Classical and quantum temperature fluctuations via holography

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Holographic methods and applications,
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Outline

- Motivation and setup of the problem
- Hydrodynamics of charged liquid on the sphere
- Quasinormal modes of the Reissner-Nordström AdS black hole
- Phase digram of classical/quantum behavior

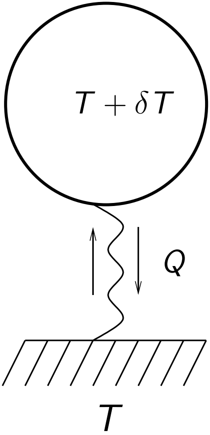
Temperature fluctuations of the quantum dot

The system (quantum dot) connected to the reservoir:

$$\langle \delta T^2 \rangle = \begin{cases} \frac{T^2}{C_v} & , T \gg \tau \quad \text{"classical"} \\ \frac{\hbar T}{\pi C_v \tau} \ln \omega_c \tau & , T \ll \tau \quad \text{"quantum"} \end{cases}$$

where τ - characteristic relaxation time

A. Balatsky and J.-X. Zhu, cond-mat/0202521



Hawking radiation of the Black Hole

Black hole in flat space

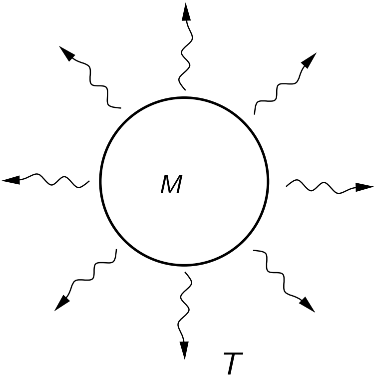
$$T_{Hawking} = \frac{1}{8\pi M}$$

In flat space the system is unstable

$$C_v < 0.$$

Hence the thermal equilibrium is never reached.

No relaxation $\Rightarrow$ No fluctuations



Black Hole in AdS

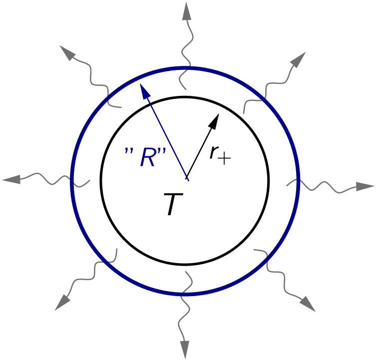
Black hole in flat space

$$T_{Hawking} = \frac{1 + 3\frac{r_+^2}{R^2}}{4\pi r_+}$$

Large BH is stable

$$C_v > 0.$$

But the radiation is not dynamical, hence there is no reservoir



Fluctuations of local temperature

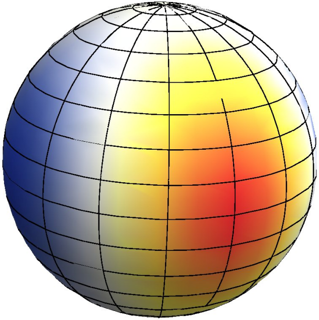
Mean temperature is fixed by the energy conservation law

$$\langle T \rangle = const,$$

but local fluctuations are allowed

$$\langle \delta T(x)^2 \rangle \neq 0$$

Hence we study spherical harmonics on the surface of the spherical black hole.



Fluctuation-dissipation theorem

Fluctuations are governed by the imaginary part of the retarded Green function

$$\langle \delta T^2 \rangle_l = \hbar \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, \mathrm{Im}\, G_l^R(\omega) \coth \frac{\hbar\omega}{2T} \, .$$

Where we take into account spherical harmonics separately

$$G_l^R(\omega) = i \int_0^\infty dt e^{i\omega t} \int d^d x \sqrt{g} Y_{l0}(x) \langle [T(t,x), T(0,0)] \rangle$$

Thus to study the fluctuations one needs to calculate G_l^R

Hydrodynamic equations for the charged liquid

With external sources

$$\nabla_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda \qquad \nabla_\mu J^\mu = 0$$

where $F^{\mu\nu}$ is a strength of external field.
Constituent equations

$$\begin{aligned} T^{\mu\nu} =& \epsilon u^\mu u^\nu + p\Delta^{\mu\nu} \\ & - \eta \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{d} g_{\alpha\beta}\nabla_\mu u^\mu\right) - \zeta \Delta^{\mu\nu}\nabla_\lambda u^\lambda, \\ J^\mu =& n u^\mu - \sigma T \Delta^{\mu\nu}\nabla_\nu(\mu/T) + \sigma \Delta^{\mu\lambda} F_{\lambda\nu} u^\nu, \\ \Delta^{\mu\nu} =& g^{\mu\nu} + u^\mu u^\nu, \end{aligned}$$

d-number of spacial dimensions.

