Entanglement Negativity in Conformal Field Theory

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P. Calabrese, J. Cardy and E.T.;	**[1206.3092]**
	[1210.5359]
P. Calabrese, L. Tagliacozzo and E.T.;	**[1302.1113]**
A. Coser, L. Tagliacozzo and E.T.;	**[1309.2189]**
P. Calabrese, J. Cardy and E.T.;	**[1408.3043]**
A. Coser, E.T. and P. Calabrese;	**[14xx.xxxx]**

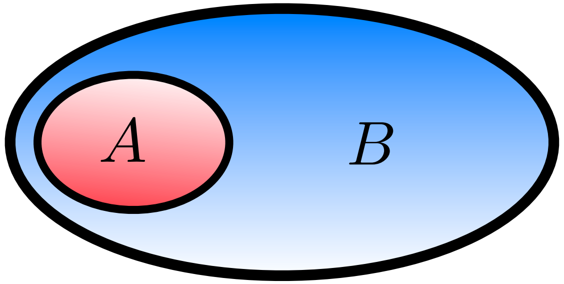
Holographic Methods and Applications (HoloGrav 2014)
Reykjavík, August 2014

Outline

- Entanglement in 2D CFT:
  - Motivations for negativity and definitions
  - Entanglement entropies for disjoint intervals
  - Entanglement negativity: pure and mixed states
  - Entanglement negativity at finite temperature
  - Entanglement negativity after a global quantum quench

Motivations for Negativity

- Ground state $\rho = |\Psi\rangle\langle\Psi|$ and *bipartite* system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$



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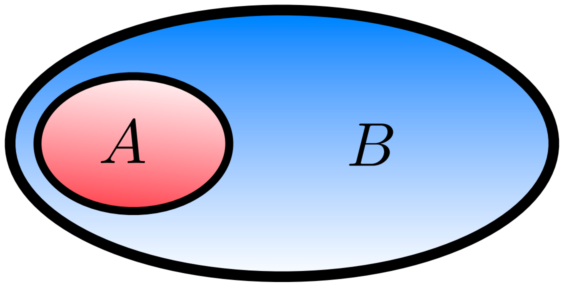

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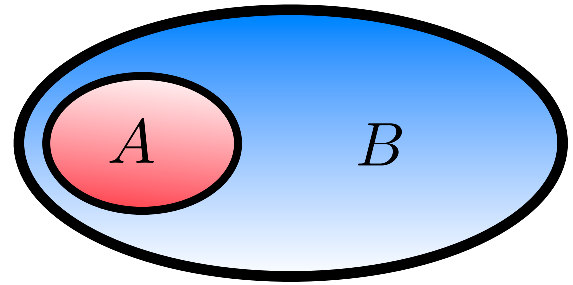

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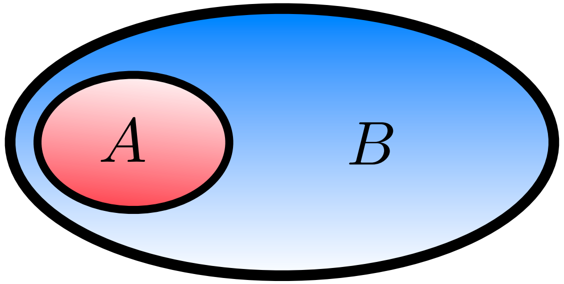

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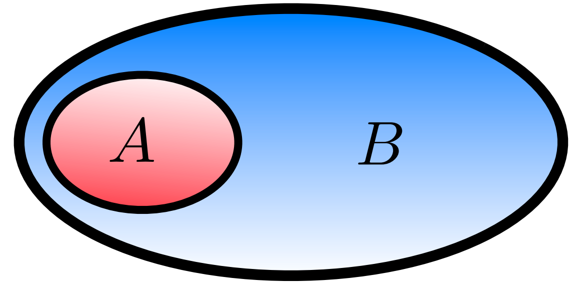

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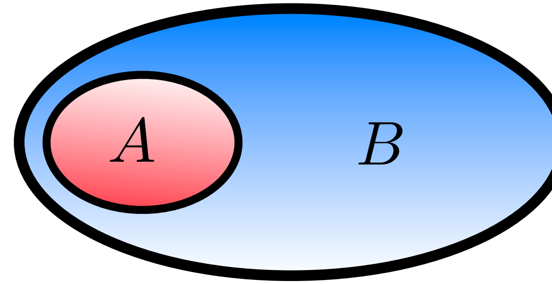

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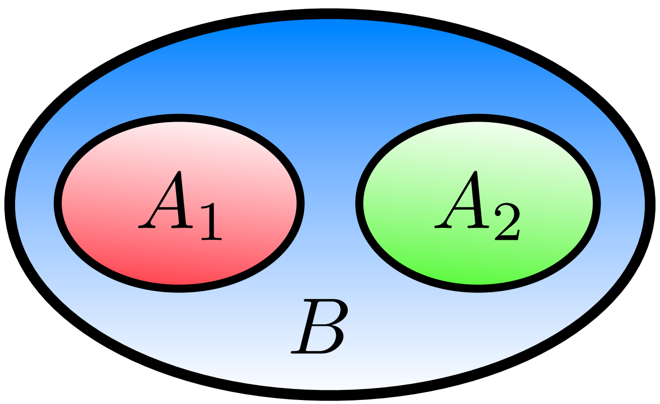

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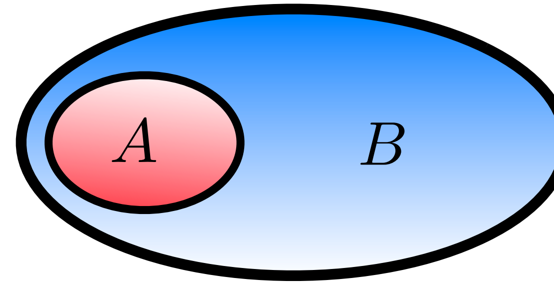

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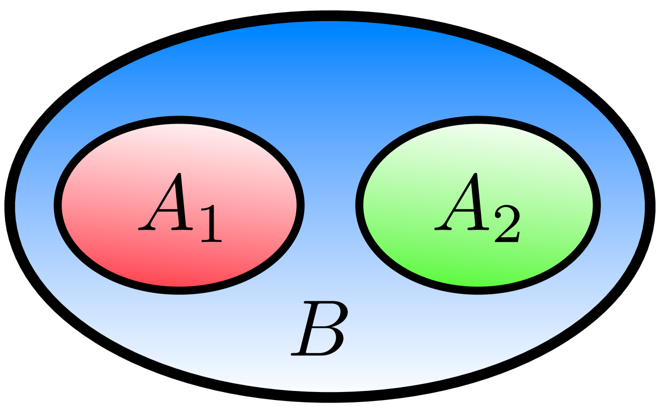

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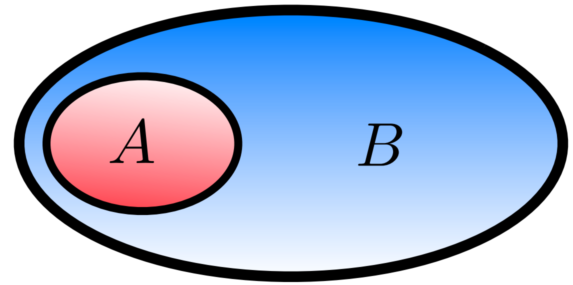

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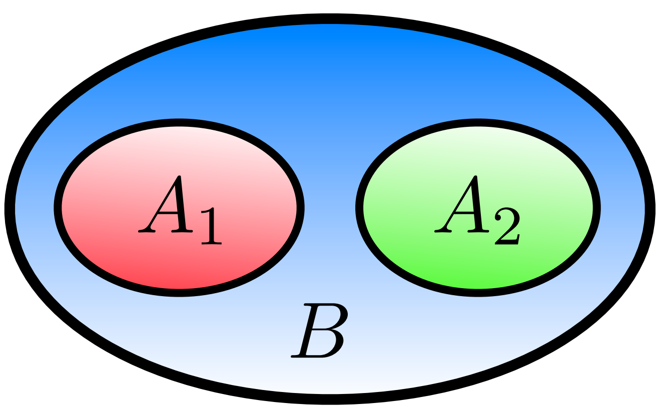

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  - The mutual information $S_{A_1} + S_{A_2} - S_{A_1\cup A_2}$ gives an upper bound
  - A *computable* measure of the entanglement is the *logarithmic negativity* [Vidal, Werner, (2002)]

Partial transpose & Negativity: definitions

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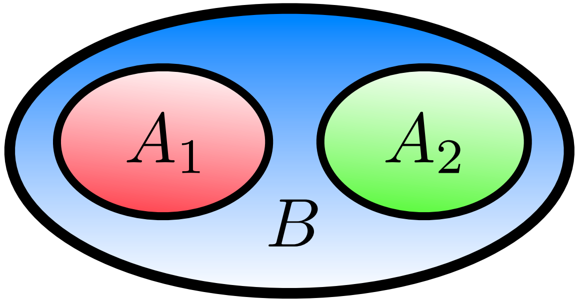

