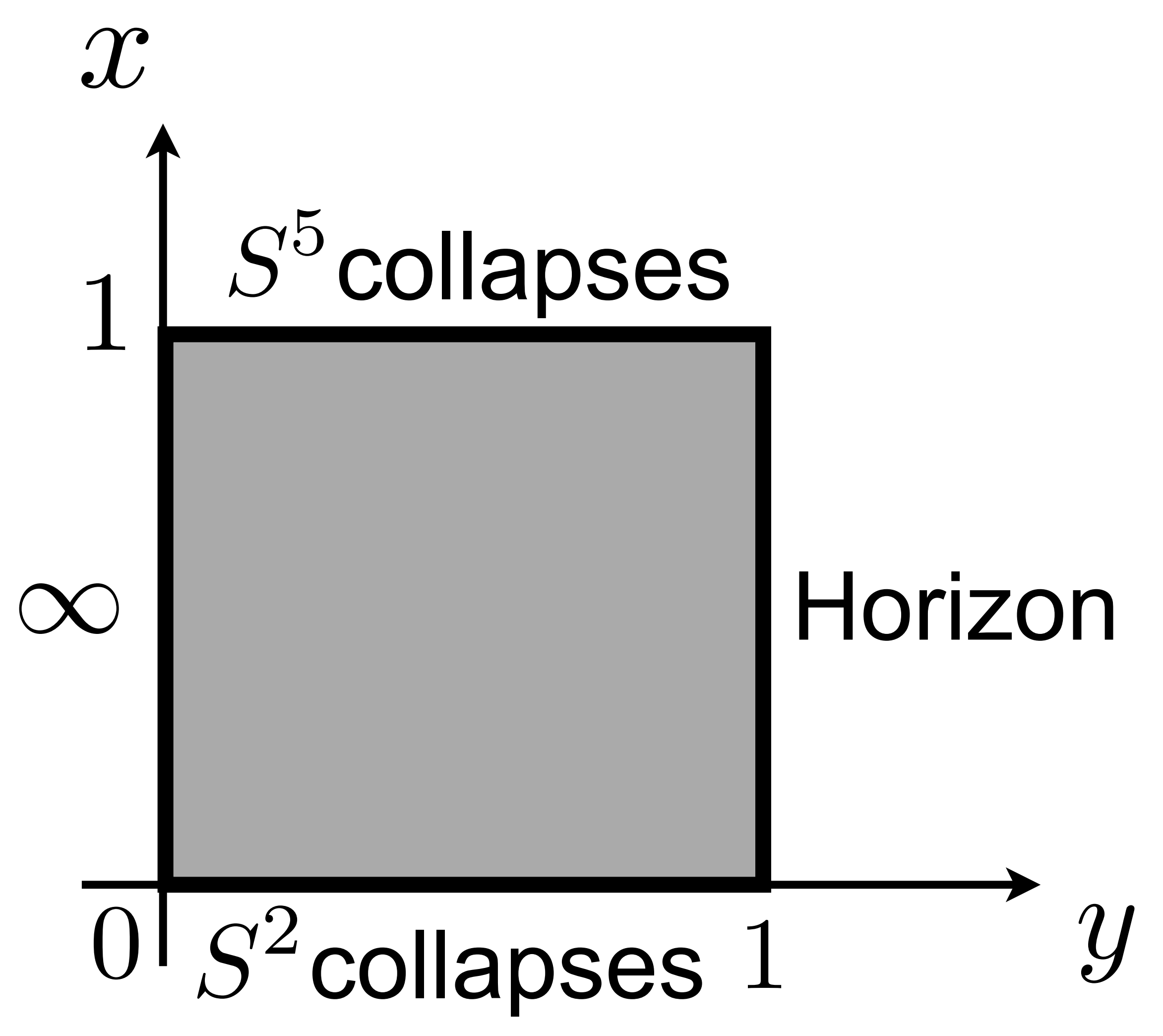

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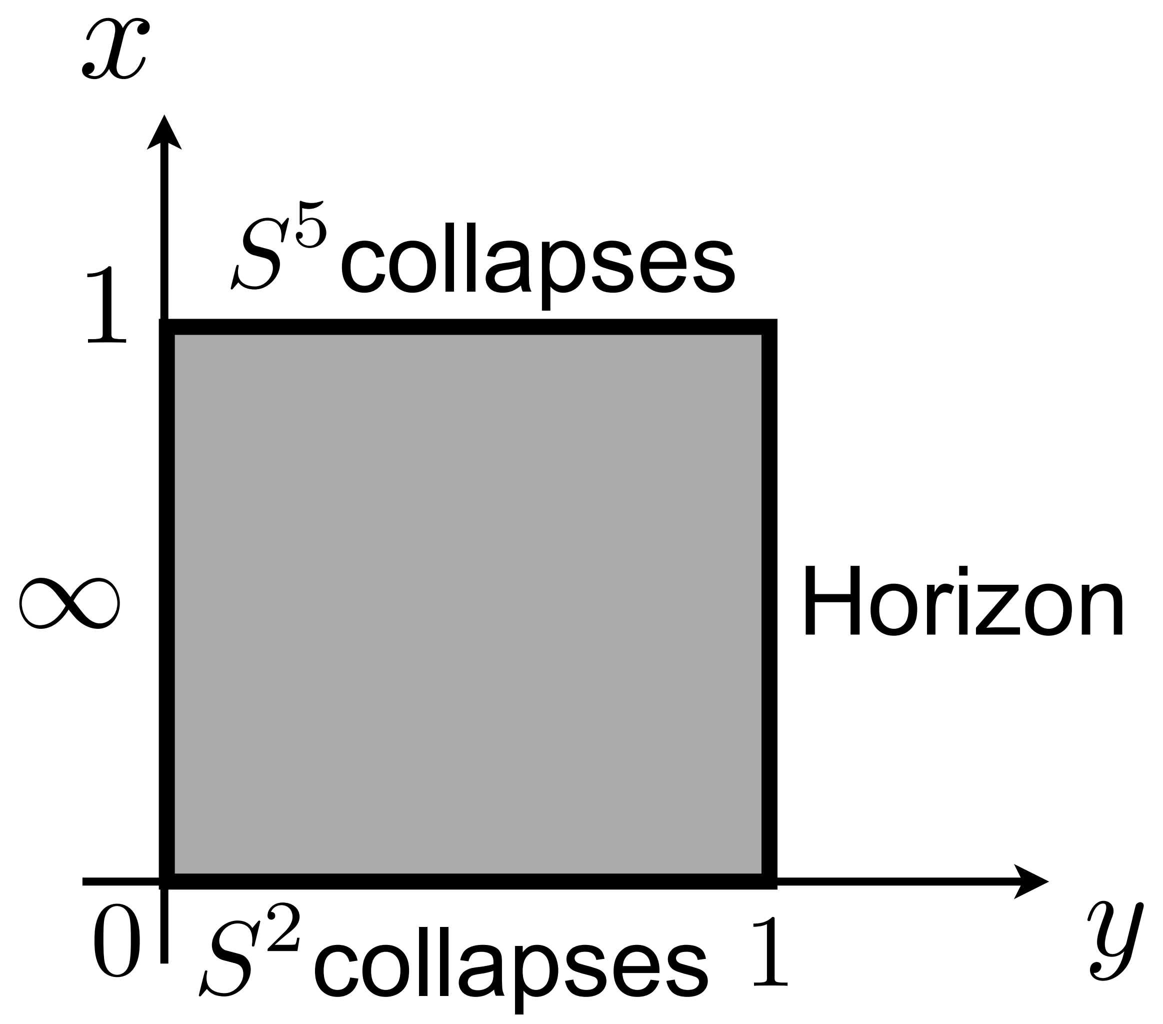

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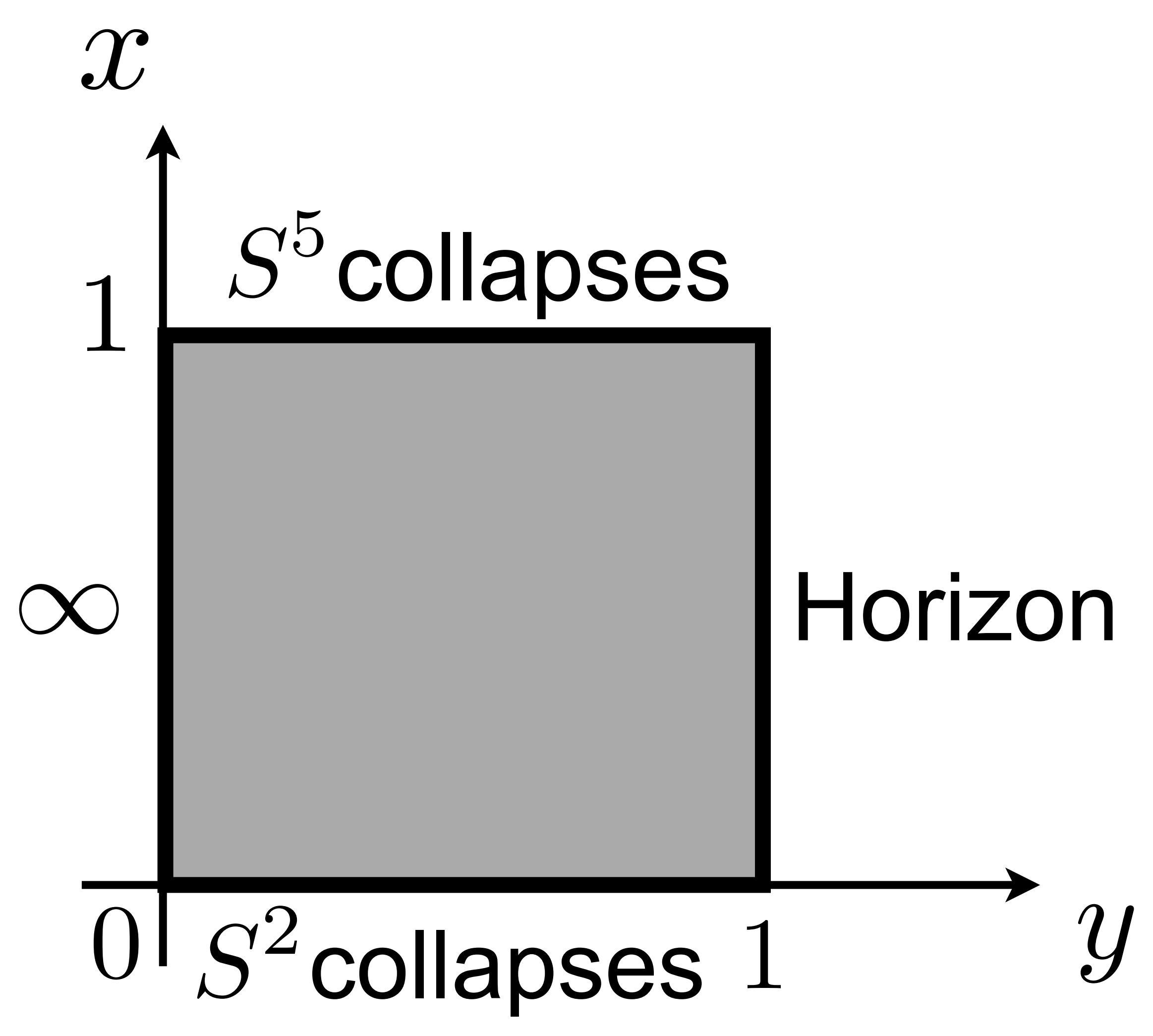

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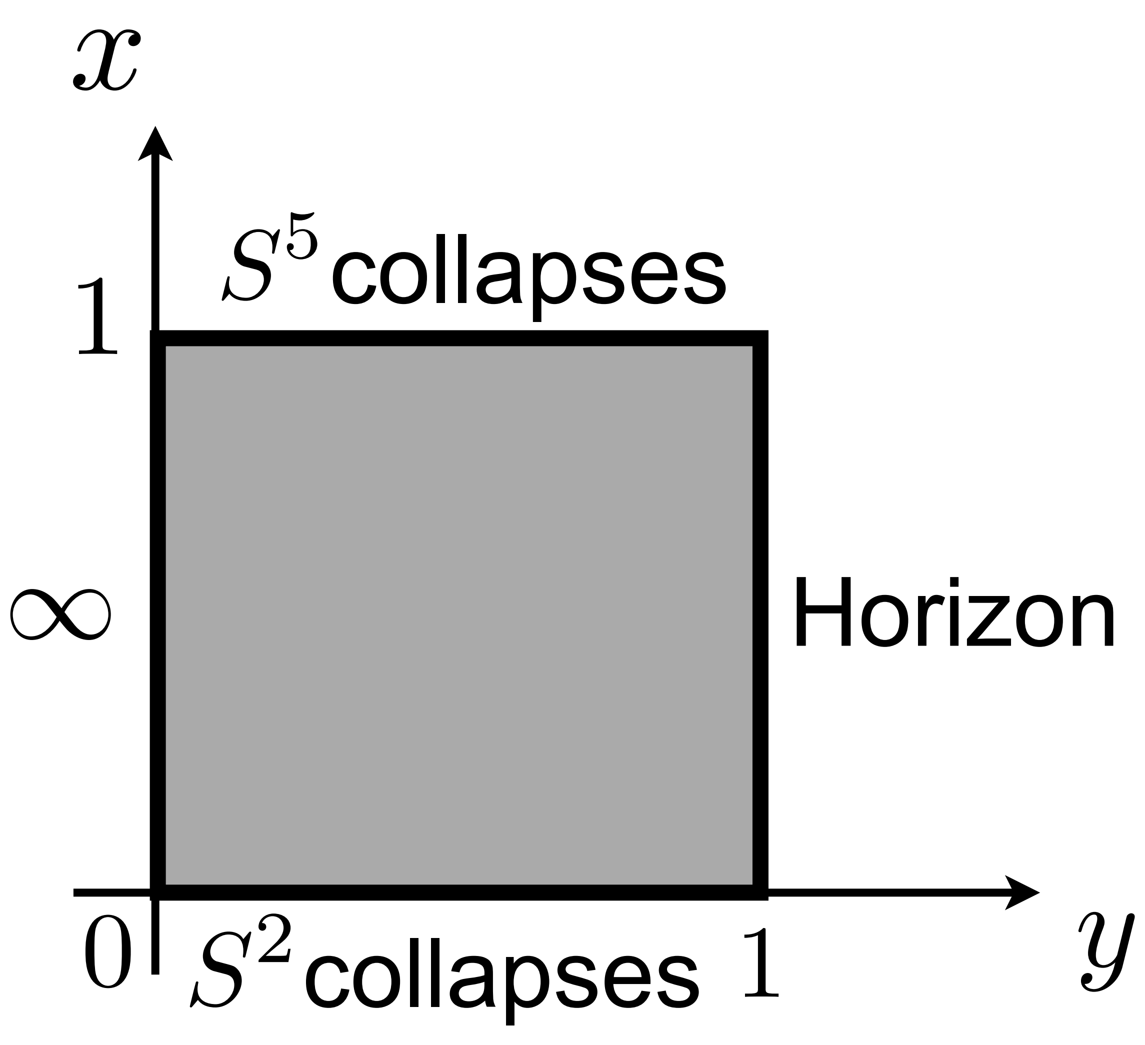