Hydrodynamic equations for the charged liquid

We introduce sources $\delta g_{tt} = -2h$, $\delta A_t = \phi$ and derive system of coupled equations for $\delta\epsilon$ and δn

$$\left[\omega^2 - \beta_1 k^2 + 2i\omega\Gamma\right]\delta\epsilon - \beta_2 k^2 \delta n = (\epsilon + p)k^2 h + nk^2\phi$$

$$\left[\frac{n}{\epsilon + p} i\omega + k^2\sigma\alpha_1\right]\delta\epsilon + \left[-i\omega + k^2\sigma\alpha_2\right]\delta n = -\sigma k^2 \phi$$

$$k^2 \equiv \frac{l(l+d-1)}{R^2},$$

$$\Gamma \equiv \frac{1}{(\epsilon+p)R^2}\left[\eta\frac{(d-1)(l-1)(l+d)}{d} + \zeta\frac{l(l+d-1)}{2}\right]$$

$$\alpha_1 = T\left.\frac{\partial(\mu/T)}{\partial\epsilon}\right|_n, \quad \alpha_2 = T\left.\frac{\partial(\mu/T)}{\partial n}\right|_\epsilon, \quad \beta_1 = \left.\frac{\partial p}{\partial\epsilon}\right|_n, \quad \beta_2 = \left.\frac{\partial p}{\partial n}\right|_\epsilon$$

Hydrodynamic Green function

The solution is expressed via the Green function $\begin{pmatrix}\delta\epsilon\\ \delta n\end{pmatrix} = G\begin{pmatrix}h\\ \phi\end{pmatrix}$:

$$G = \frac{1}{\mathcal{P}}\begin{pmatrix} k^2 w\omega + i\sigma k^4 w\alpha_2 & k^2 n\omega + i\sigma k^4(n\alpha_2 - \beta_2) \\ k^2 n\omega - i\sigma k^4 w\alpha_1 & -i\sigma k^2\omega^2 + k^2\omega\left(\frac{n^2}{w} + 2\sigma\Gamma\right) - i\sigma k^4(n\alpha_1 - \beta_1)\end{pmatrix}$$

where enthalpy density $w = \epsilon + p$ and denominator accounts for the "sound" and "diffusion" poles

$$\mathcal{P} = (\omega^2 - \Omega^2 + 2i\omega\hat{\Gamma})(\omega + i\mathcal{D}k^2)$$

$$\Omega^2 = k^2 \left.\frac{\partial p}{\partial\epsilon}\right|_s \equiv k^2 v_s^2, \qquad \hat{\Gamma} = \Gamma + \frac{1}{2}\frac{k^2}{w}\frac{\sigma}{v_s^2}\left.\frac{\partial p}{\partial n}\right|_\epsilon^2$$

$$\mathcal{D} = \frac{\sigma}{v_s^2}\left(T\left.\frac{\partial(\mu/T)}{\partial n}\right|_\epsilon \left.\frac{\partial p}{\partial \epsilon}\right|_n - T\left.\frac{\partial(\mu/T)}{\partial \epsilon}\right|_n \left.\frac{\partial p}{\partial n}\right|_\epsilon\right)$$

Hydrodynamics of CFT

In CFT thermodynamic relations simplify because $\epsilon = 3p$

$$\langle \delta\epsilon(k)\delta\epsilon(-k)\rangle = \frac{w}{v_s^2}\frac{\Omega^2}{\omega^2 - \Omega^2 + 2i\Gamma\omega}$$

$$\langle \delta\epsilon(k)\delta n(-k)\rangle = \frac{n}{v_s^2}\frac{\Omega^2}{\omega^2 - \Omega^2 + 2i\Gamma\omega}$$

$$\langle \delta n(k)\delta n(-k)\rangle = \frac{n^2}{v_s^2 w}\frac{\Omega^2}{\omega^2 - \Omega^2 + 2i\Gamma\omega} + \frac{-i\sigma k^2}{\omega + i\mathcal{D}k^2}$$

There is no diffusive pole in the correlators of energy.
Sound mode is the same as in the uncharged case.