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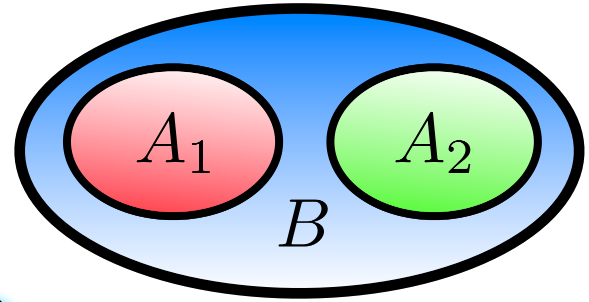

$$\boxed{\langle e_i^{(1)} e_j^{(2)} | \, \rho^{T_2} \, | e_k^{(1)} e_l^{(2)} \rangle \; = \; \langle e_i^{(1)} e_l^{(2)} | \, \rho \, | e_k^{(1)} e_j^{(2)} \rangle}$$

($|e_i^{(k)}\rangle$ base of $\mathcal{H}_{A_k}$)

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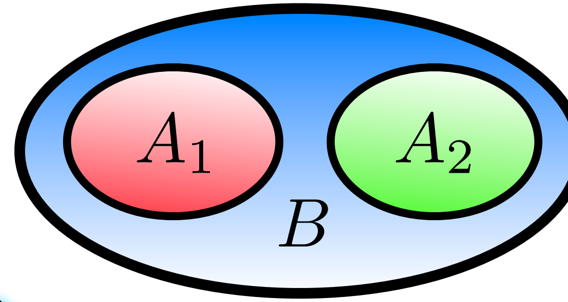

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$$\boxed{||\rho^{T_2}|| = \mathrm{Tr}|\rho^{T_2}| = \sum_i |\lambda_i| = 1 - 2 \sum_{\lambda_i < 0} \lambda_i}$$

λ_j eigenvalues of ρ^{T_2}

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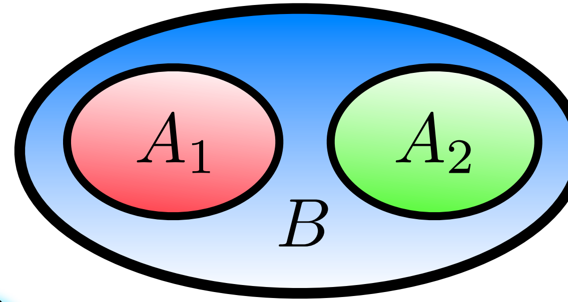

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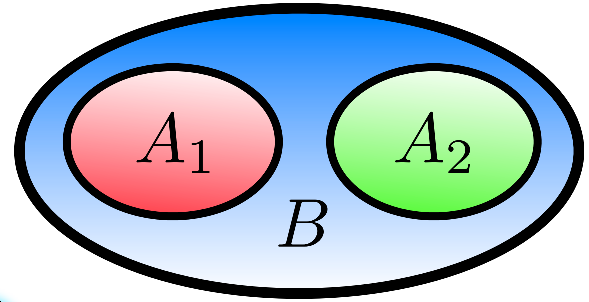

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- *Bipartite* system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ in *any* state ρ $\longrightarrow$ $\mathcal{E}_1 = \mathcal{E}_2$

Replica approach to Negativity

[Calabrese, Cardy, E.T., (2012)]

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- Taking $n_e \to 1$ we have $\boxed{\mathcal{E} = 2\log \mathrm{Tr}\rho_2^{1/2}}$ (Renyi entropy 1/2)

2D CFT: Renyi entropies as correlation functions

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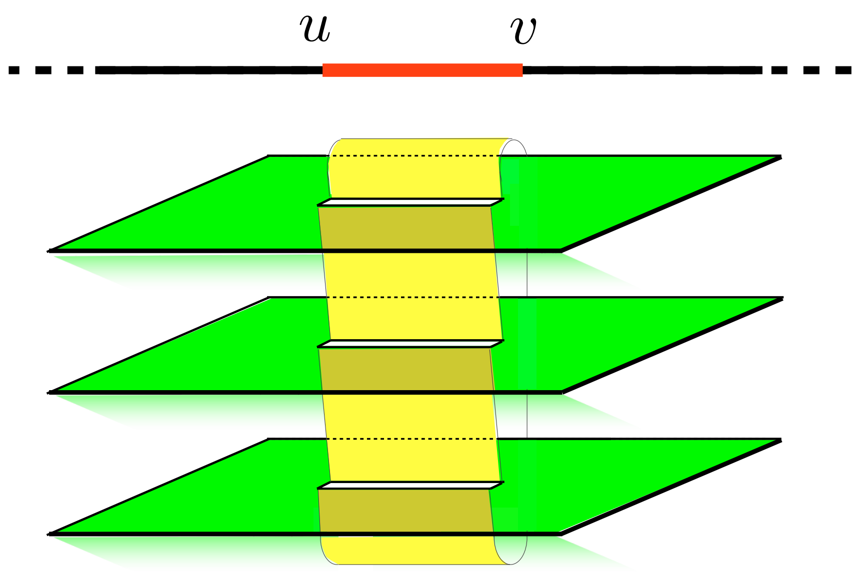

$\mathcal{R}_{1,n}$

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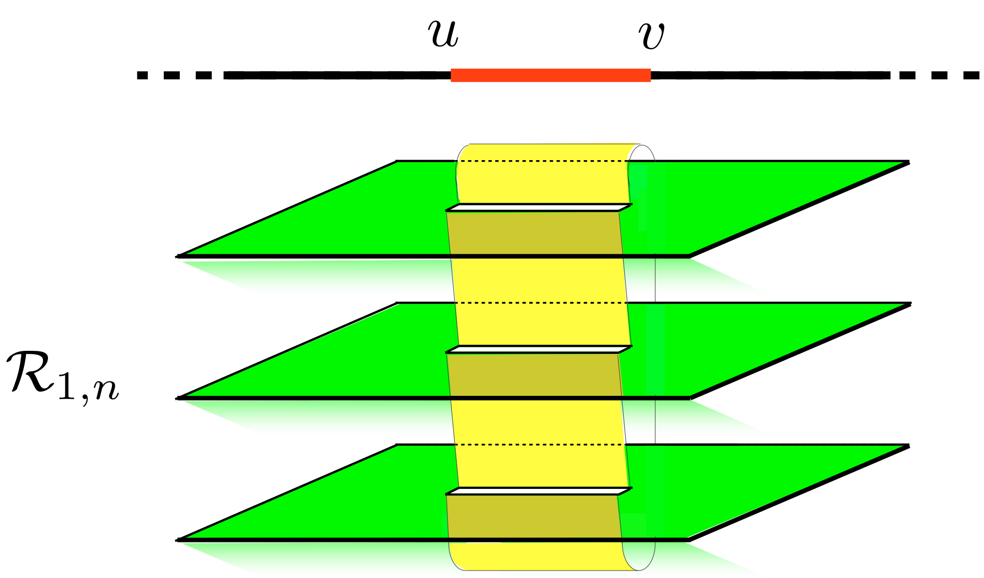
2D CFT: Renyi entropies as correlation functions

- One interval ($N = 1$): the Renyi entropies can be written as a two point function of *twist fields* on the sphere [Calabrese, Cardy, (2004)]



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$$\boxed{\Delta_n = \frac{c}{12}\left(n - \frac{1}{n}\right)}$$

- Twist fields have been largely studied in the 1980s
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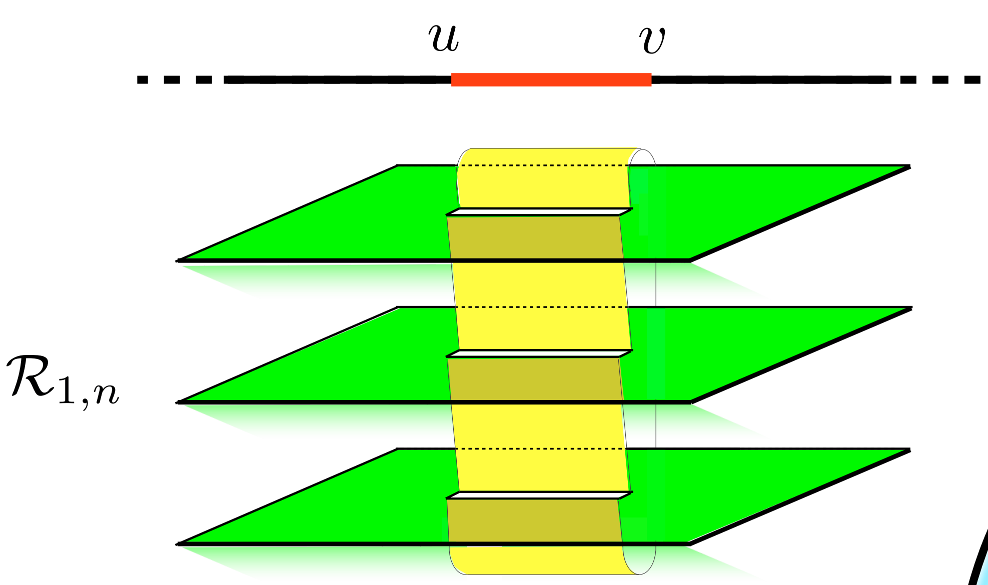

[Holzhey, Larsen, Wilczek, (1994)]

$$S_A = \frac{c}{3} \log \frac{\ell}{\epsilon} + c'_1$$

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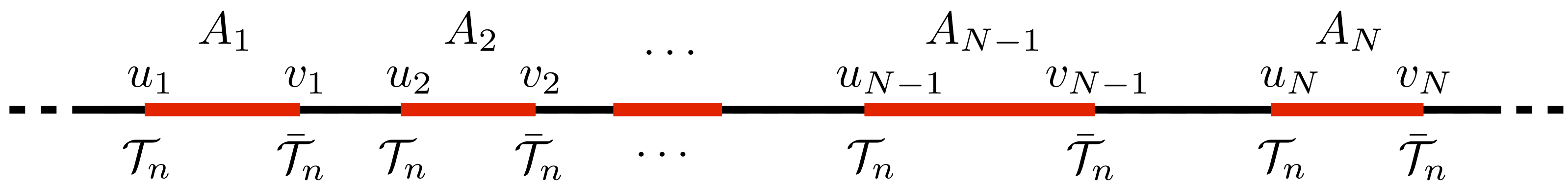