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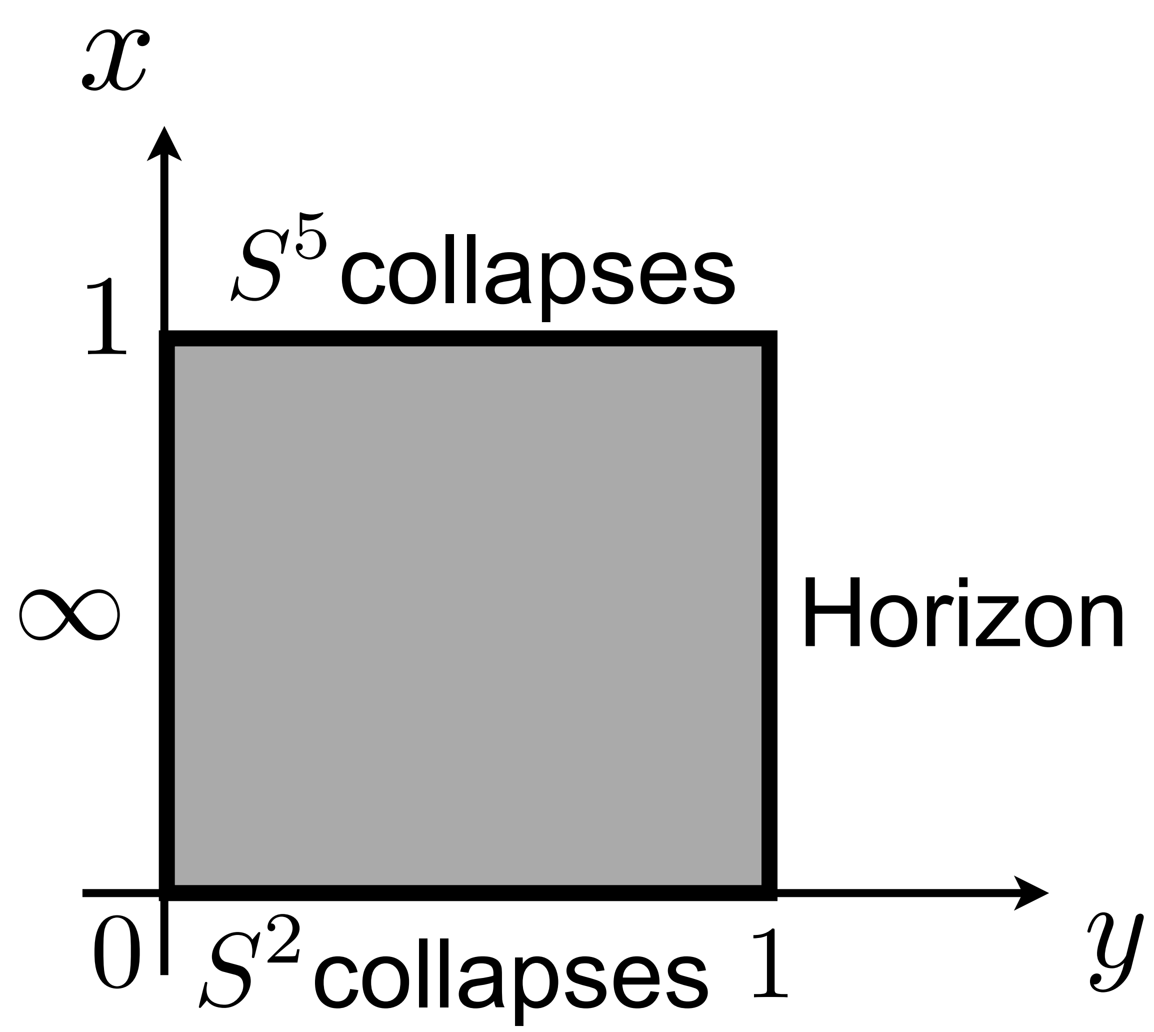

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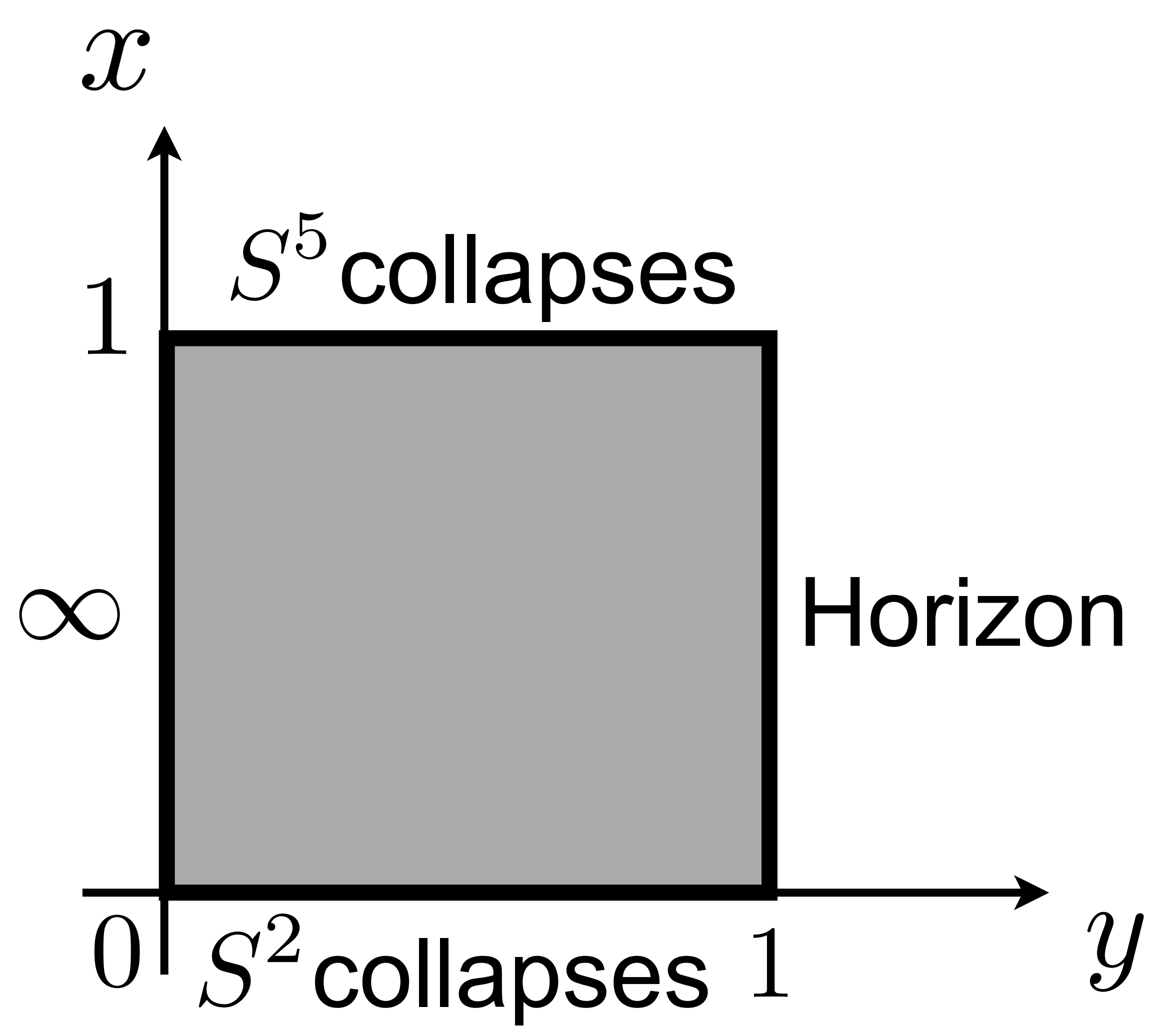
$SO(6)\times SO(3)$ invariant tensor hamonic



Regularity at the axis of symmetry: horizon ($y=1$), S^2 pole ($x=1$) and S^5 equator ($x=0$).

Perform above scalings, then geometry has asymptotics of non-extremal D0-brane with temperature T and mass deformation turned on. There is a single parameter

$$\hat{\mu} = \frac{7}{12\pi}\frac{\mu}{T}$$

Important! Just learned that

$$I = \frac{s^9 s'}{16\pi G_N}\,\hat{I}\left(\frac{\mu}{T}\right) = \frac{15}{28}\left(\frac{15}{14^2\pi^8}\right)^{\frac{2}{5}} N^2 \left(\frac{T}{\lambda^{\frac{1}{3}}}\right)^{\frac{9}{5}} \hat{I}\left(\frac{\mu}{T}\right)$$

$$S = \frac{s^9 s'}{4G_N}\,\hat{S}\left(\frac{\mu}{T}\right) = \frac{15\pi}{7}\left(\frac{15}{14^2\pi^8}\right)^{\frac{2}{5}} N^2 \left(\frac{T}{\lambda^{\frac{1}{3}}}\right)^{\frac{9}{5}} \hat{S}\left(\frac{\mu}{T}\right)$$

Smarr formulae (good to check numerics)

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- Let v^{μ} be a killing vector. From field equations it follows that

$$(K_v)^{\mu\nu} = \nabla^{\mu} v^{\nu} + \frac{1}{3} F^{\mu\nu\alpha\beta} v^{\gamma} C_{\alpha\beta\gamma} + \frac{1}{6} v^{[\mu} F^{\nu]\alpha\beta\gamma} C_{\alpha\beta\gamma}$$

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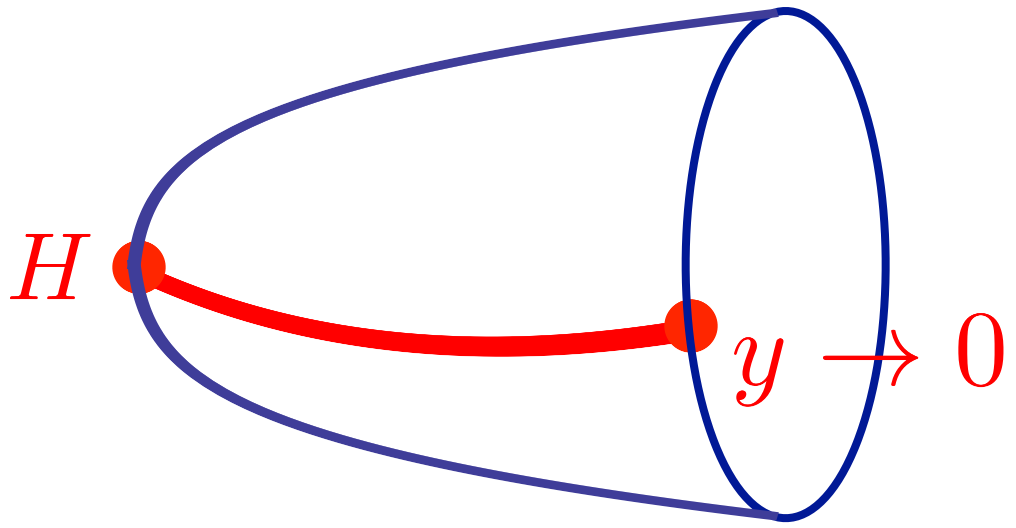

- Integrate $d\left(\star K_v\right) = 0$ over surface of constant time with $y_1 < y < y_2$

$$0 = \int_{\Sigma_{12}} d\left(\star K_v\right) = \int_{\partial\Sigma_{12}} \star K_v = \int_H \star K_v - \int_{y\to 0} \star K_v$$

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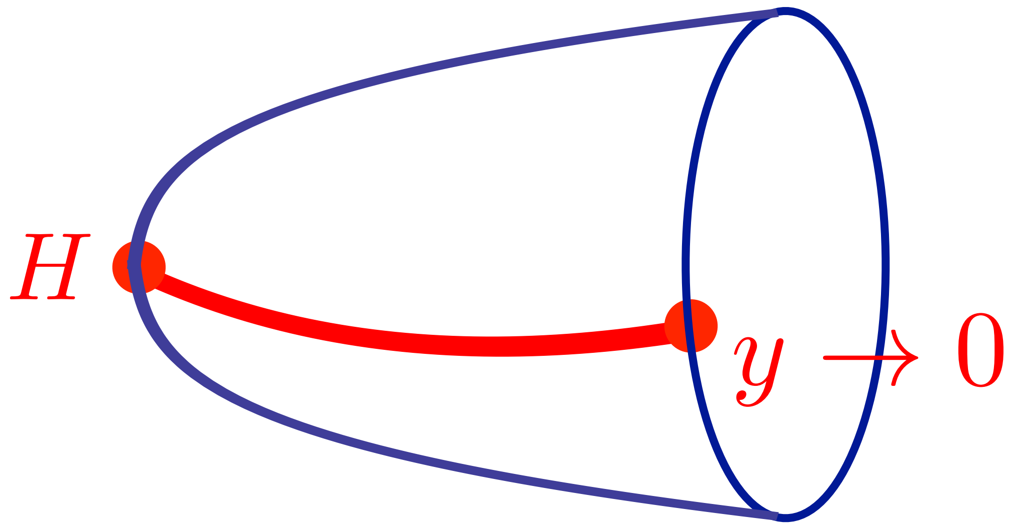

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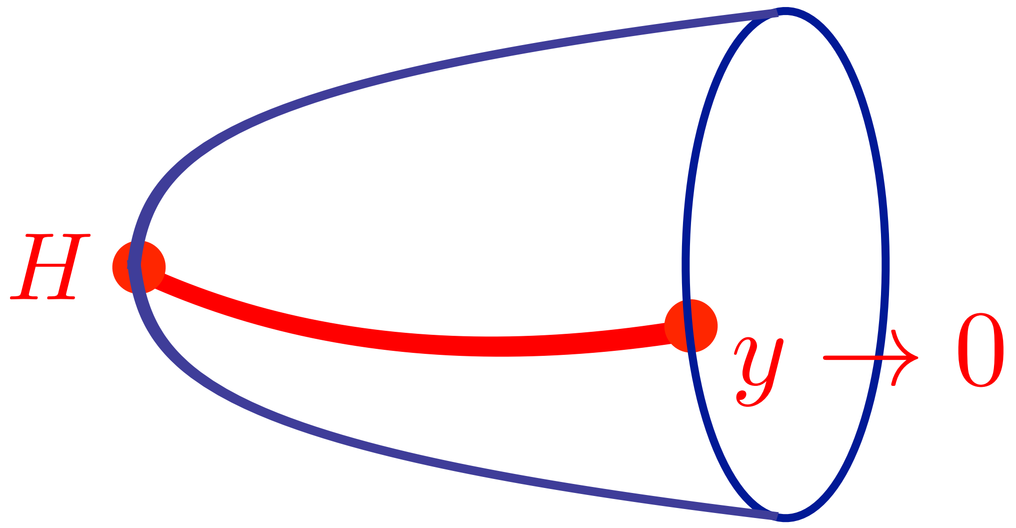