Temperature correlation function

Expanding $\delta T = \frac{\partial T}{\partial \epsilon}\Big|_n \delta\epsilon + \frac{\partial T}{\partial n}\Big|_\epsilon \delta n$ we write the correlation function as

$$\langle \delta T \delta T \rangle_I = \frac{w}{v_s^2}\left(\frac{\delta T}{\delta \epsilon}\bigg|_n + \frac{n}{w}\frac{\delta T}{\delta n}\bigg|_\epsilon\right)^2 \frac{\Omega^2}{\omega^2 - \Omega^2 + 2i\Gamma\omega} - \frac{\delta T}{\delta n}\bigg|_\epsilon^2 \frac{i\sigma k^2}{\omega + i\mathcal{D}k^2}$$

or after some thermodynamics

$$\langle \delta T \delta T \rangle_I = \frac{T^2}{3w}\frac{\Omega^2}{\omega^2 - \Omega^2 + 2i\Gamma\omega} + \frac{n^2 T^2}{w(3n^2 - w\chi)}\frac{i\mathcal{D}k^2}{\omega + i\mathcal{D}k^2}$$

where $\chi = \frac{\partial n}{\partial \mu}\Big|_T$ is susceptibility

Classical limit

At large temperatures the $coth\frac{\omega}{2T}$ in FDT may be substituted by $\frac{2T}{\omega}$

$$\langle\delta T^2\rangle \xrightarrow{T\to\infty} T\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\frac{Im\langle\delta T(k)\delta T(-k)\rangle}{\omega}$$

And using Maxwell relations one shows that

$$\langle\delta T^2\rangle = \frac{T^2}{3}\frac{\chi}{w\chi - 3n^2} = \frac{T^2}{C_v}$$

which proves that the normalization of Green functions is right, as it agrees with the classical result

Quantum limit

In the sound mode the quantum limit is applicable when $T \ll T_s = \frac{v_s k}{2\pi} \sim \frac{1}{R}$

$$\langle \delta T^2 \rangle_{sound} \simeq \hbar \frac{T\Omega}{2} \frac{T}{3w} =\sim \frac{1}{R}$$

The diffusion mode becomes quantum when $T \ll T_d = \frac{\mathcal{D}k^2}{2\pi} \sim \frac{1}{R^2}$ and

$$\langle \delta T^2 \rangle_{diffuse} \simeq \frac{\hbar}{\pi} \frac{T^2 n^2 \mathcal{D}}{w(3w\chi - n^2)} \frac{1}{2} k^2 \log\left[\frac{\Omega}{\mathcal{D}k^2}\right] \sim \frac{1}{R^2}$$

These results show, that in hydrodynamic approximation the transition to the quantum regime ($T \ll R^{-1}$) never occurs. One need to consider full gravitational problem with finite R.

AdS Reissner-Nordström black hole

The metric of the charged black hole in AdS is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{R^2}$$

Thermodynamic quantities are

$$T = \frac{1 + 3\frac{r_+^2}{R^2}}{4\pi r_+}\left(1 - \frac{Q^2}{Q_{\rm ext}^2}\right), \qquad Q_{\rm ext} = r_+\sqrt{1 + 3\frac{r_+^2}{R^2}},$$

$$S = \pi r_+^2, \qquad \mu = \frac{Q}{r_+}$$

Quasinormal modes of AdS-RN BH

We look for a spectrum of quasinormal modes of the spherical black hole, which include

$$\delta g_{00} \leftrightarrow \delta\epsilon, \qquad \delta A_0 \leftrightarrow \delta n$$

For nontrivial spherical harmonics the perturbation (polar mode) involves other metric components as well

$$\delta g \sim \begin{pmatrix} \times & \times & 0 & 0 \\ \times & \times & 0 & 0 \\ 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \end{pmatrix}$$

Quasinormal modes of AdS-RN BH

Decoupling of modes leads to the characteristic equation
(F. Mellor and I. Moss, Phys.Rev. D41, 403 (1990))

$$\frac{\Delta}{r^2}\frac{d}{dr}\left(\frac{\Delta}{r^2}\frac{d}{dr}Z_i^+\right)+\omega^2 Z_i^+ = V_i^+ Z_i^+ \qquad (i=1,2)$$

with Schrödinger-type potentials

$$V_1^+ = \frac{\Delta}{r^5}\left[U+\frac{1}{2}(p_1-p_2)W\right], \qquad V_2^+ = \frac{\Delta}{r^5}\left[U-\frac{1}{2}(p_1-p_2)W\right]$$