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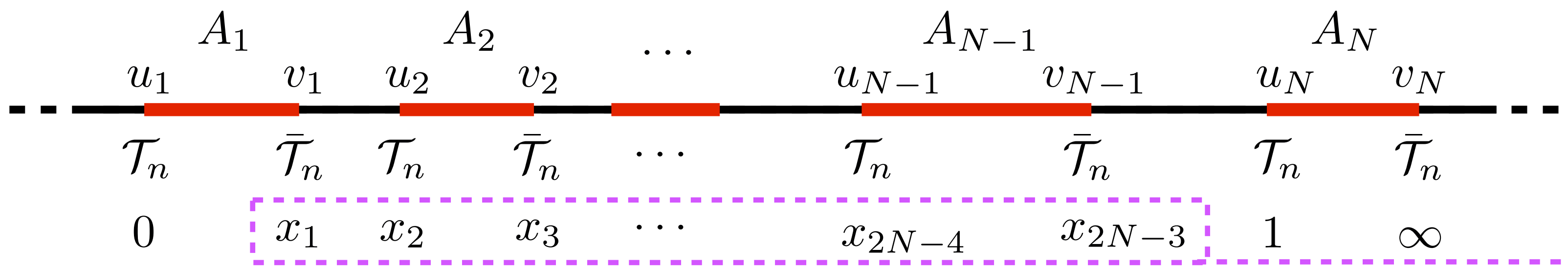

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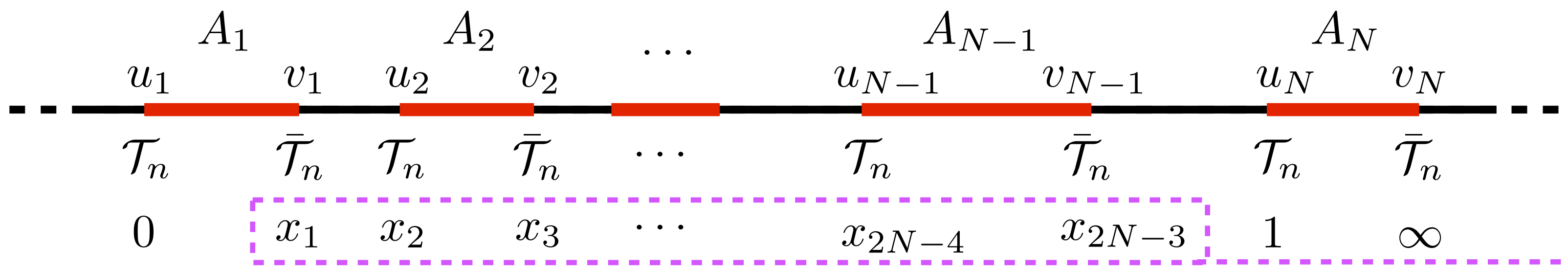
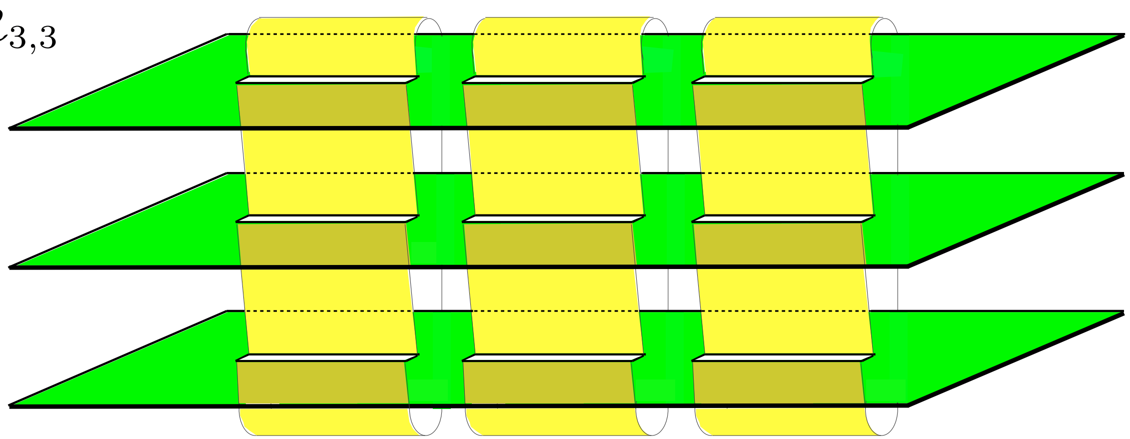


$$\mathrm{Tr}\rho_A^n = \frac{\mathcal{Z}_{N,n}}{\mathcal{Z}^n} = \langle \prod_{i=1}^{N} \mathcal{T}_n(u_i)\bar{\mathcal{T}}_n(v_i)\rangle = c_n^N \left| \frac{\prod_{i<j}(u_j-u_i)(v_j-v_i)}{\prod_{i,j}(v_j-u_i)} \right|^{2\Delta_n} \mathcal{F}_{N,n}(\boldsymbol{x})$$

- $\mathcal{Z}_{N,n}$ partition function of $\mathcal{R}_{N,n}$, a particular Riemann surface of genus $g=(N-1)(n-1)$ obtained through replication

$\mathcal{R}_{3,3}$



2D CFT: Renyi entropies for many disjoint intervals

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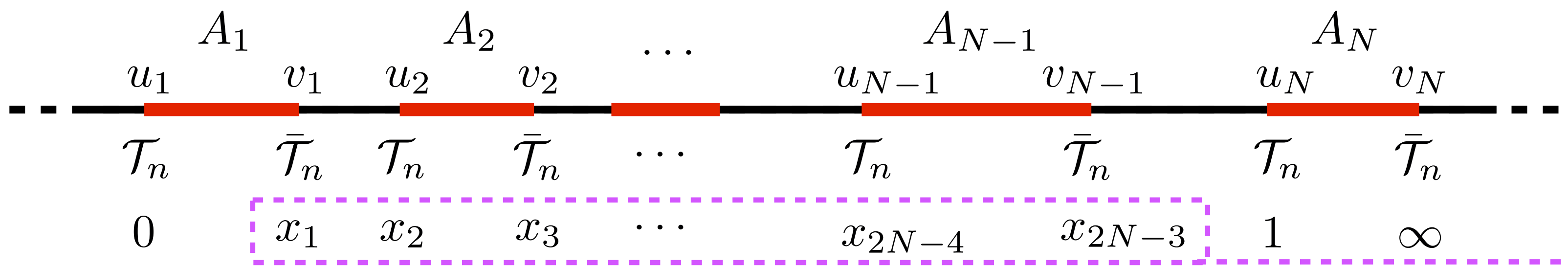

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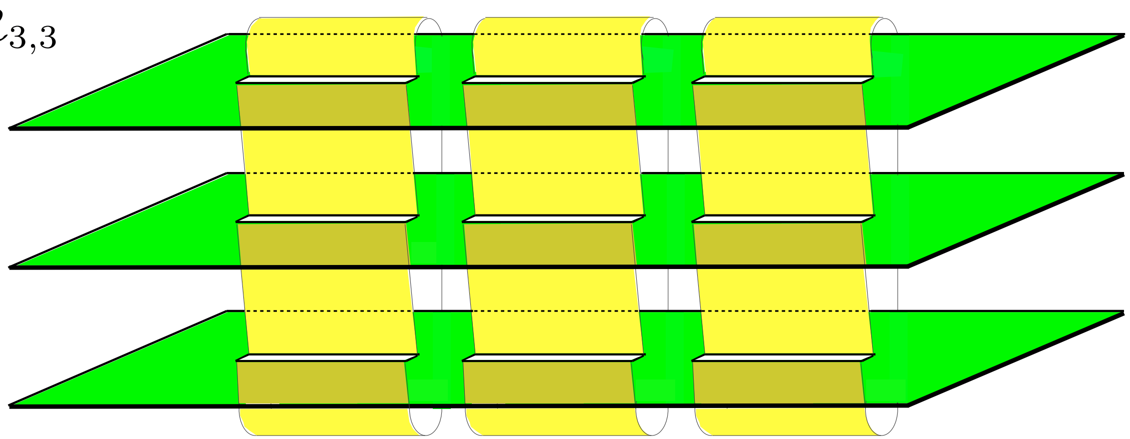
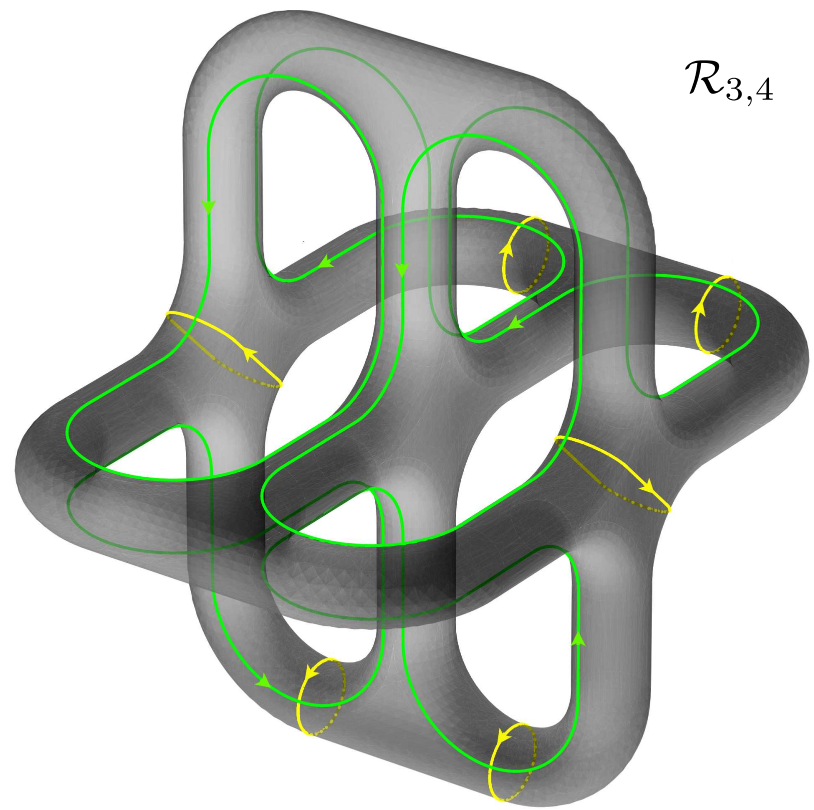