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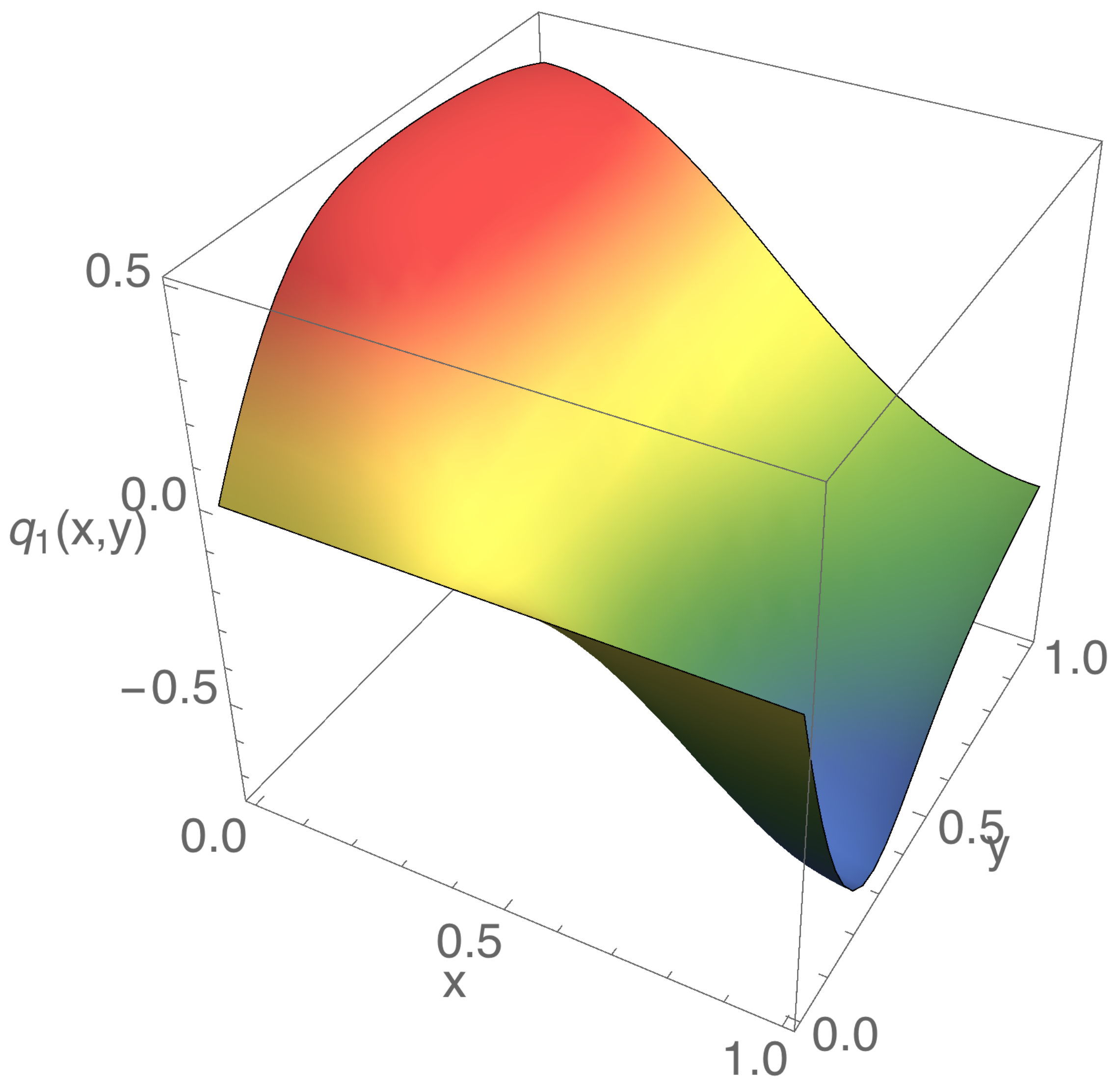
- Can also consider generator of 11D translations $v = \dfrac{\partial}{\partial \zeta}$

The solution



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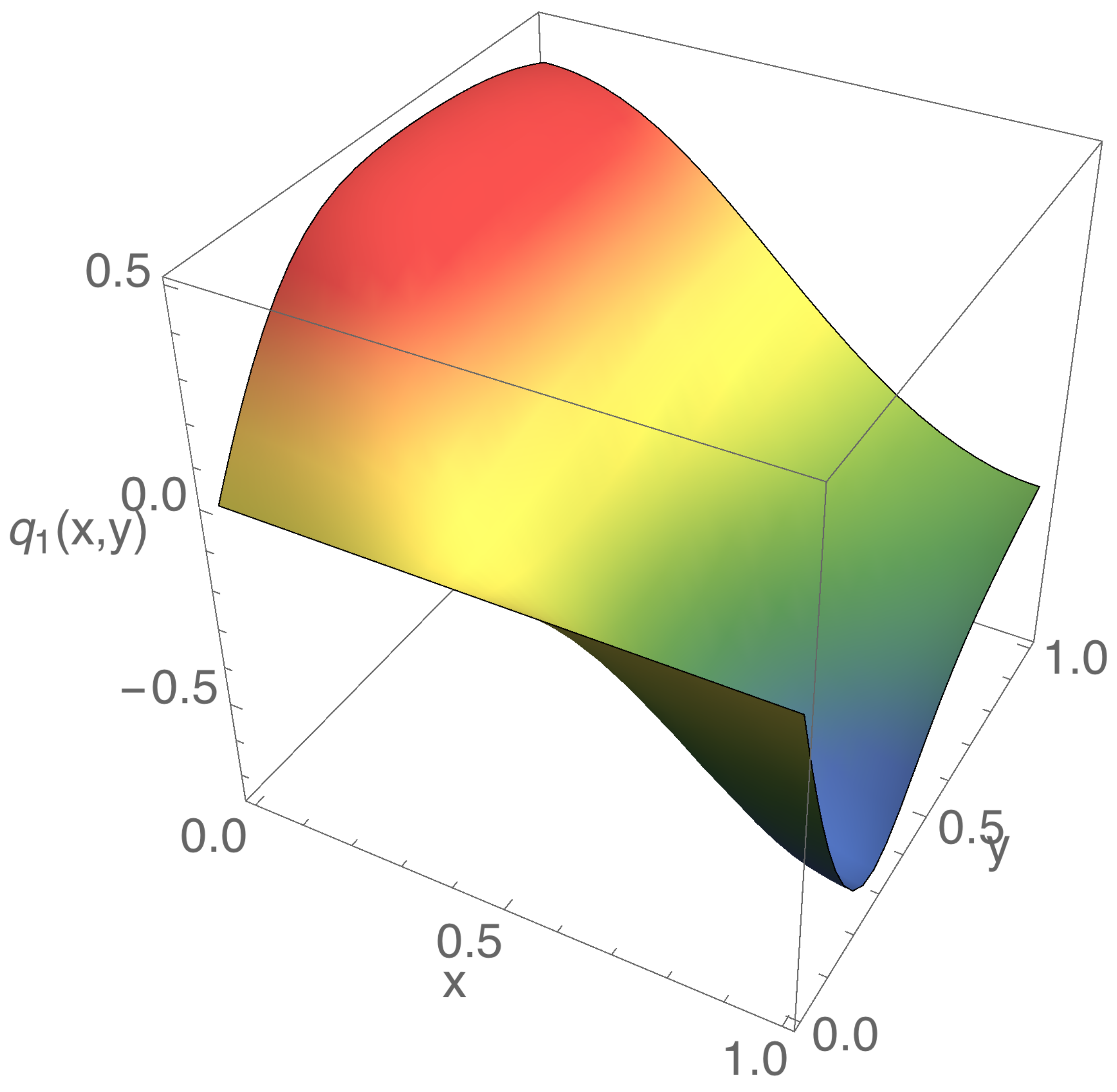

- Einstein-deTurck equations [Headrick, Kitchen, Wiseman '09]

$$R_{\mu\nu} - \nabla_{(\mu}\xi_{\nu)} = \frac{1}{12}\left(F_{\mu\alpha\beta\gamma}F_{\mu}{}^{\alpha\beta\gamma} - \frac{1}{12}g_{\mu\nu}F^2\right)$$

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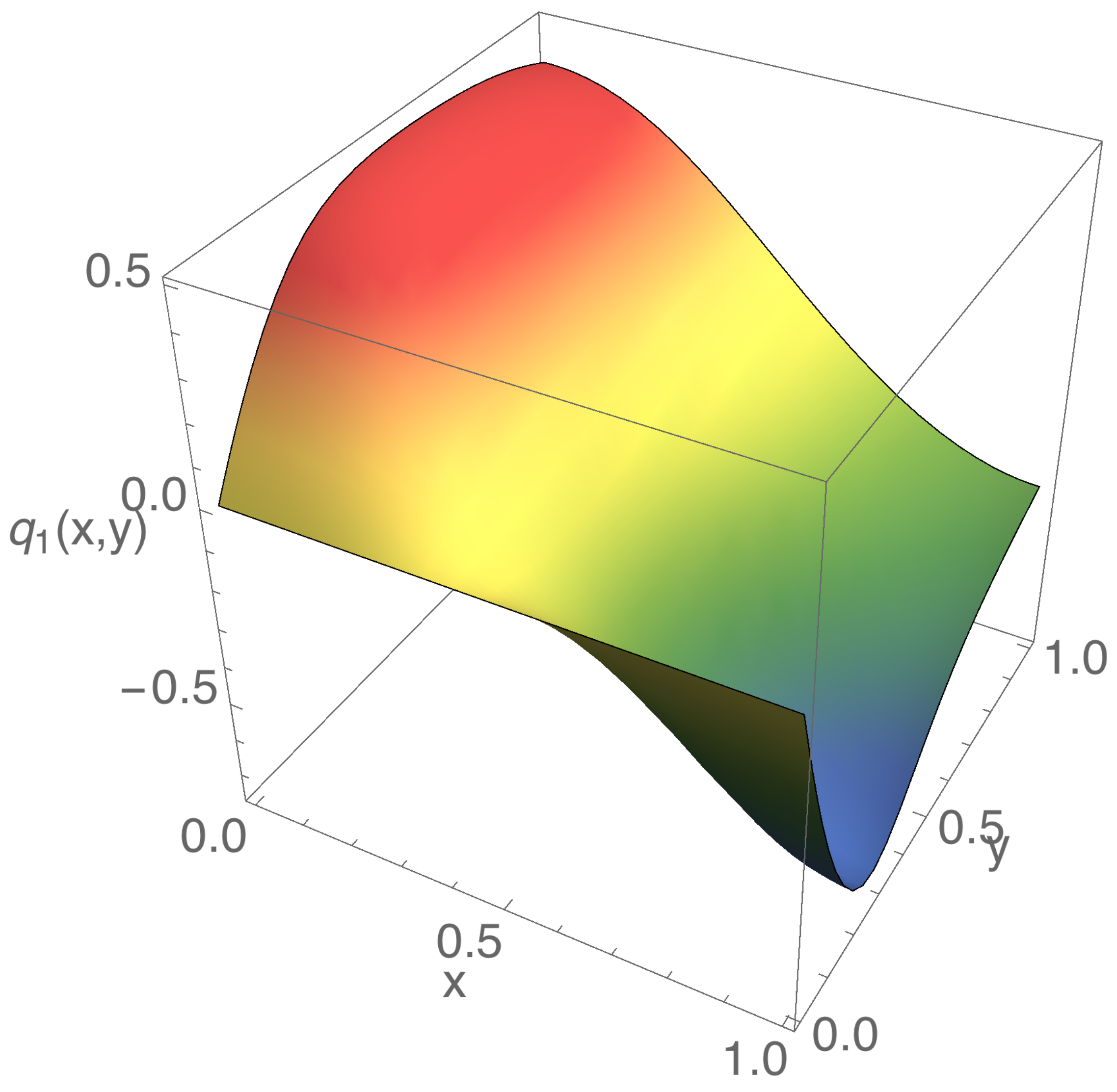

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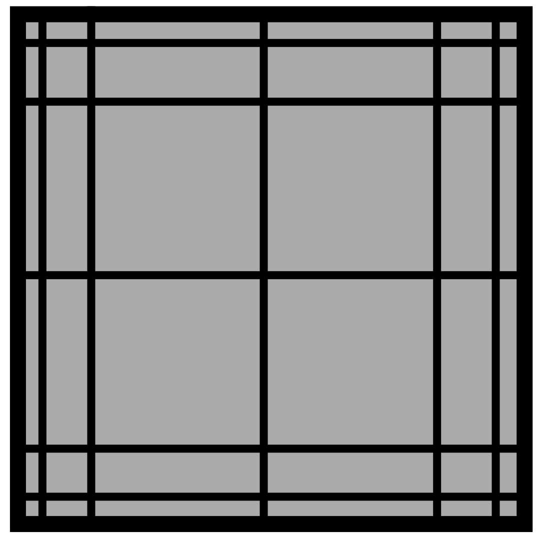
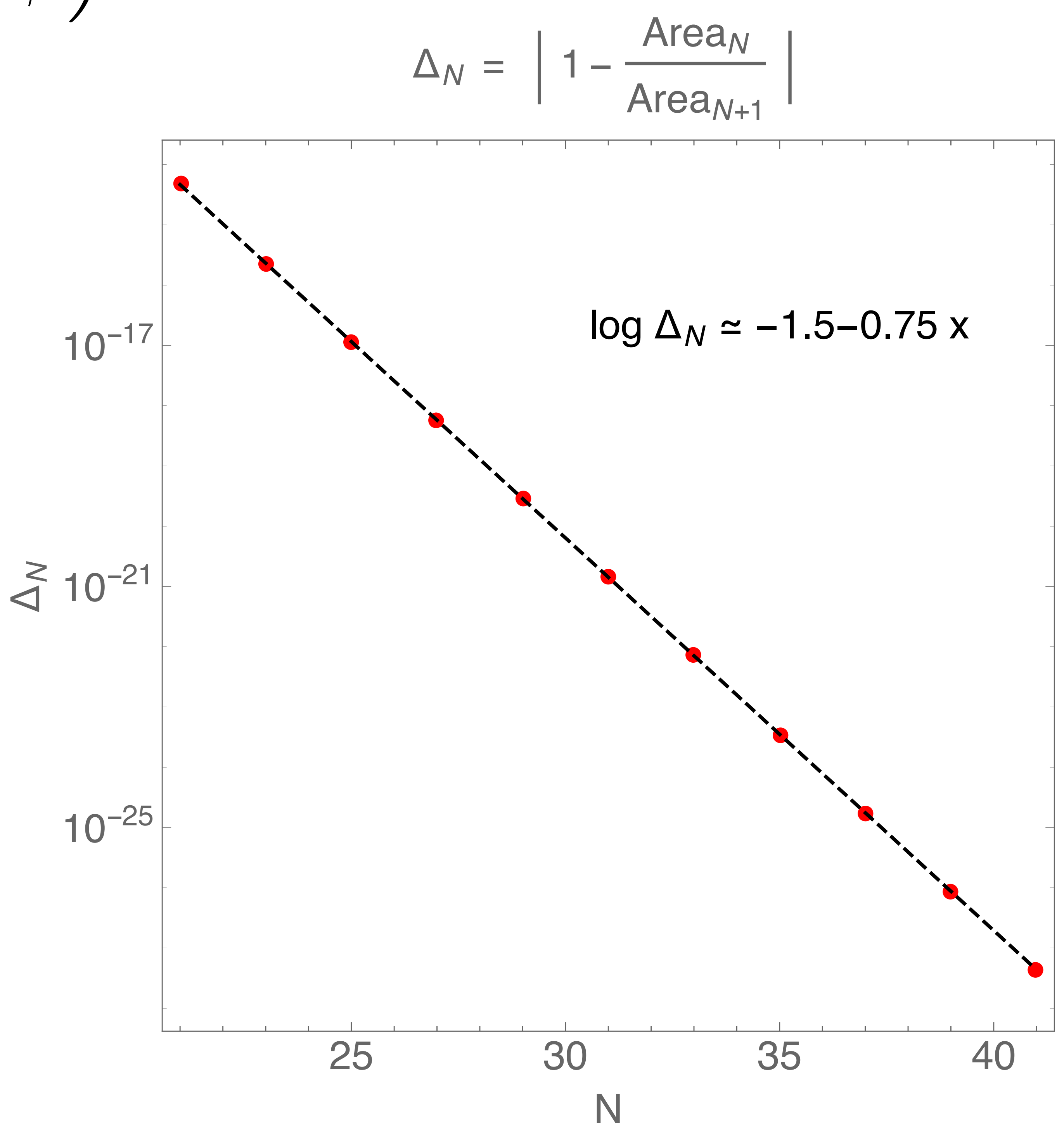
- Descretize PDEs with $N \times N$ Chebyshev grid