$$U = (2nr+3M)\,W + \left(\bar{\omega}-nr-M+2\frac{r^3}{R^2}\right)-\frac{2n\Delta}{\bar{\omega}},$$

$$W = \frac{\Delta}{r\bar{\omega}^2}(2nr+3M)+\frac{1}{\bar{\omega}}\left(nr+M-2\frac{r^3}{R^2}\right)$$

$$\Delta = r^2-2Mr+Q^2+\frac{r^4}{R^2}, \qquad \bar{\omega} = nr+3M-\frac{2Q^2}{r}$$

$$n=\frac{1}{2}(l-1)(l+2), \quad p_1 = 3M+(9M^2+4Q^2\mu^2)^{1/2}, \quad p_2 = 3M-(9M^2+4Q^2\mu^2)^{1/2}$$

Robin boundary conditions

To obtain the spectrum one solves the equation with appropriate boundary conditions (J. J. Friess et al., hep-th/0611005)

$$\frac{\Delta}{r^2}\frac{d}{dr}\left(\frac{\Delta}{r^2}\frac{d}{dr}Z_i^+\right)+\omega^2 Z_i^+ = V_i^+ Z_i^+ \qquad (i=1,2)$$

On the boundary

$$Z_2^+\Big|_{r\to\infty} \sim 1-\left(\frac{3M}{n}+\frac{4Q^2}{3M+\sqrt{9M^2+4Q^2\mu^2}}\right)\frac{1}{r},$$

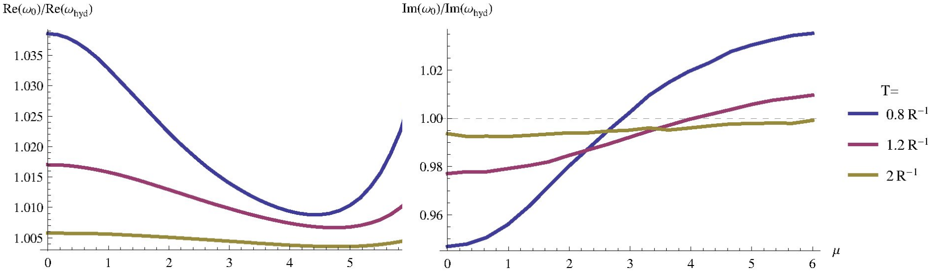
$$Z_1^+\Big|_{r\to\infty} \sim 1-\left(\frac{3M}{n}-\frac{3M+\sqrt{9M^2+4Q^2\mu^2}}{2n}\right)\frac{1}{r}.$$

And purely infalling on the horizon

Comparison with hydrodynamics

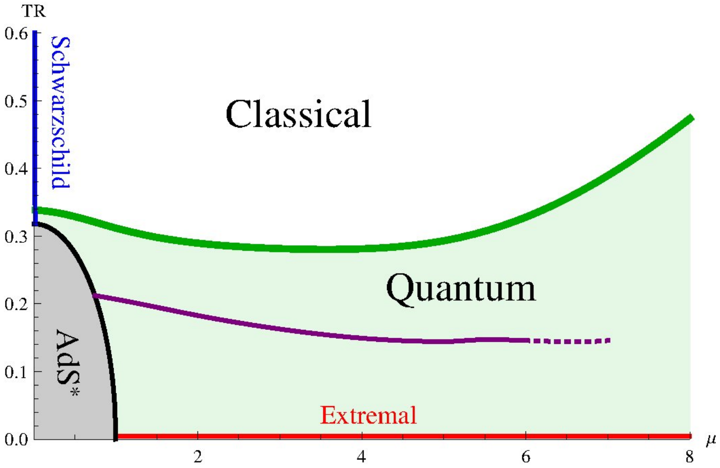
We compare the numerical results with hydrodynamic expression

$$\omega_{\mathrm{hyd}} = \pm \frac{1}{R}\sqrt{\frac{l(l+d-1)}{d}} - i\frac{(d-1)(l+d)l(l-1)}{R^2 d}\frac{s}{4\pi(\epsilon+p)},$$



Reasonable agreement at relatively low temperatures $T \sim R^{-1}$

Phase digram of the global RN-AdS BH



Conclusion

- CFT on the sphere experiences local temperature fluctuations
- Depending on the temperature and chemical potential the fluctuations may be either in quantum or in classical regime
- Charged black hole in AdS undergoes quantum-classical transition at low temperatures
- Extremal black hole can not be probed in the linear responce aproach.
- The gravitational calculus for charged black holes with compact horizons agrees with hydrodynamics if the proper boundary conditions are taken into account.