$\mathcal{R}_{3,4}$



Periodic harmonic chain

- Harmonic chain on a circle (critical for $\omega = 0$)

$$\boxed{H = \frac{1}{2}\sum_{j=1}^{L}\left[\, p_j^2 + \omega^2 q_j^2 + (q_{j+1} - q_j)^2 \,\right]}$$

[Peschel, Chung, (1999)] [Botero, Reznik, (2004)]
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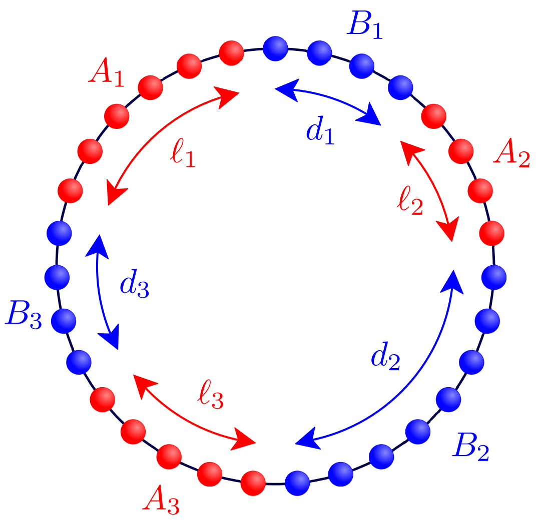

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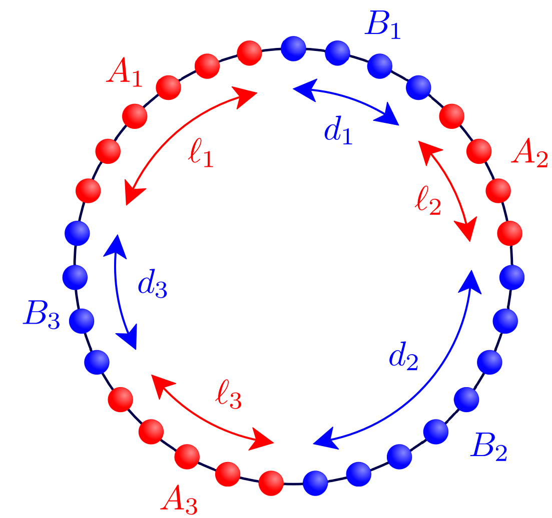

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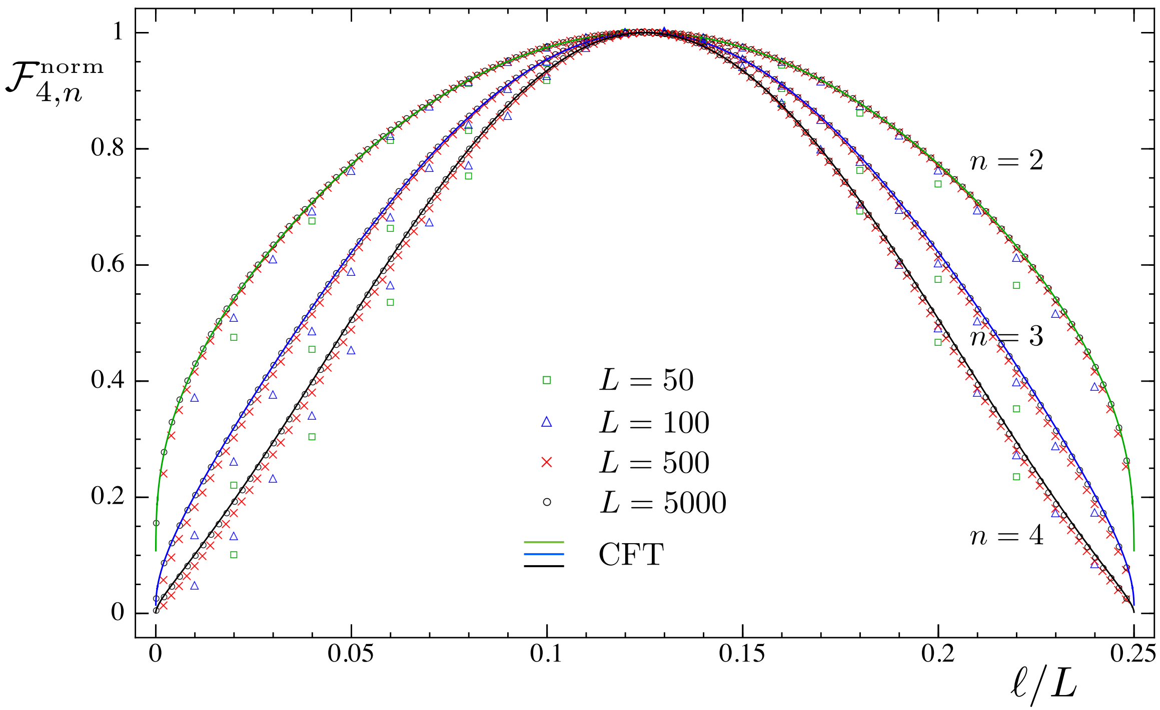
- Decompactification regime

[Dijkgraaf, Verlinde, Verlinde, (1988)] [. . .]
[Coser, Tagliacozzo, E.T., (2013)]

$$\mathcal{F}^{\text{dec}}_{N,n}(\boldsymbol{x}) = \frac{\eta^{g/2}}{\sqrt{\det(\mathcal{I})}\,|\Theta(\mathbf{0}|\tau)|^2}$$

- period matrix $\tau = \mathcal{R} + \mathrm{i}\,\mathcal{I}$
  [Enolski, Grava, (2003)]
- Riemann theta function Θ

→ Nasty n dependence



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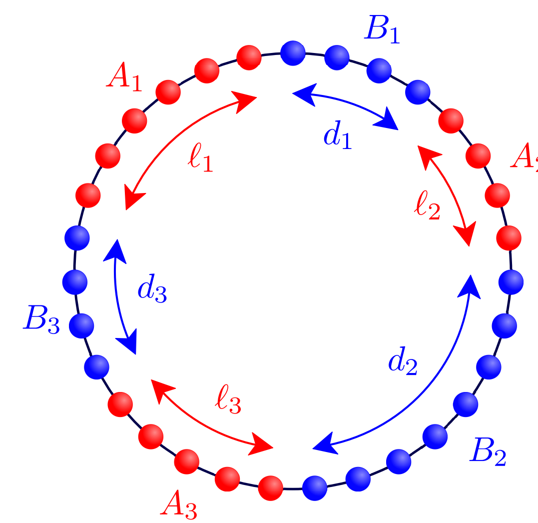