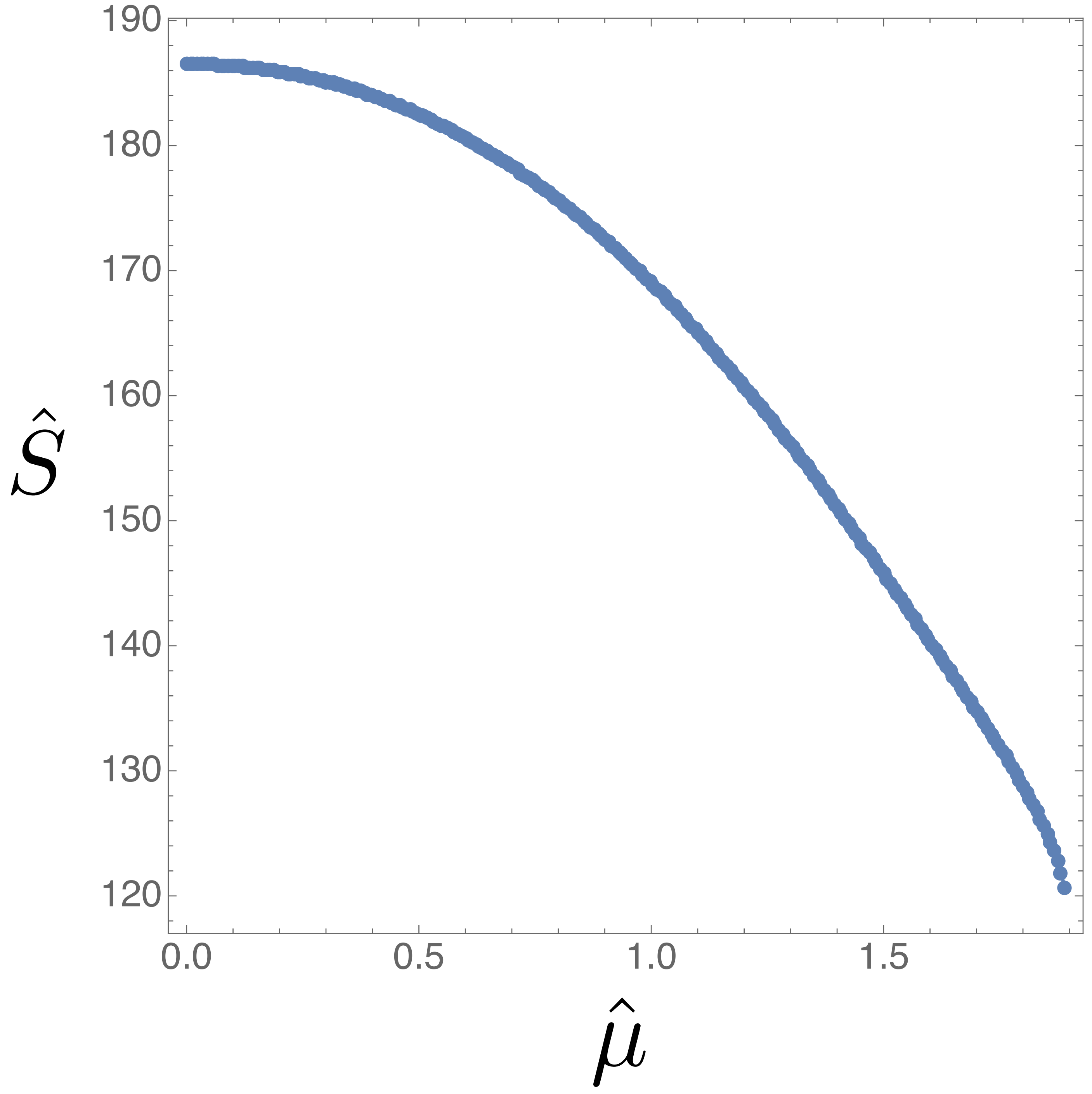
Derivatives are estimated using
polynomial approximation that
involves all points in the grid
spectral methods - exponential convergence



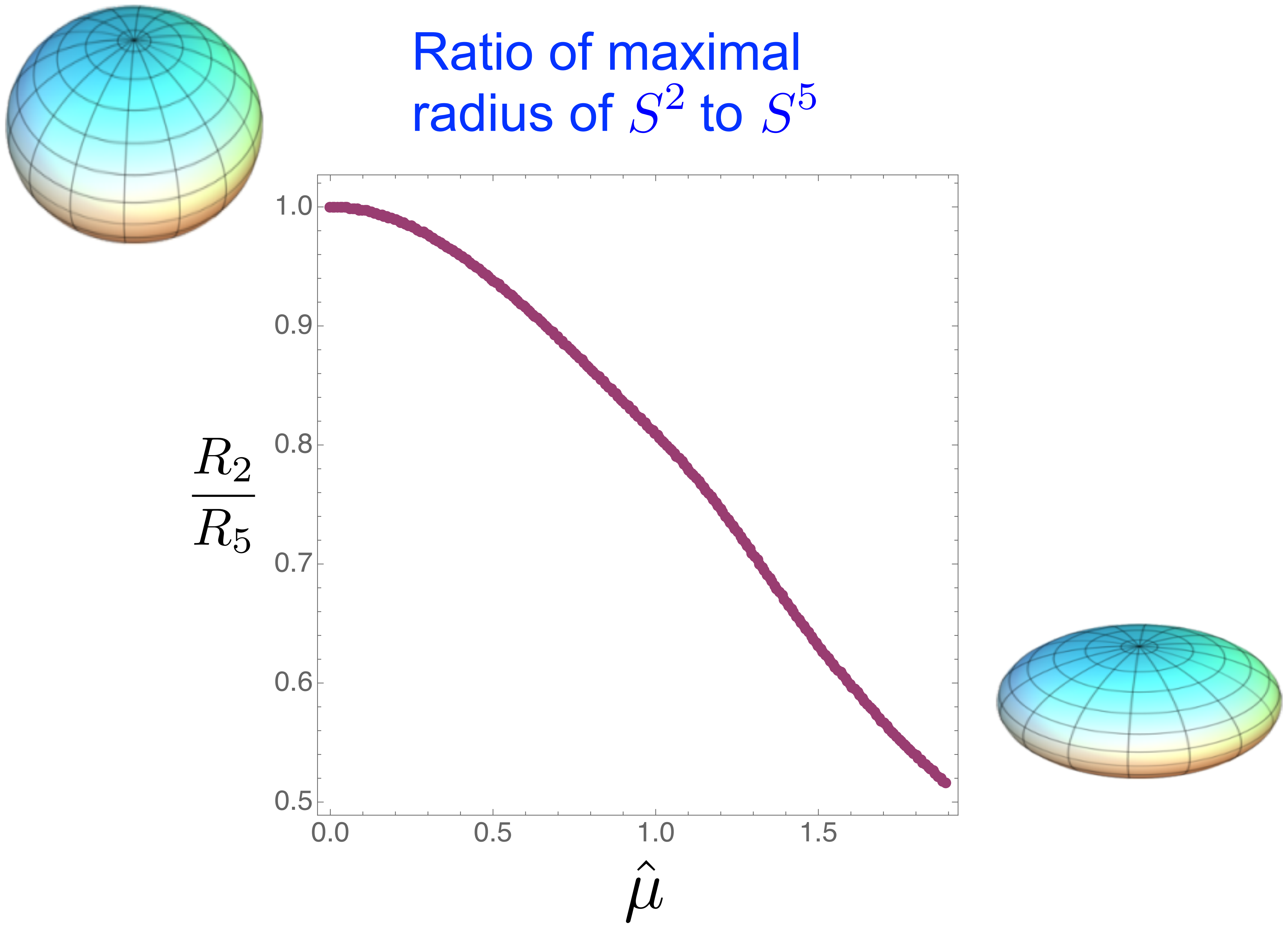
$$\Delta_N = \left| 1 - \frac{\text{Area}_N}{\text{Area}_{N+1}} \right|$$

- Horizon area and shape

Horizon area



Ratio of maximal radius of S^2 to S^5



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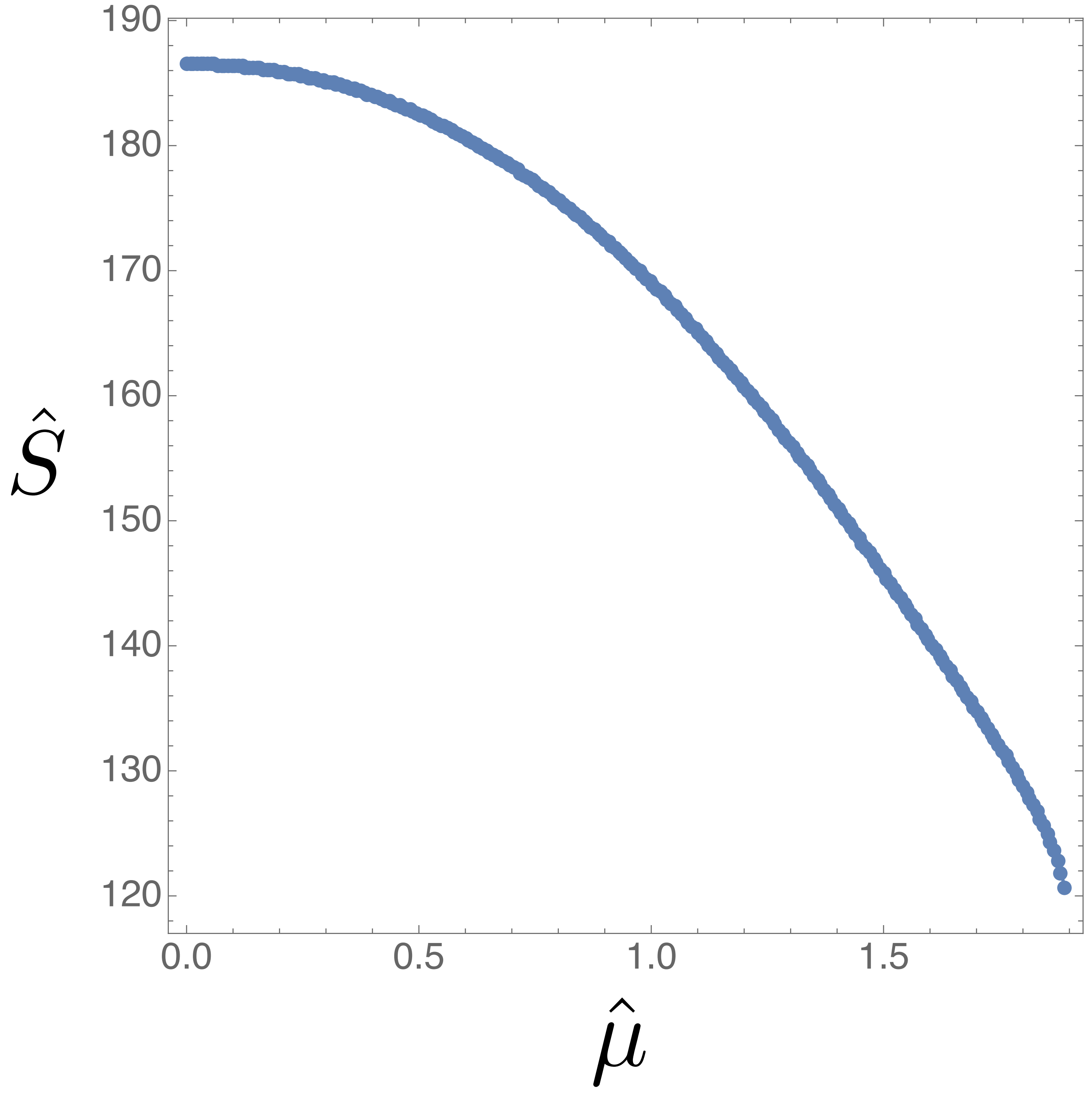

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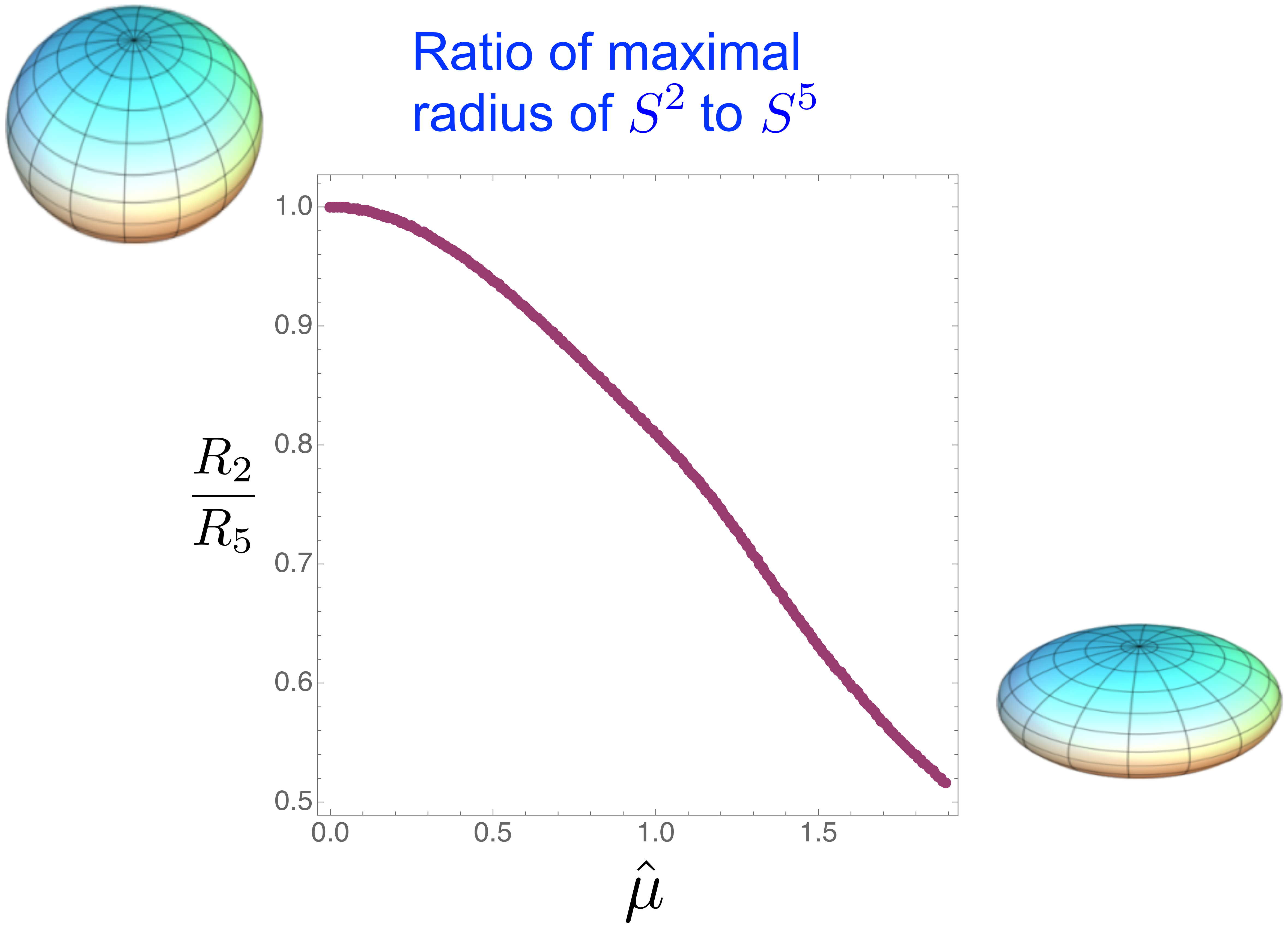

After scaling symmetry to obtain physical metric:

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$$R_i = a_i \left(\frac{T}{\lambda^{\frac{1}{3}}}\right)^{\frac{2}{5}} \hat{R}_i\left(\frac{\mu}{T}\right)$$

Reproduces scalings predicted from strongly coupled low energy moduli estimate [Wiseman ’13]