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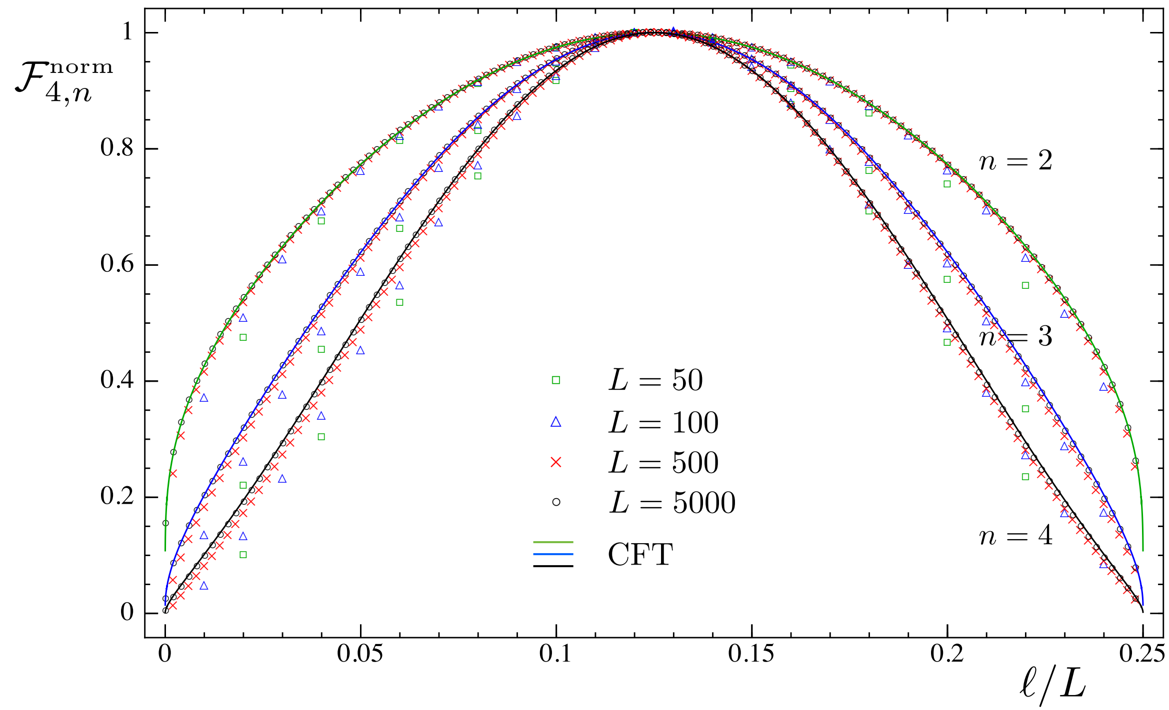
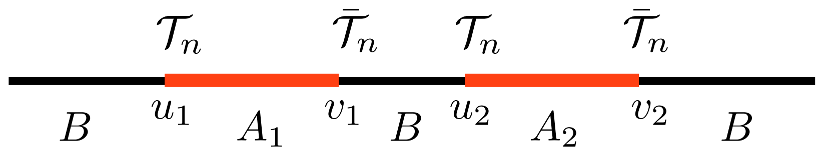
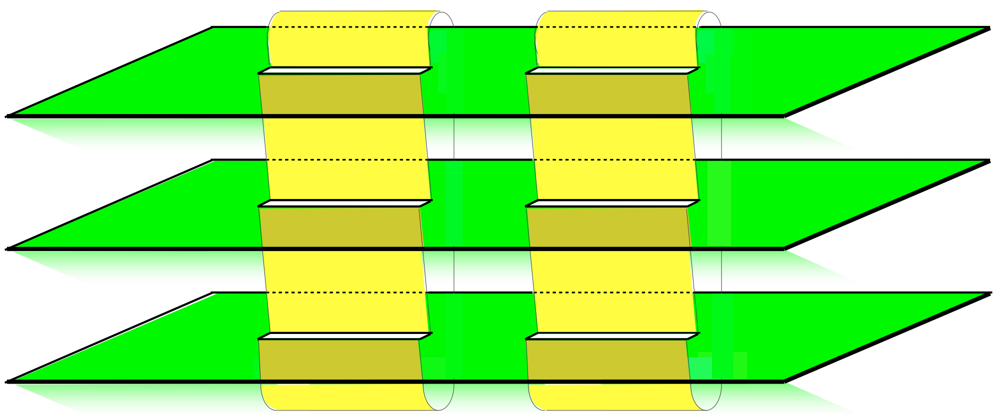


- Numerical checks for the Ising model through Matrix Product States

Partial transposition: two disjoint intervals

$$\mathrm{Tr}\rho^n_{A_1\cup A_2}$$





$$\mathrm{Tr}\rho^n_A \;=\; \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\mathcal{T}_n(u_2)\bar{\mathcal{T}}_n(v_2)\rangle$$

[Caraglio, Gliozzi, (2008)]
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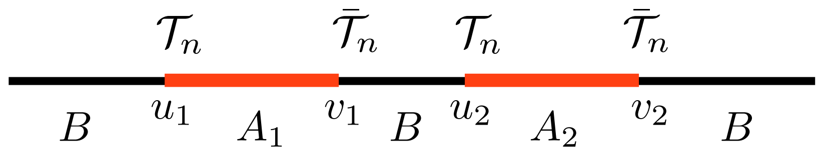

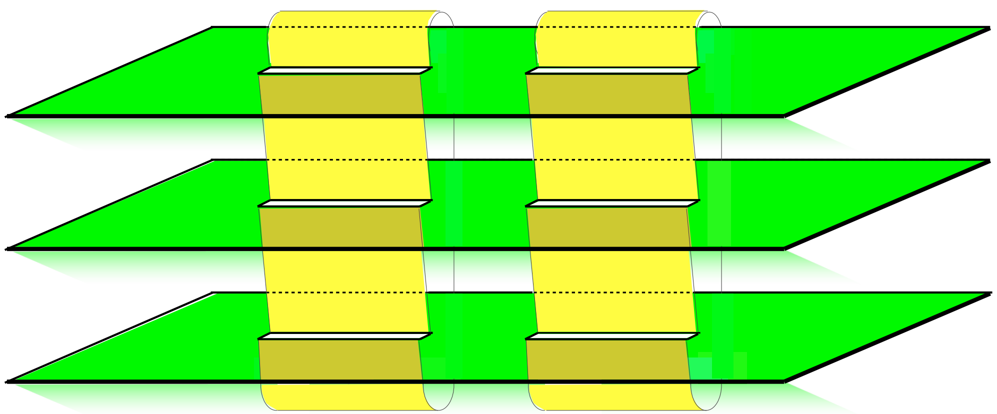

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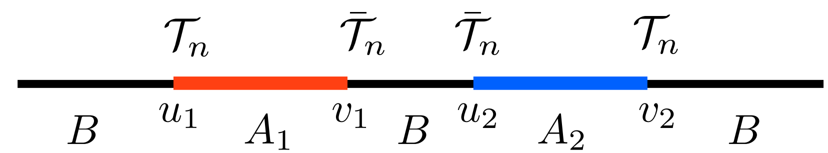
$$\mathrm{Tr}\left(\rho^{T_2}_{A_1 \cup A_2}\right)^n$$



- The partial transposition exchanges $\mathcal{T}_n$ and $\bar{\mathcal{T}}_n$

[Calabrese, Cardy, E.T., (2012)]

Partial transposition: two disjoint intervals

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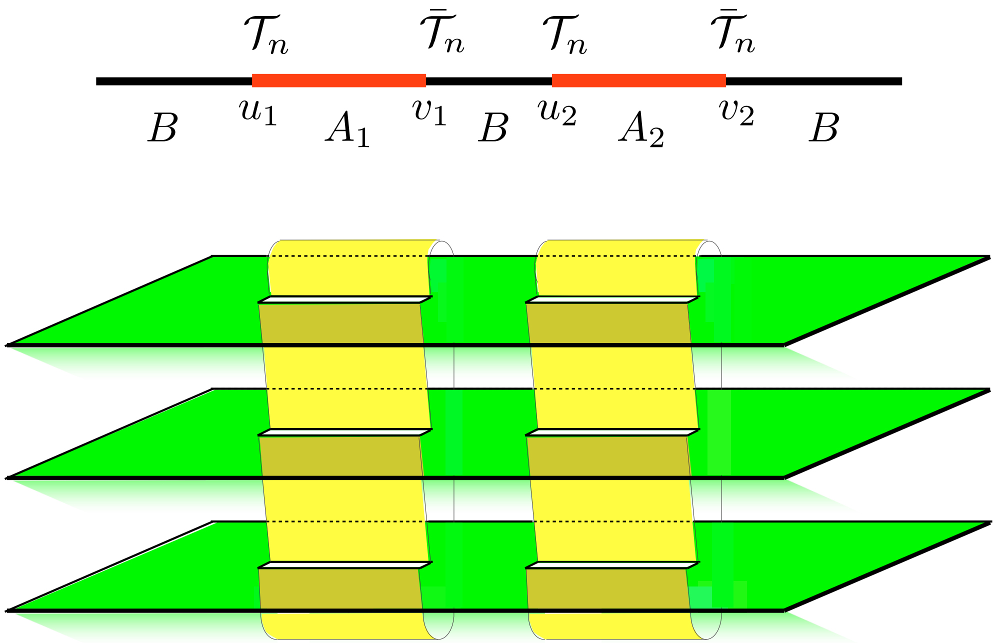

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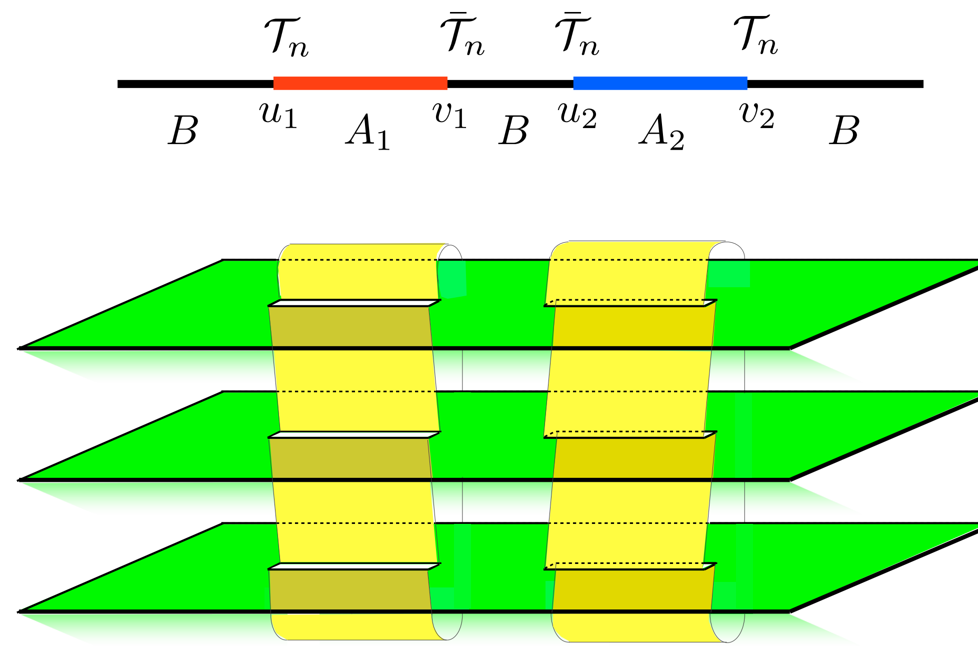