Black hole thermodynamics

Black hole thermodynamics

- From scaling symmetry we saw that

$$F(T,\mu) = -c_0\, T^{\frac{14}{5}}\, \hat{I}\Big(\frac{\mu}{T}\Big)\,, \qquad S(T,\mu) = c_0\, \frac{14}{5}\, T^{\frac{9}{5}}\, \hat{S}\Big(\frac{\mu}{T}\Big)$$

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Therefore ratio of free energies and entropies

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Analyticity $\Rightarrow \quad s(\hat{\mu}) = \sum_{n=0}^{\infty} s_n\, \hat{\mu}^n\,, \qquad f(\hat{\mu}) = \sum_{n=0}^{\infty} \frac{14 s_n}{14 - 5n}\, \hat{\mu}^n$

Critical temperature

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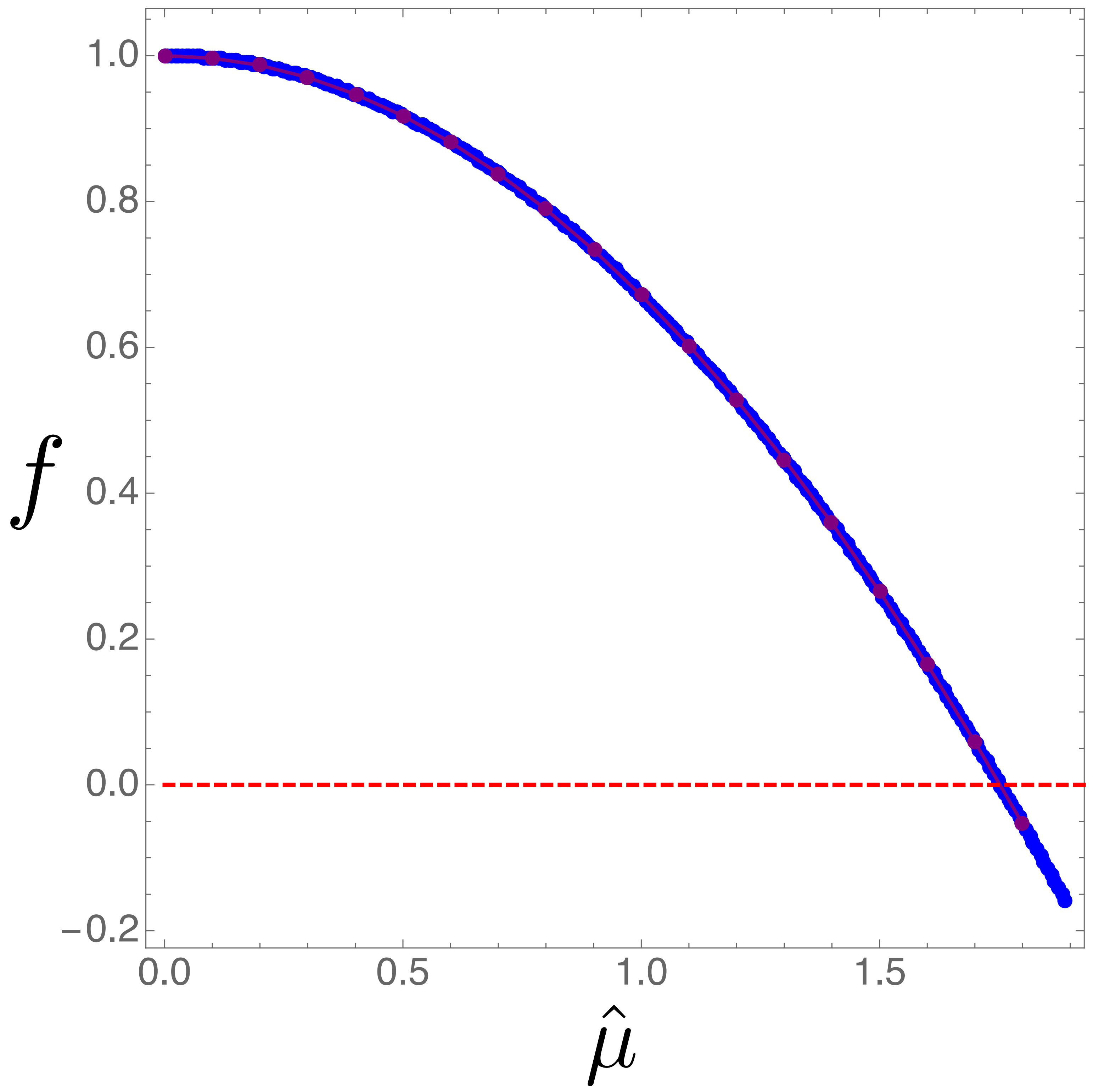

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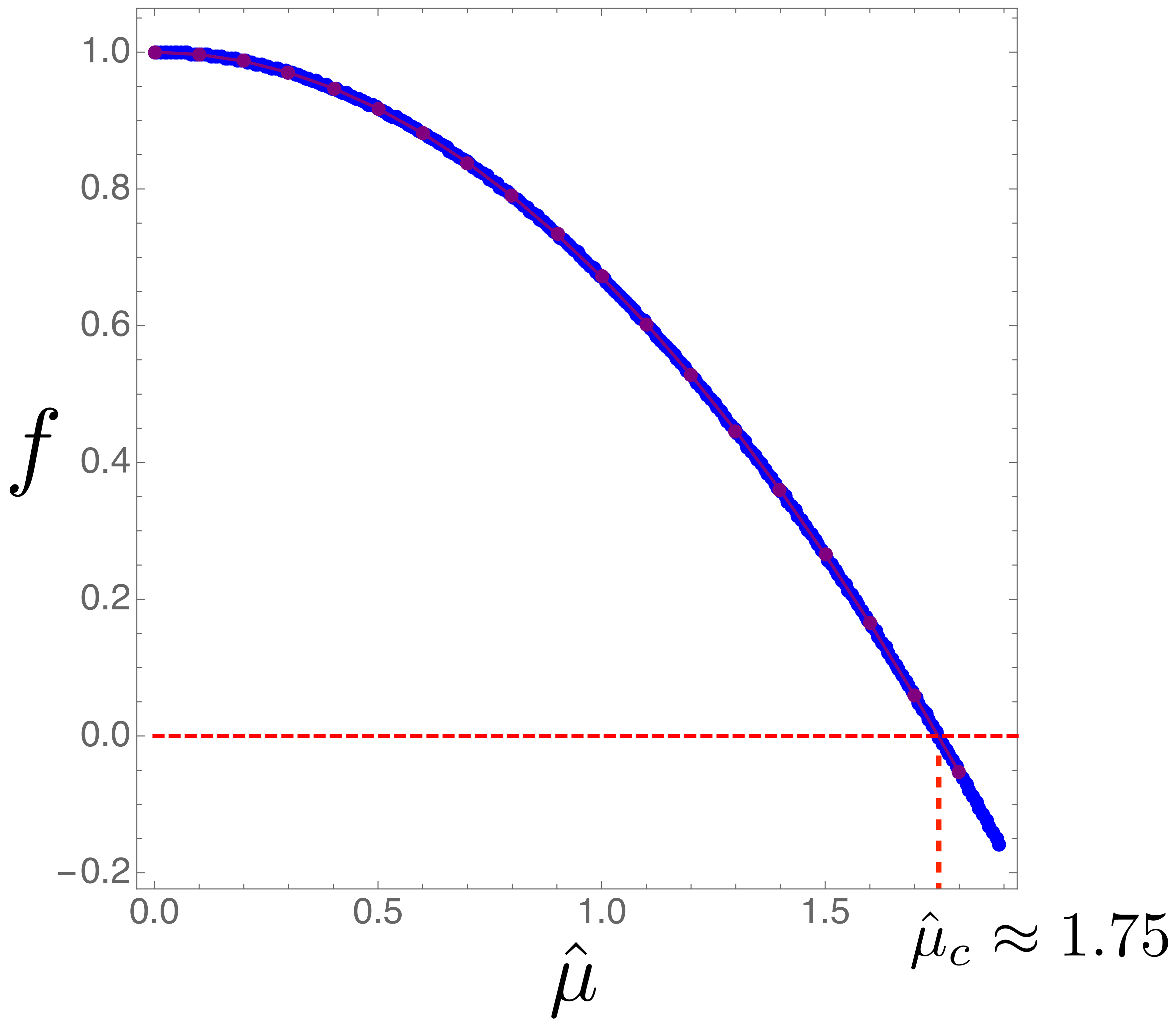

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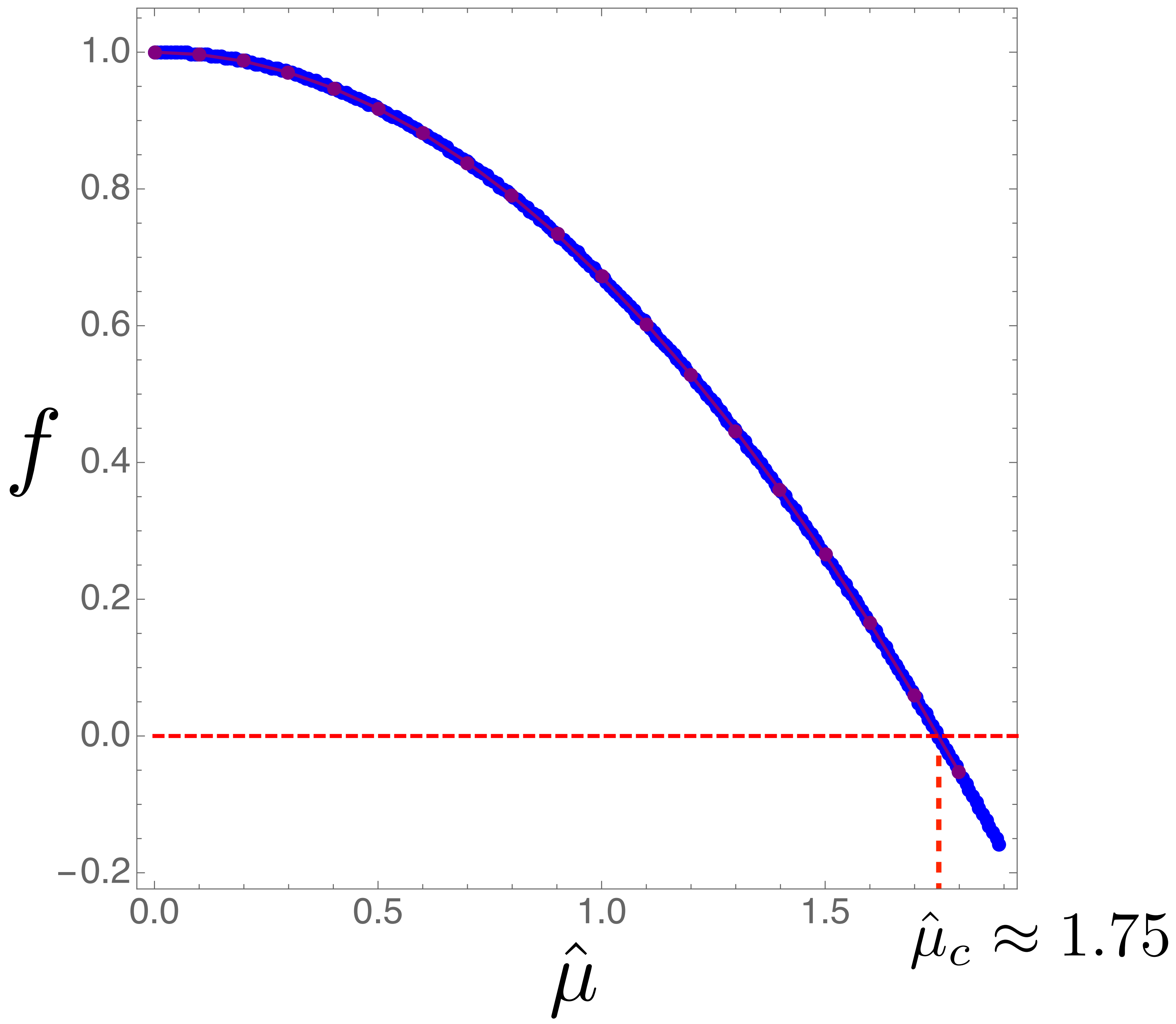

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- BH is thermodynamically stable for $\hat{\mu} < \hat{\mu}_c$ $c = T\left(\frac{\partial S}{\partial T}\right)_{\mu} \Rightarrow \frac{c}{S} = \frac{9}{5} - \hat{\mu}\frac{\partial}{\partial\hat{\mu}}\log s(\hat{\mu}) > 0$

Phase diagram at large N

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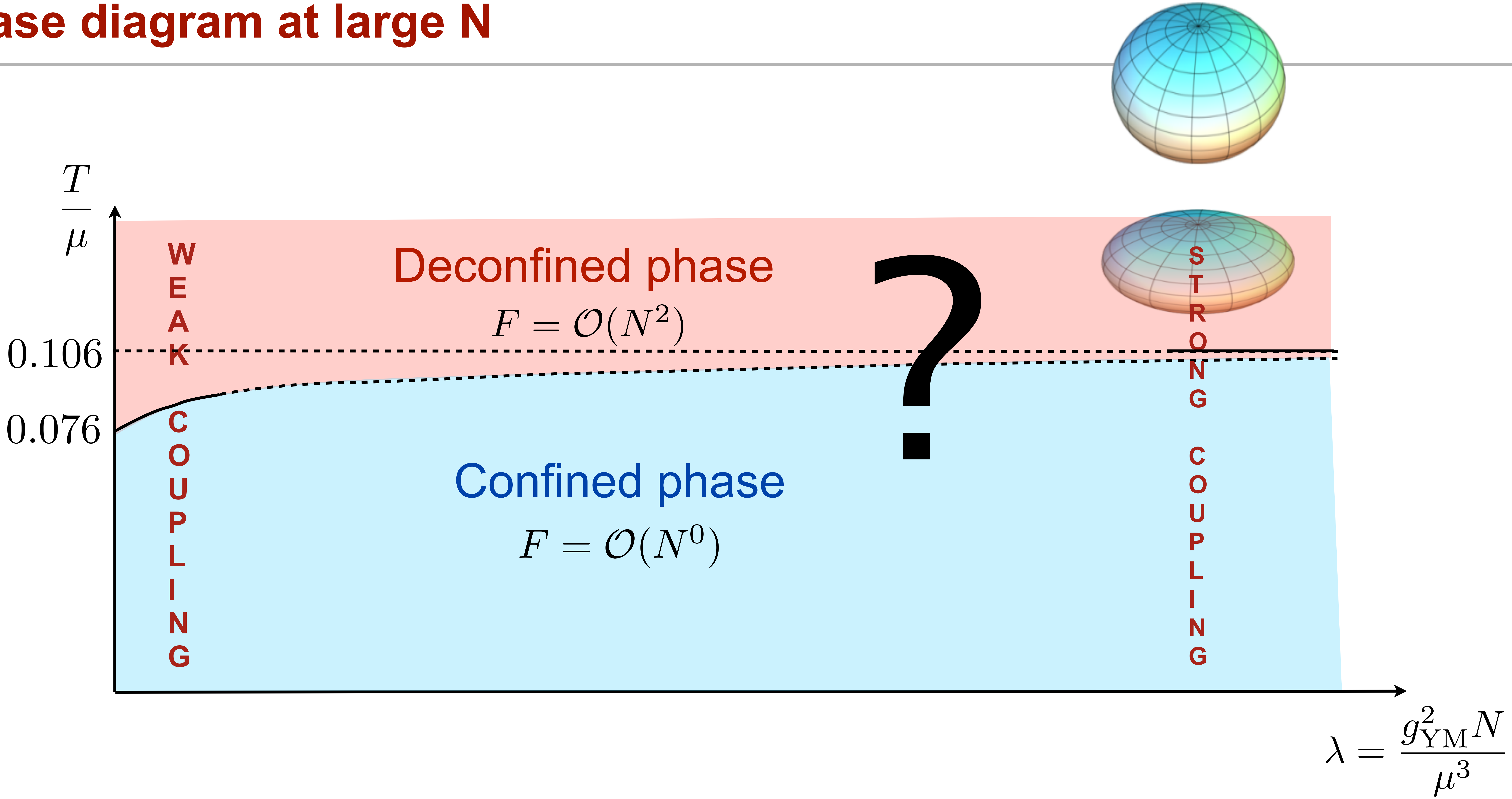

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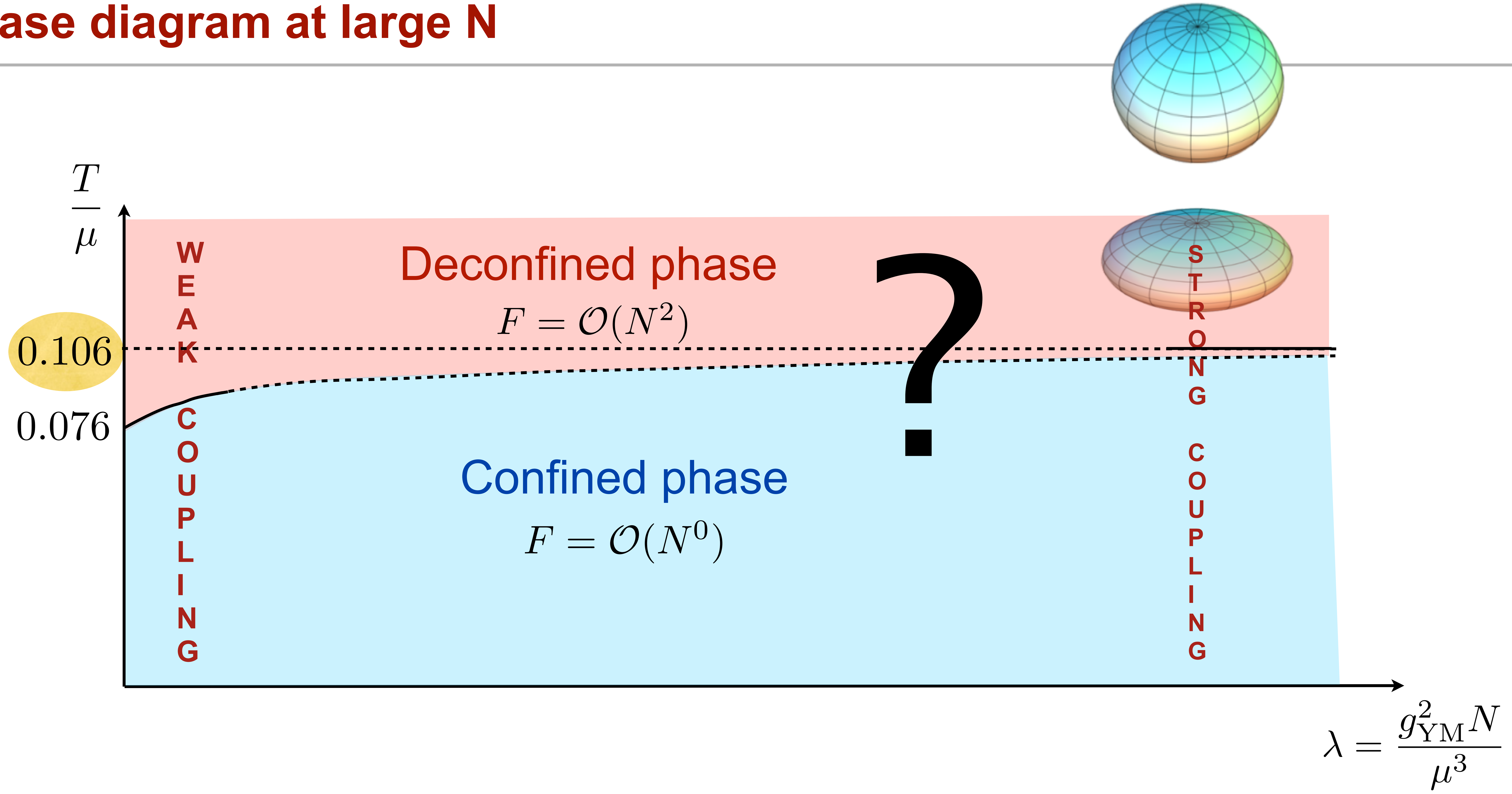

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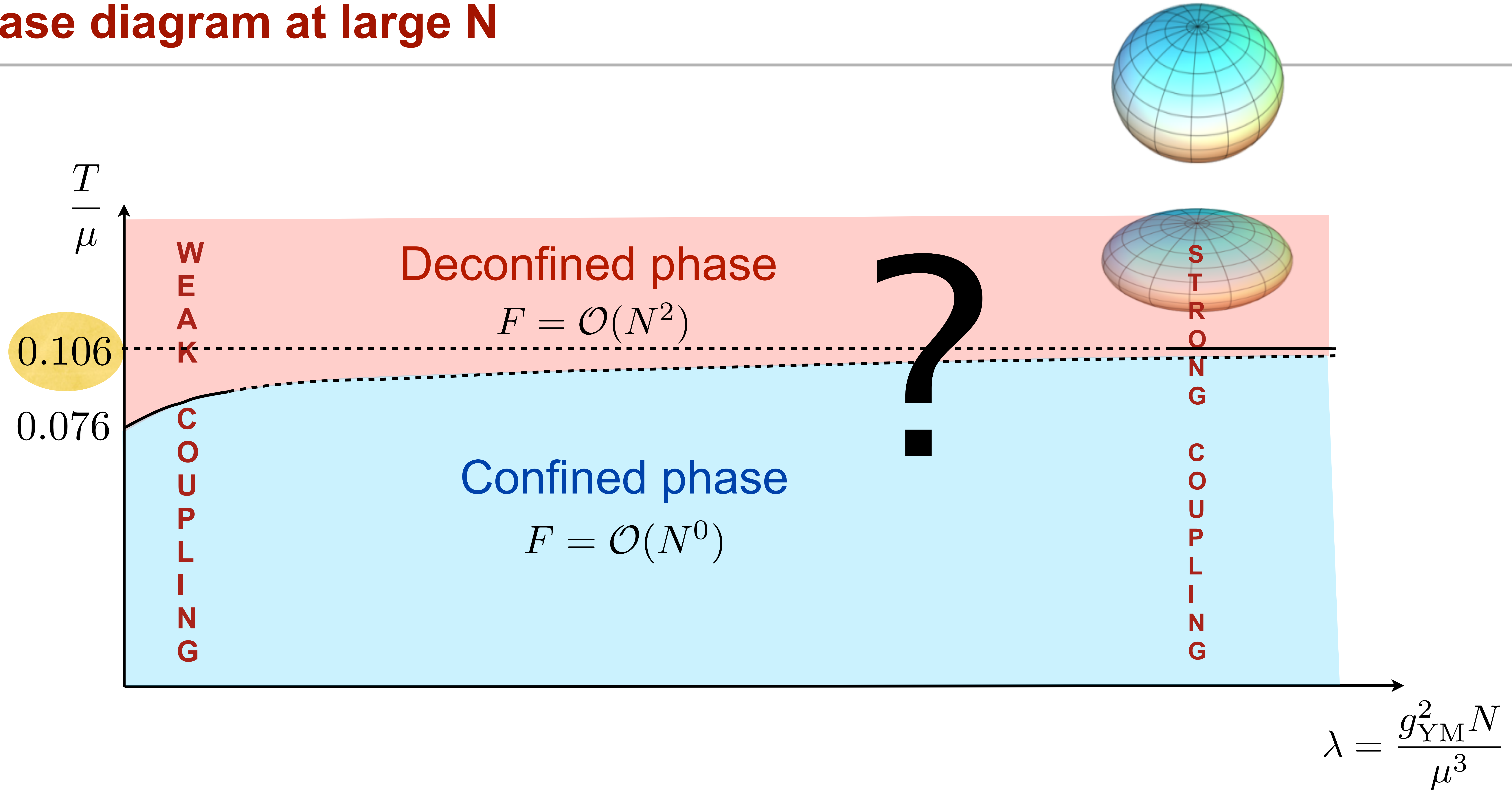

Very similar to SYM on a 3-sphere $(\mu \equiv 1/R)$

[Aharony, Marsano, Minwalla, Papadodimas, van Raamsdonk ’03]

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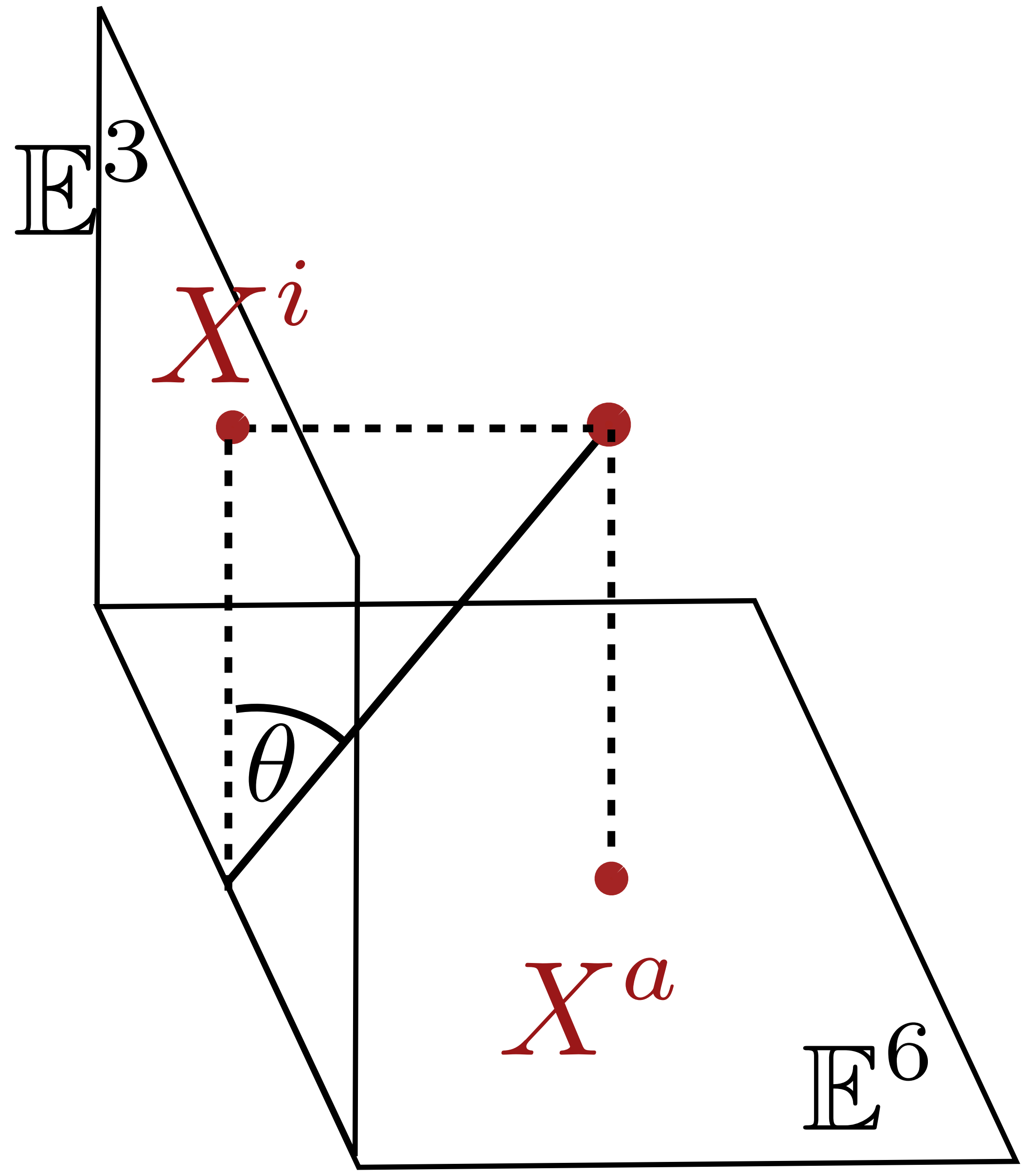

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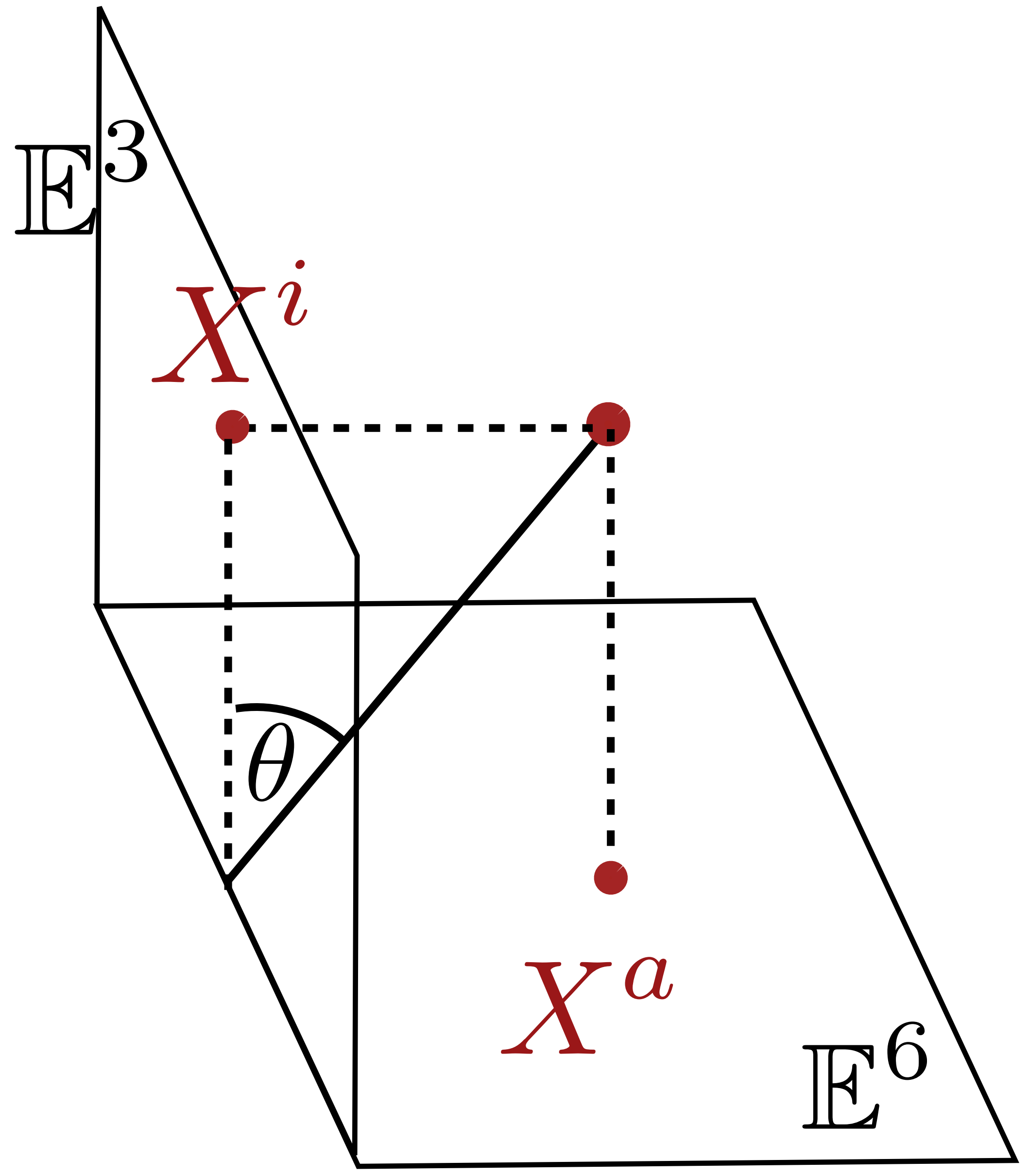

- Boundary metric has $SO(9)$ symmetry, so $\tilde{Q}_i^j(x)$ are harmonic functions on S^8 . Thus we can classify the $SO(6)\times SO(3)$ invariant perturbations according to $SO(9)$ spin. This helps to establish bulk field / operator correspondence.