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Renyi entropies vs traces of the Partial Transpose

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Partial Transposition for bipartite systems: pure states

$$\mathcal{H} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$$

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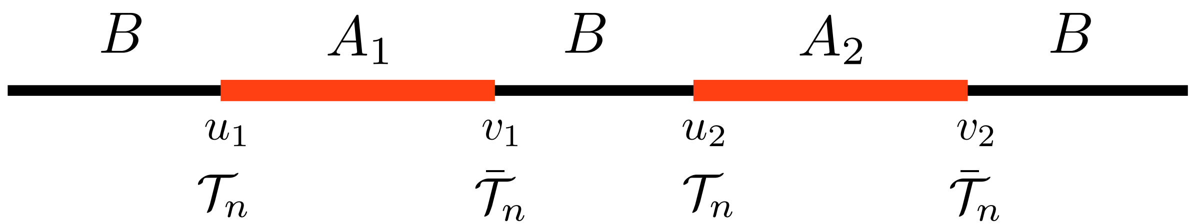

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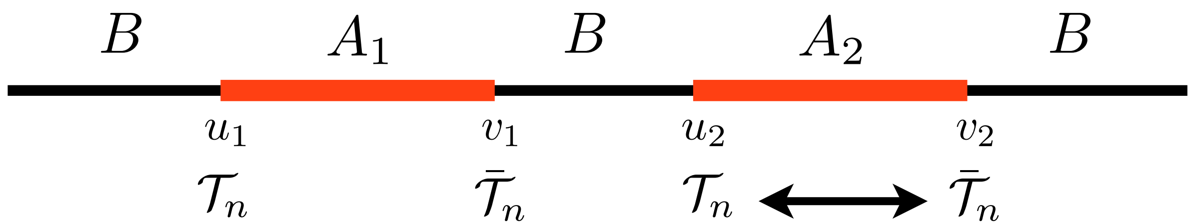

$$\boxed{\text{Partial Transposition} = \text{exchange two twist fields}}$$

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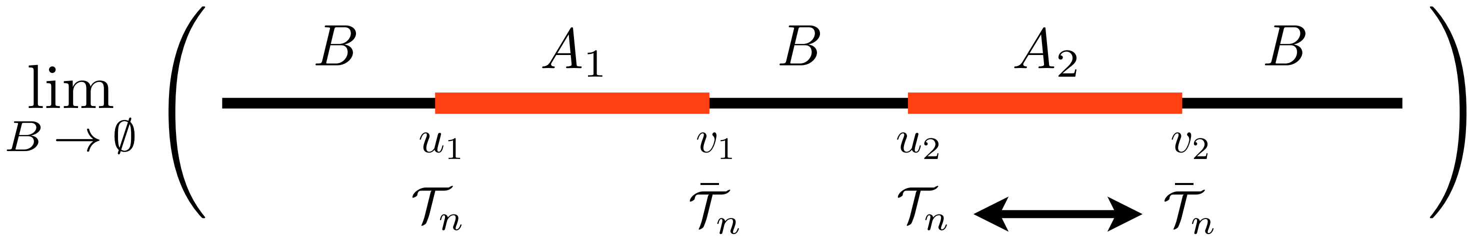

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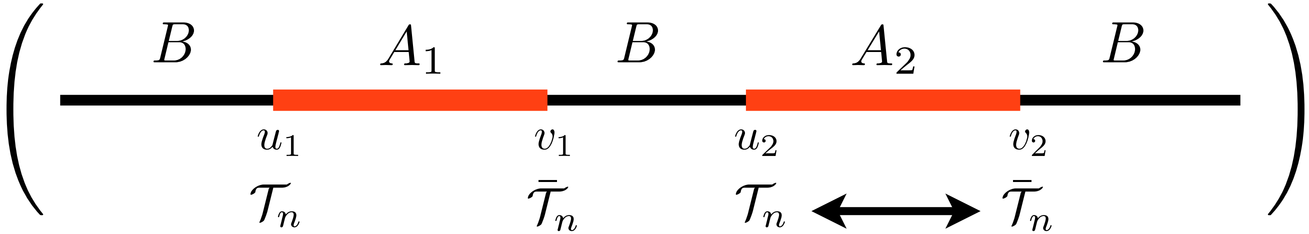

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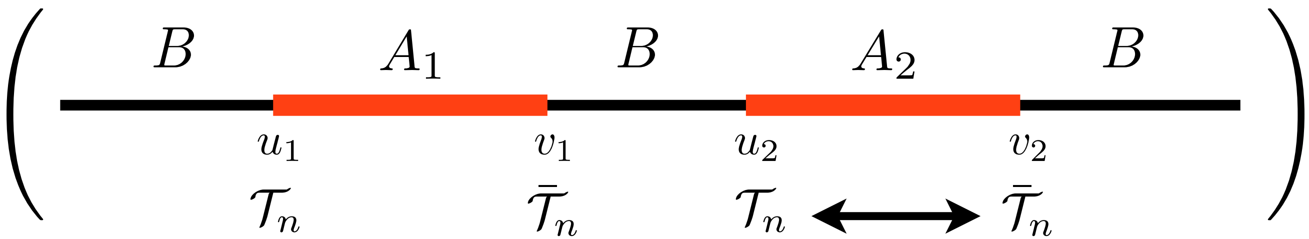
- $\lim_{B \to \emptyset}$ 

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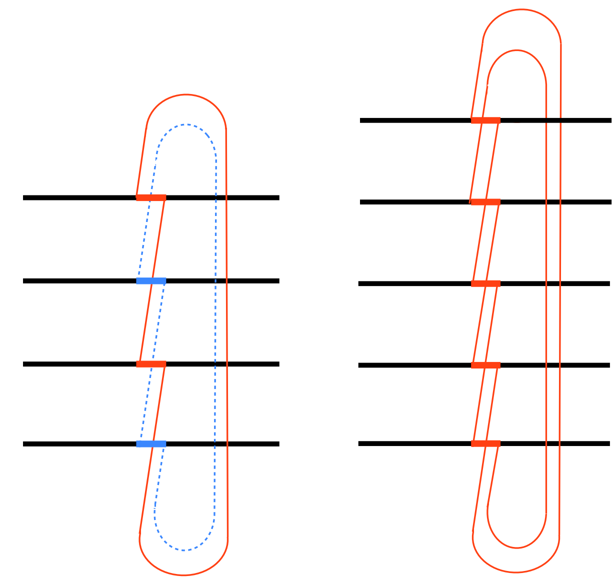

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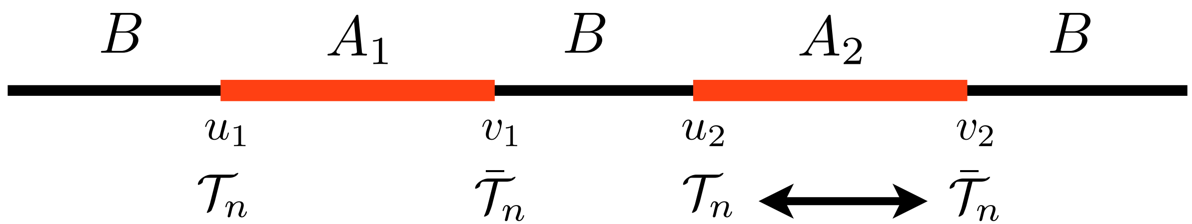
- $\lim_{B \to \emptyset}$ ()

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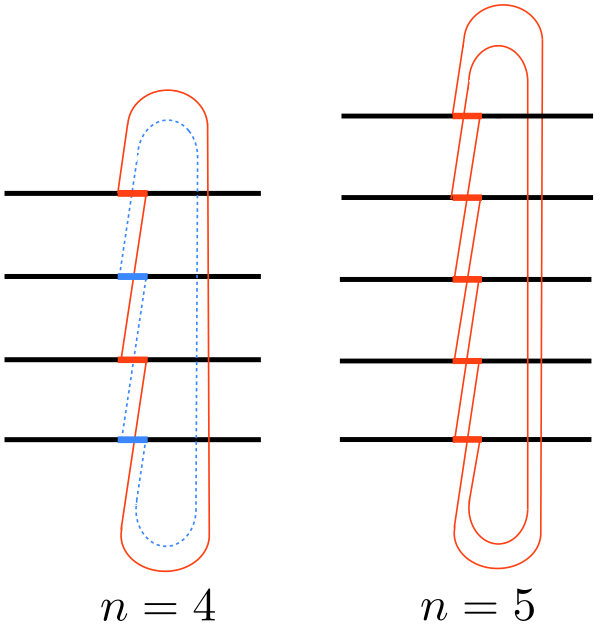