- 2- form modes in the asymptotic expansion $C = (M\, d\eta + L\, d\zeta) \wedge d^2\Omega_2$

$$v(x,y) = \sum_{l\, odd} \Big(\alpha_l\, f_l(y) + \tilde{\alpha}_l\, \tilde{f}_l(y)\Big)\, \mathbb{H}_l(x) \;\; + \;\; \text{back reaction}$$

$$f_l(y) \sim y^{1+l}$$

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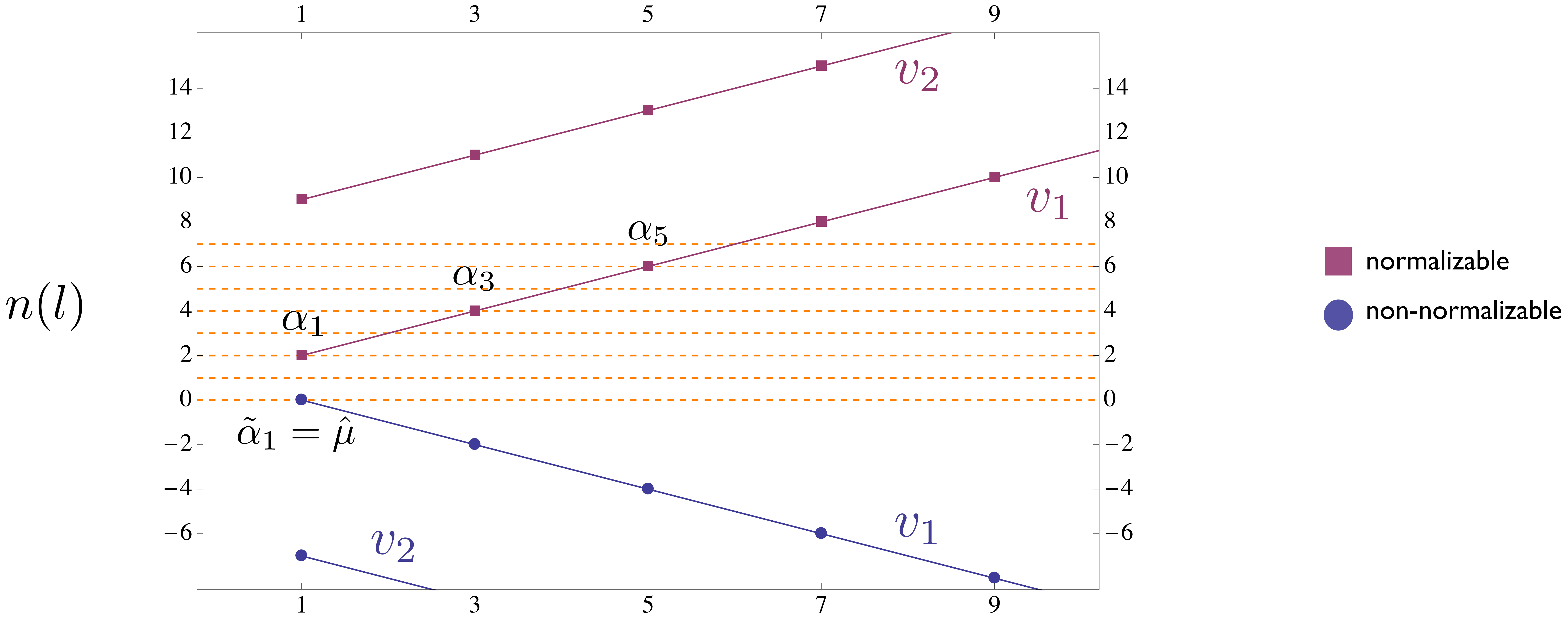

$$\mathcal{O} \sim \epsilon^{ijk} \mathrm{Tr}\left(X_i X_j X_k\, X_{A_1} \ldots X_{A_{l-1}}\right), \qquad l \geq 1 \text{ odd}$$

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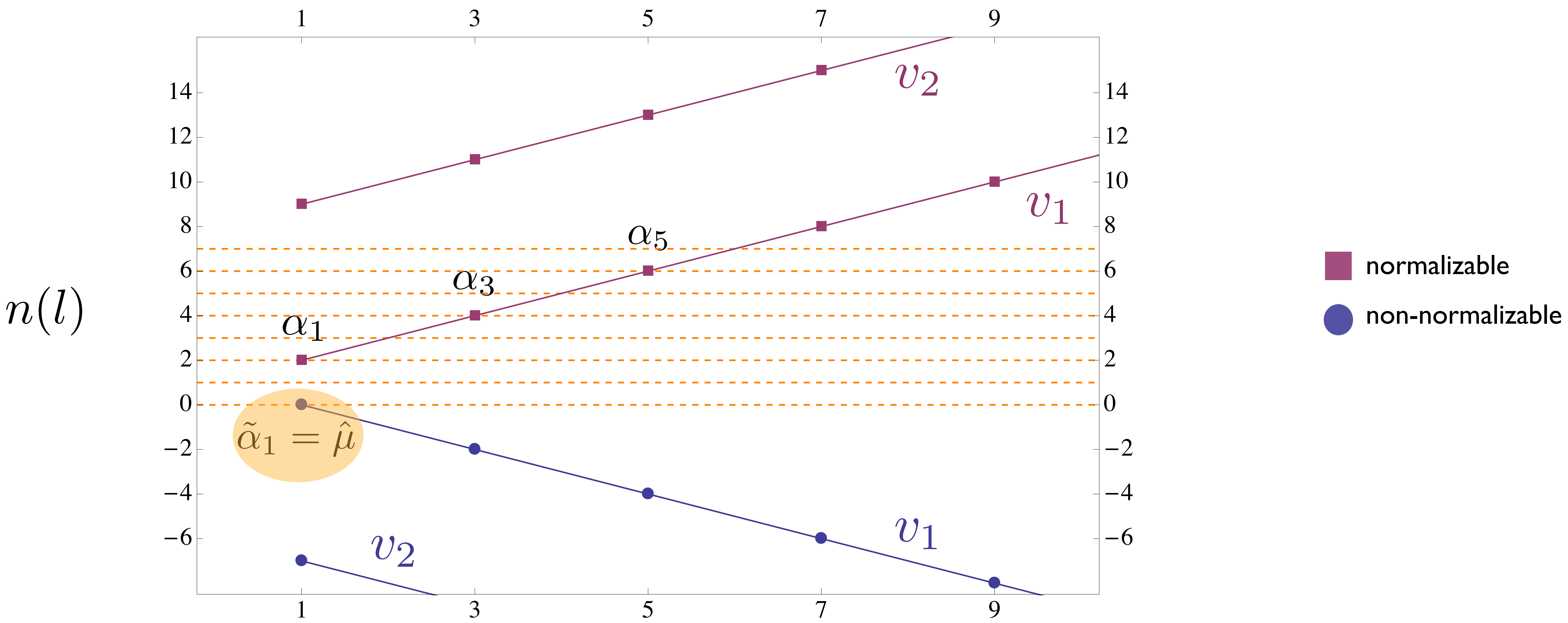

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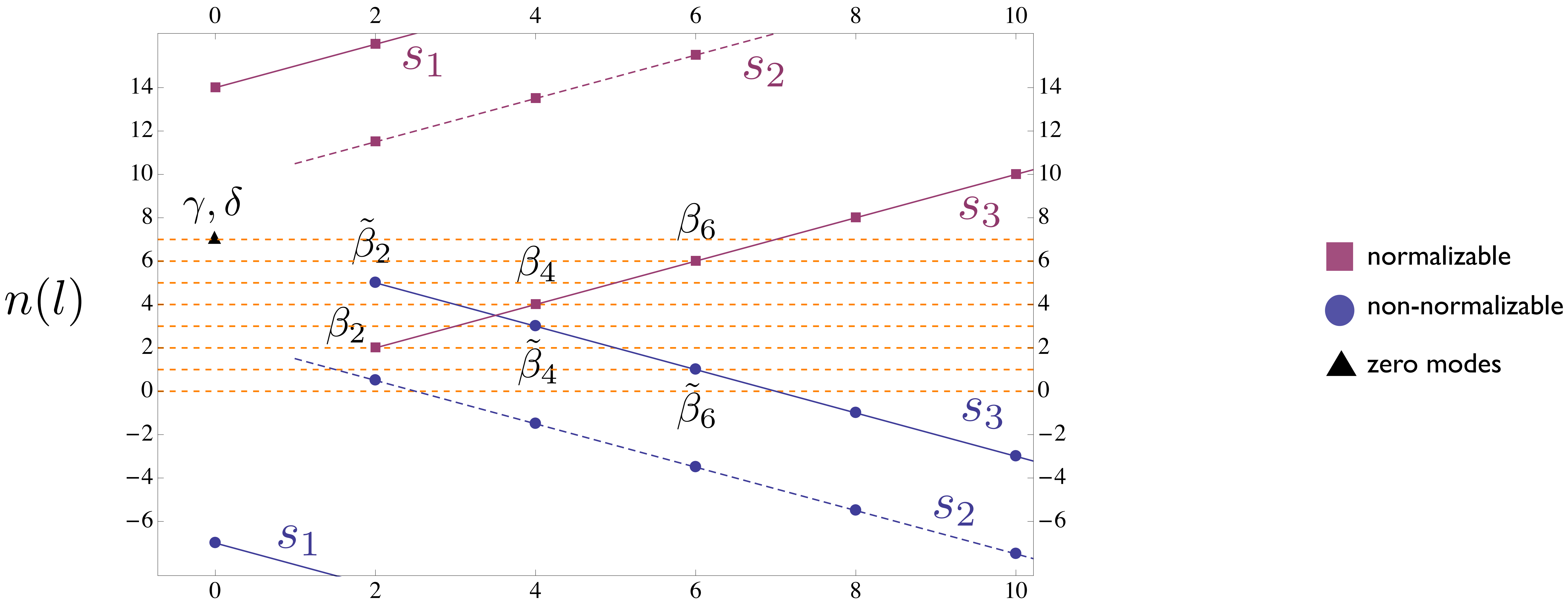

$$\mathcal{O} \sim \mathrm{Tr}\,(X_{A_1} \ldots X_{A_l})\ , \qquad l \geq 2 \ \ \text{even}$$

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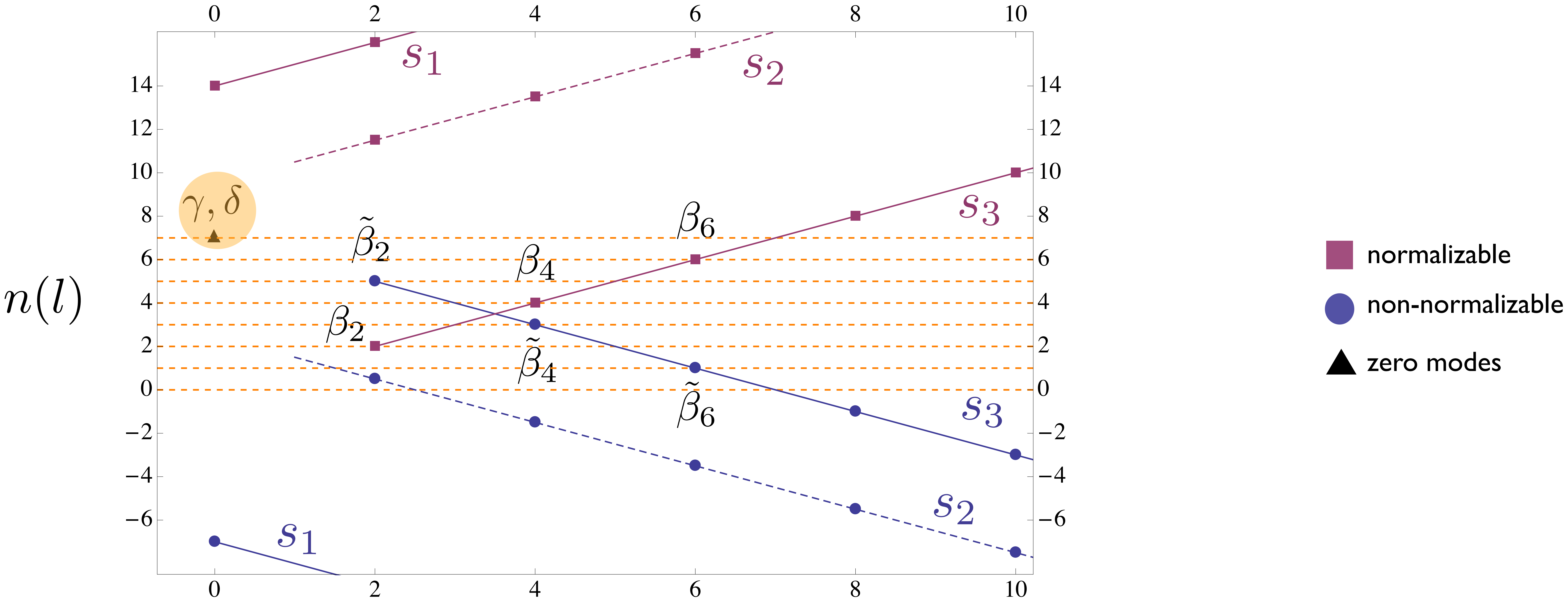
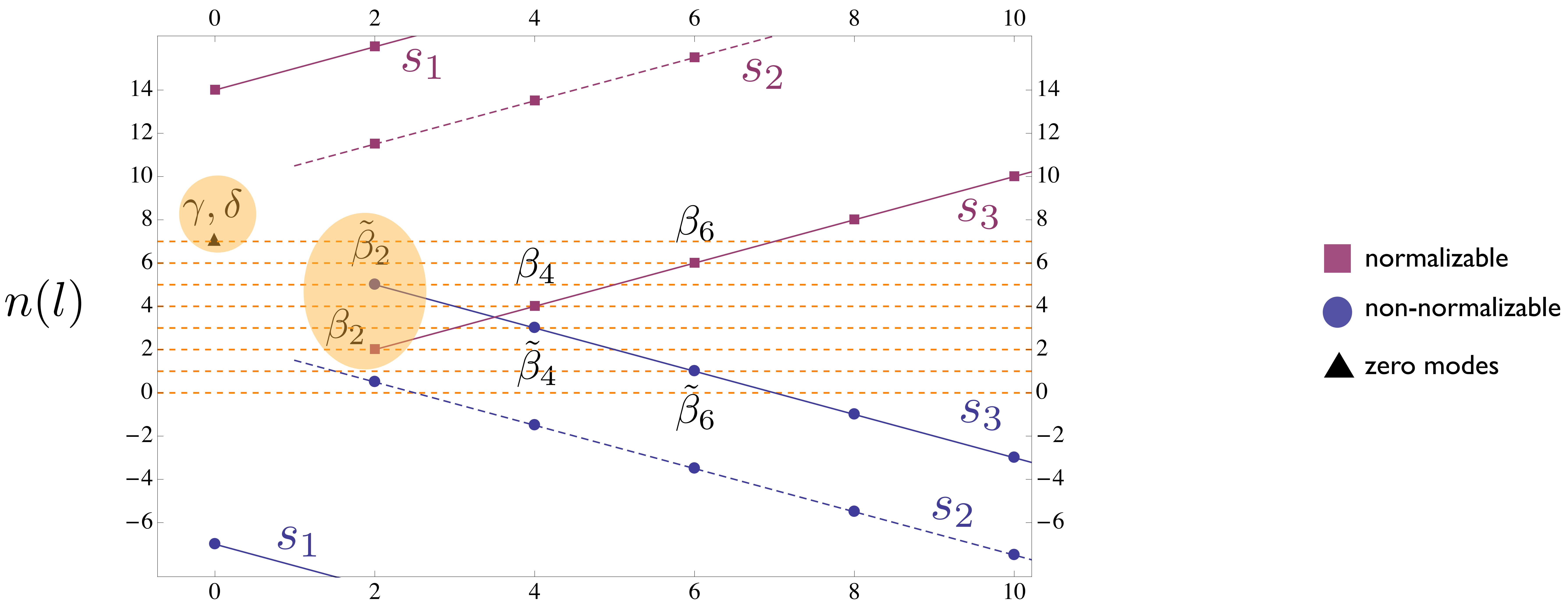


$$\mathcal{O} \sim \mathrm{Tr}\left(X_{A_1} \ldots X_{A_l}\right)\,, \qquad l \geq 2 \ \ \text{even}$$

- scalar modes in the asymptotic expansion

$$s(x,y) = \sum_{l\,even} \Big(\beta_l\, g_l(y) + \tilde{\beta}_l\, \tilde{g}_l(y)\Big)\, \mathbb{S}_l(x) \ + \ \text{back reaction}$$

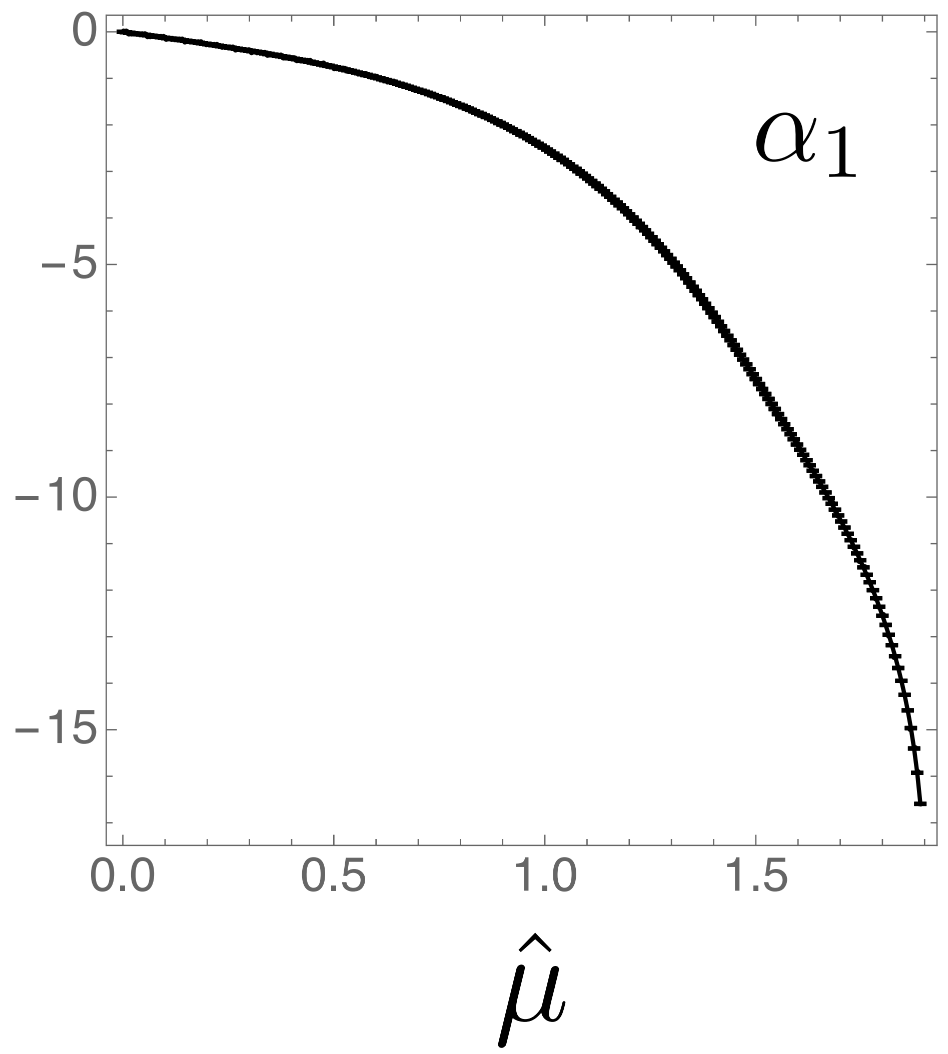
$SO(6) \times SO(3)$ invariant harmonic scalar



$$\mathcal{O} \sim \mathrm{Tr}\,(X_{A_1} \dots X_{A_l})\ , \qquad l \geq 2 \ \ \text{even}$$

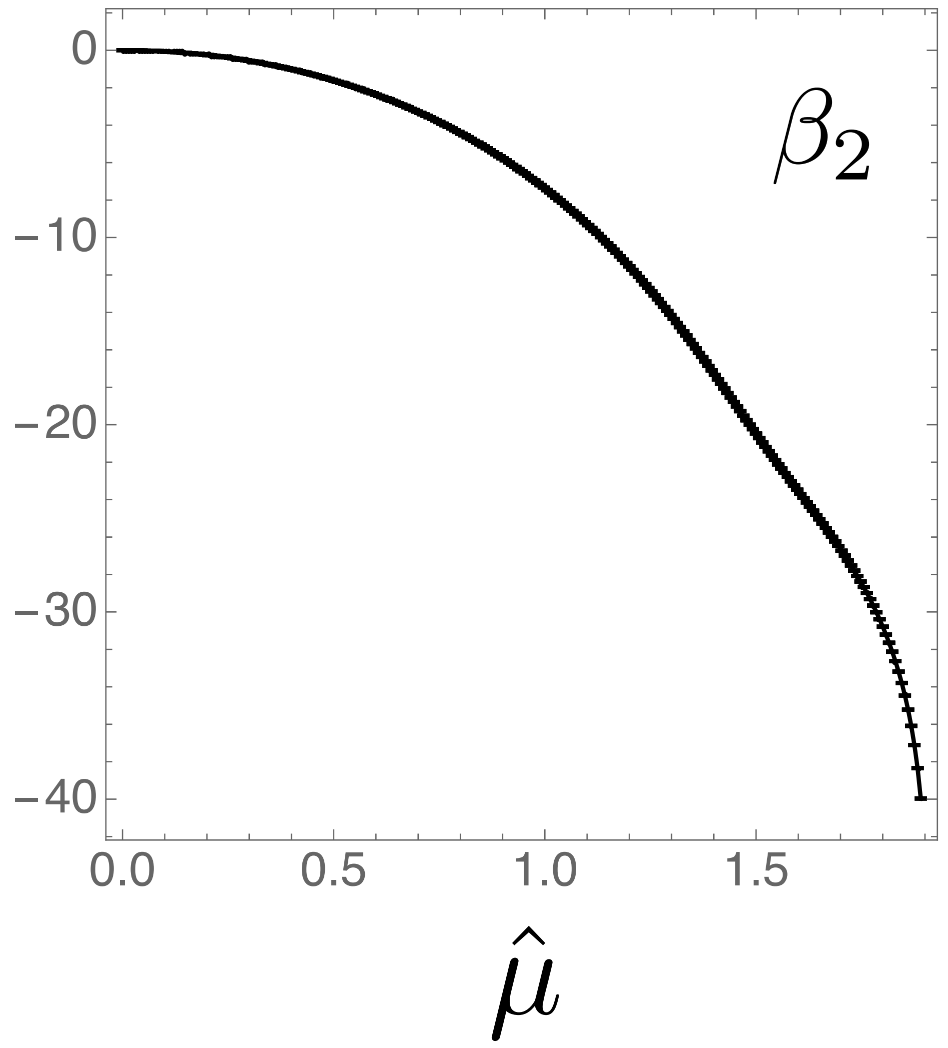
- Vevs read from normalizable modes appear first at order y^2

2-form $(l = 1)$



$\epsilon^{ijk}\langle \mathrm{Tr}\,(X_i X_j X_k)\rangle$

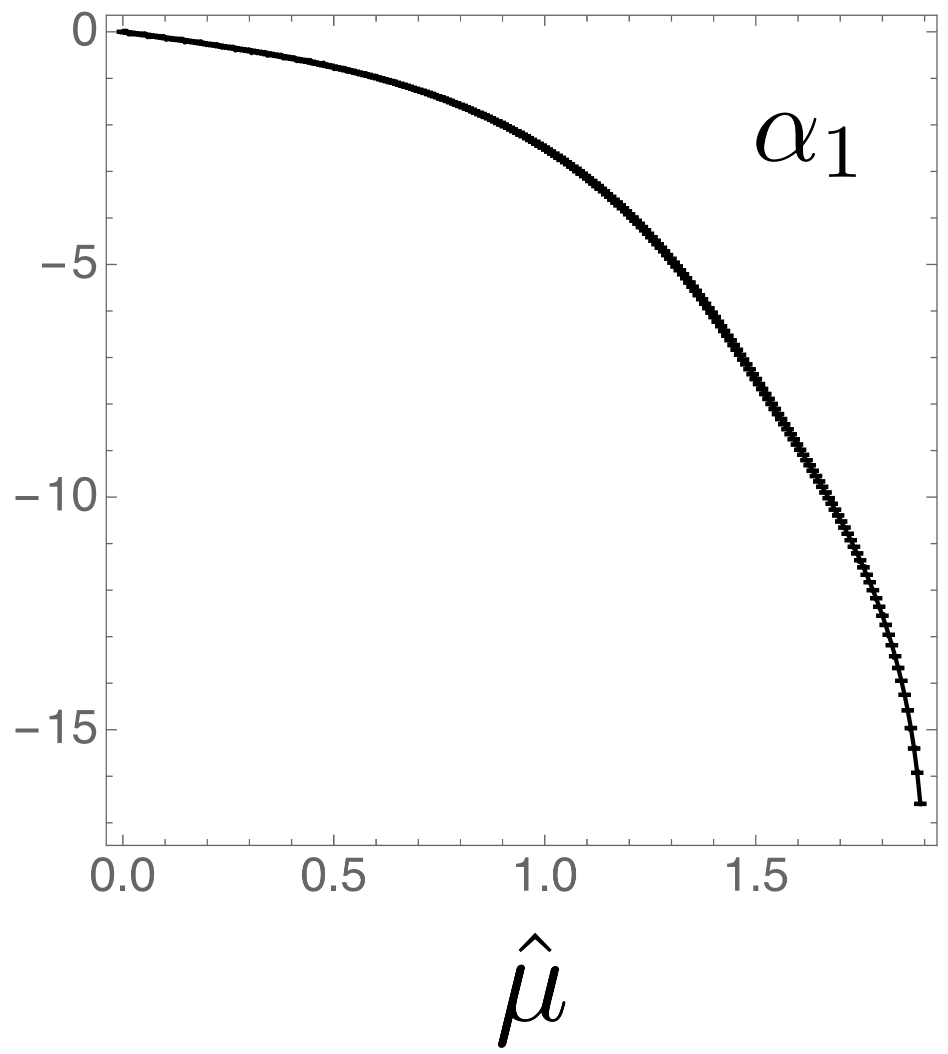
Scalar $(l = 2)$



$\langle \mathrm{Tr}\left(2X_i X^i - X_a X^a\right)\rangle$

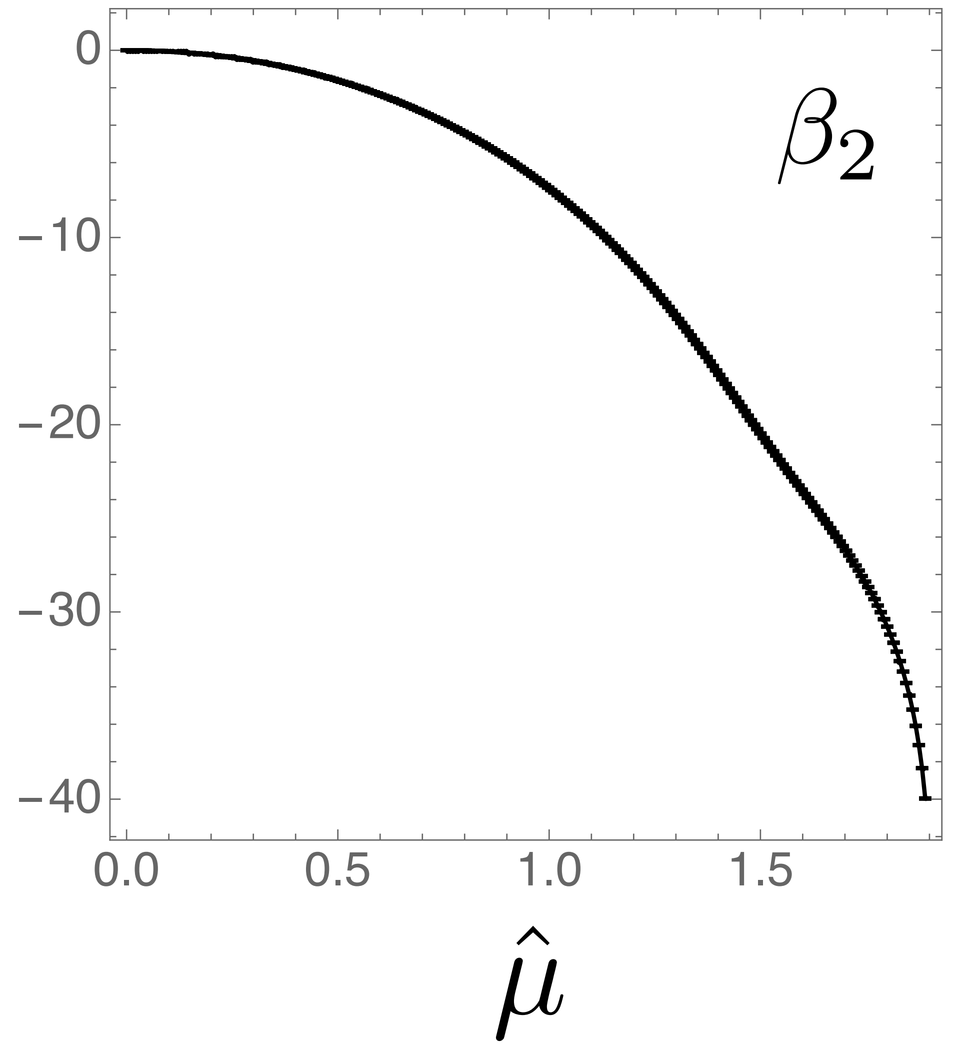
- Vevs read from normalizable modes appear first at order y^2

2-form $(l = 1)$



$\epsilon^{ijk}\langle \mathrm{Tr}\,(X_i X_j X_k)\rangle$

Scalar $(l = 2)$



$\langle \mathrm{Tr}\left(2X_i X^i - X_a X^a\right)\rangle$

- Smarr formulae involve coefficients in asymptotic expansion up to order y^7

Numerics pass this highly non-trival check with 0.05% accuracy

Future work

- Confirm phase diagram with Monte-Carlo simulations of PWMM; confirm predictions for expectation values of operators dual to normalizable modes that are turned on

- Study dynamical stability of our BH

- Construct BH duals of other vacua (different horizon topology)
  (caveat: we really only determined upper limit on critical temperature)

- **Deeper question:** What makes the PWMM special?
  What are the minimal ingredients of a quantum mechanical system such that it gives rise to classical gravity in the limit of many degrees of freedom?

THANK YOU