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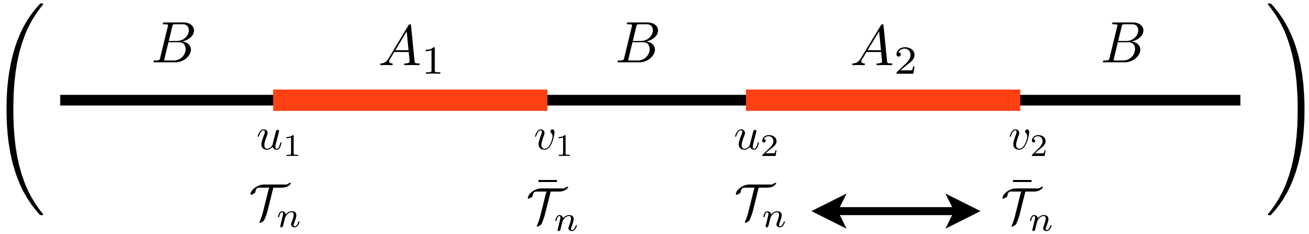

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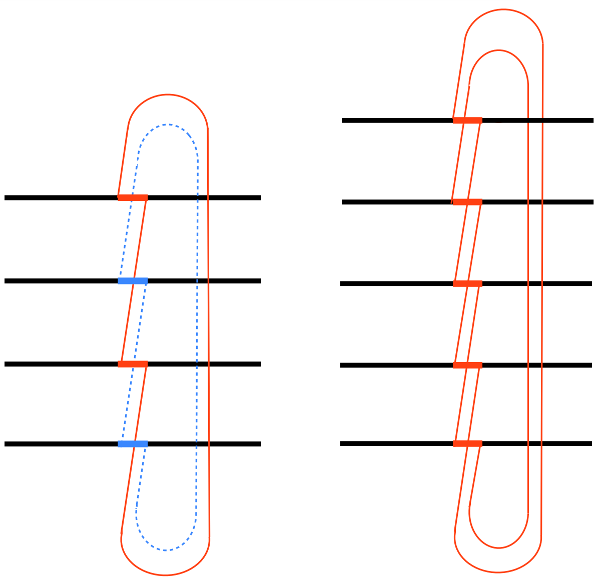

$n = 4$ $\quad$ $n = 5$

- Two dimensional CFTs

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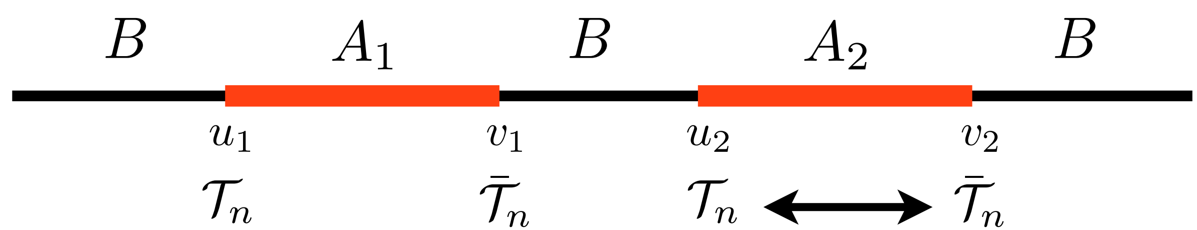

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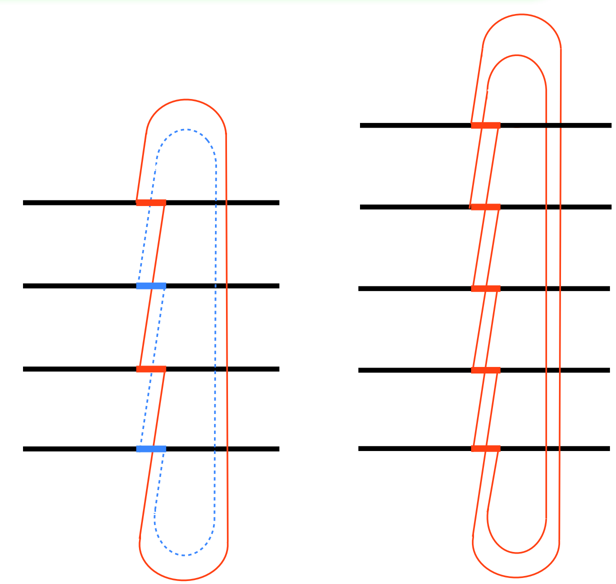

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$$\Delta_{\mathcal{T}_{n_o}^2} = \frac{c}{12}\left(n_o - \frac{1}{n_o}\right) = \Delta_{\mathcal{T}_{n_o}}$$

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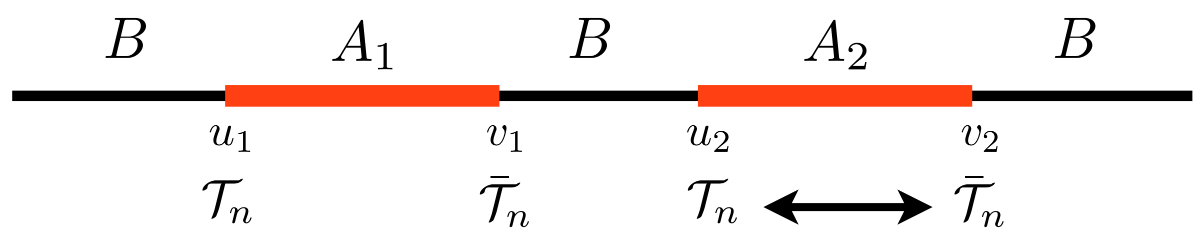
- $\lim_{B \to \emptyset}$ 

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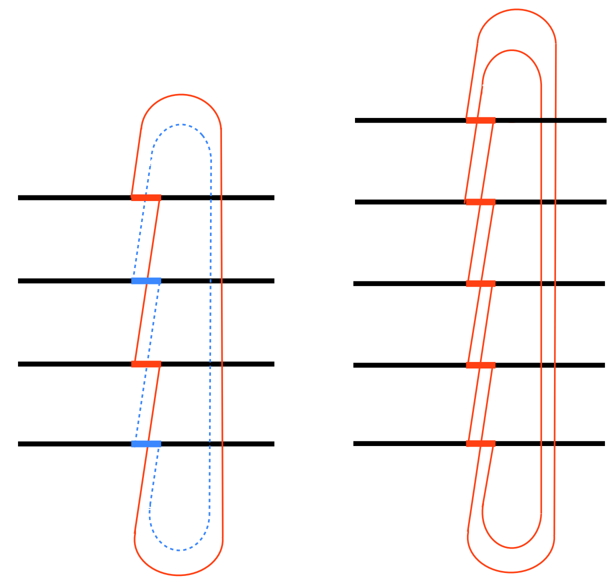

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Partial Transposition for bipartite systems: pure states

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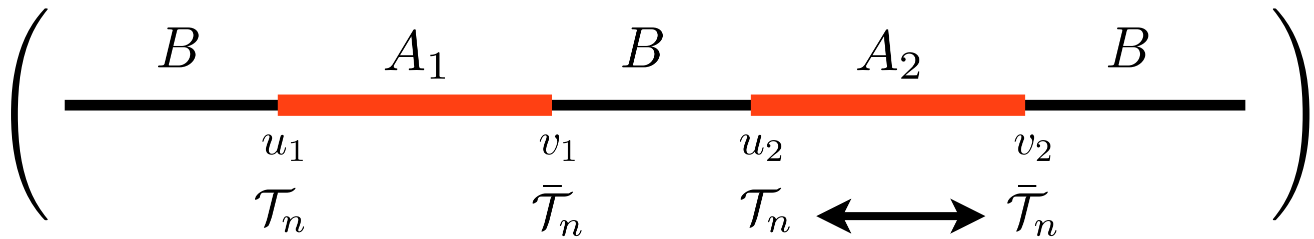

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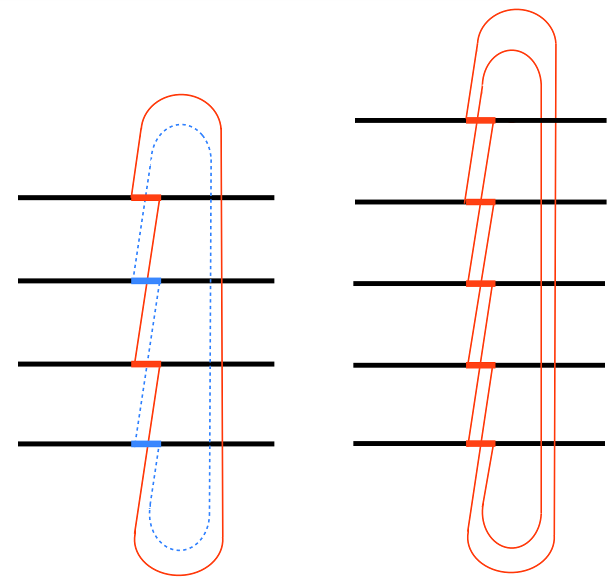

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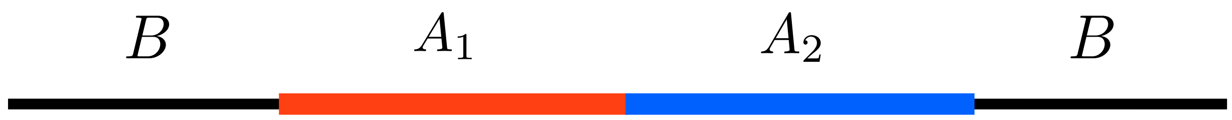
$$\boxed{\mathcal{E} = \frac{c}{2} \ln \ell + \mathrm{const}}$$

Partial Transpose in 2D CFT: two adjacent intervals



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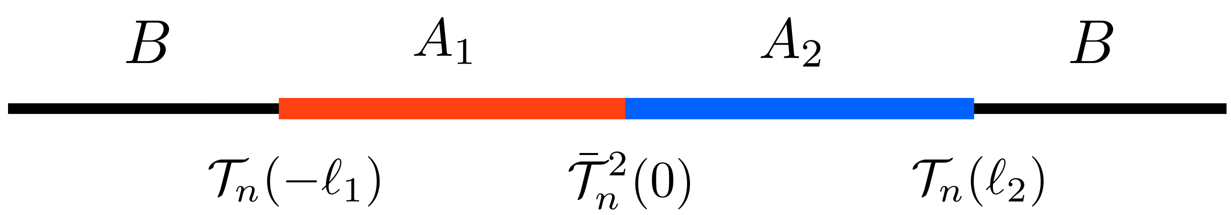
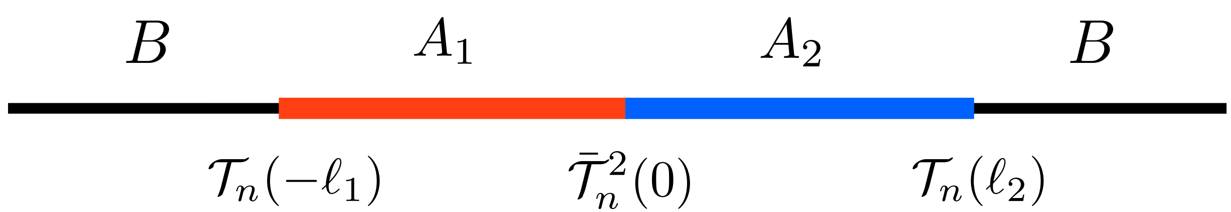


- Three point function

$$\boxed{\mathrm{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n(-\ell_1)\bar{\mathcal{T}}_n^2(0)\mathcal{T}_n(\ell_2)\rangle}$$

Partial Transpose in 2D CFT: two adjacent intervals



- Three point function

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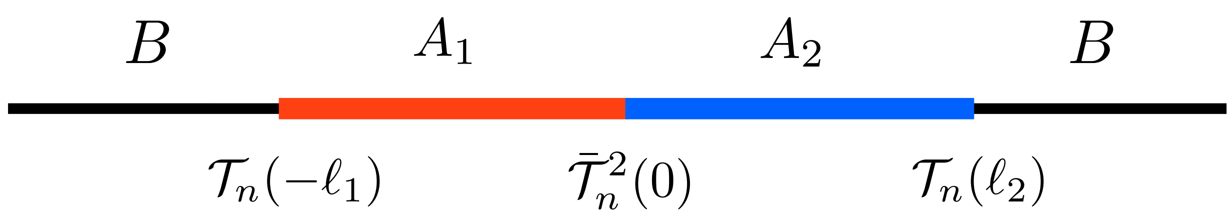

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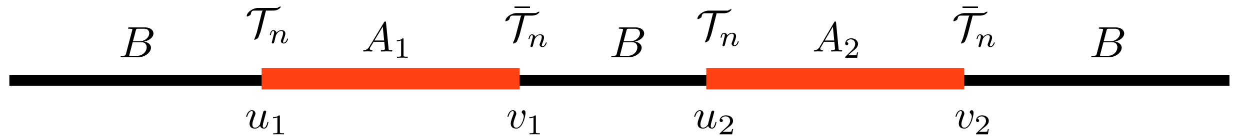
Partial Transpose in 2D CFT: two disjoint intervals

$\mathrm{Tr}\rho^n_{A_1 \cup A_2}$



Partial Transpose in 2D CFT: two disjoint intervals

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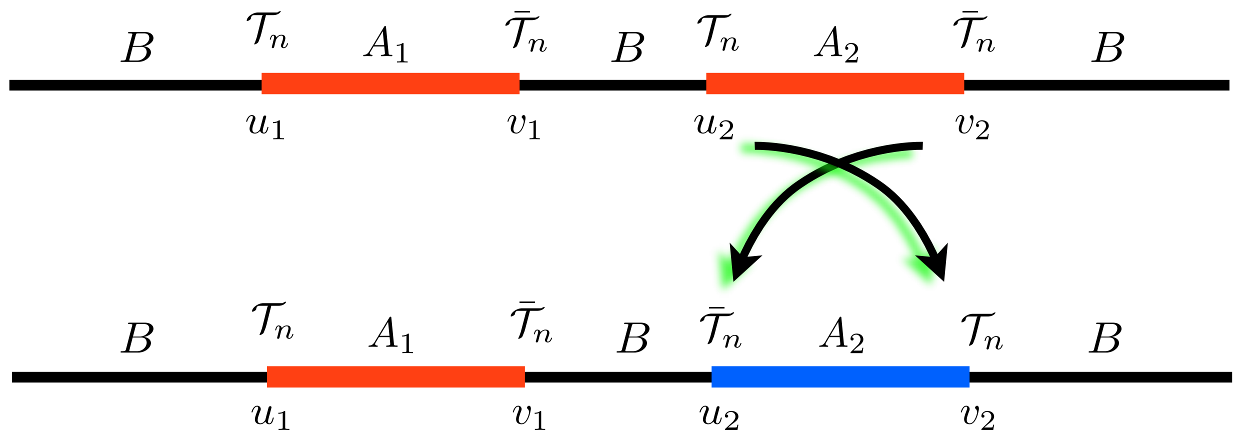
$$\mathrm{Tr}\left(\rho^{T_2}_{A_1\cup A_2}\right)^n$$



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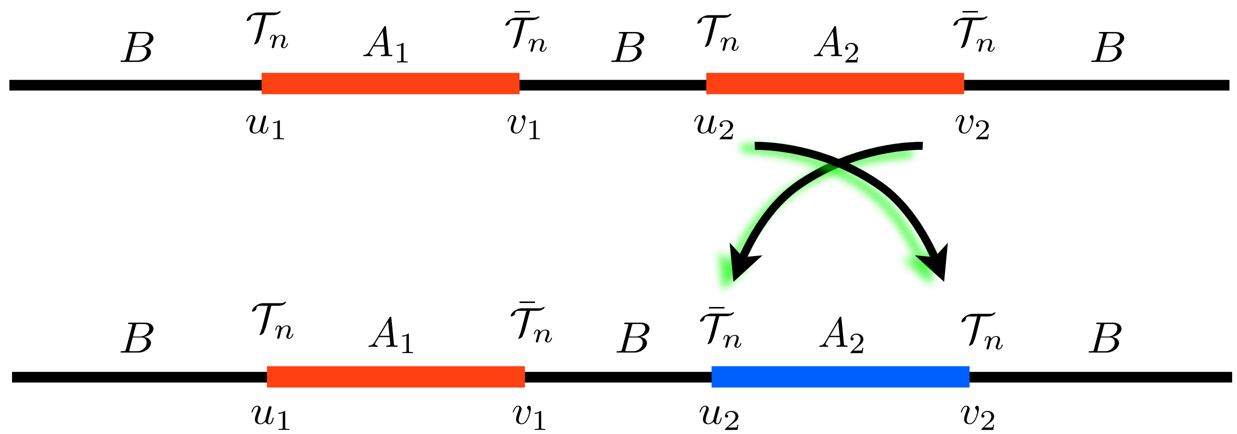

$$\boxed{\mathrm{Tr}\big(\rho^{T_2}_{A_1\cup A_2}\big)^n = c_n^2\left[\ell_1\ell_2(1-y)\right]^{-\frac{c}{6}(n-\frac{1}{n})}\mathcal{G}_n(y)}$$

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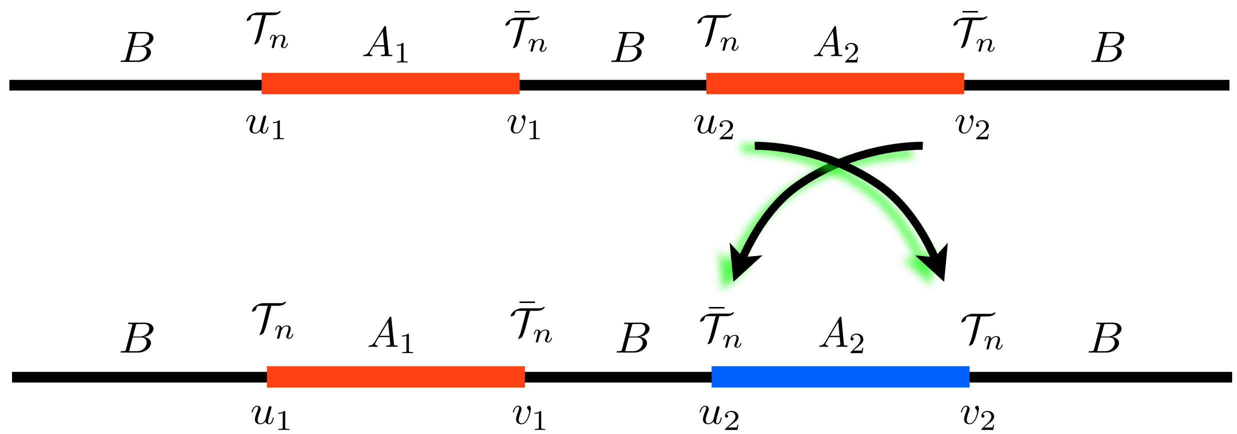

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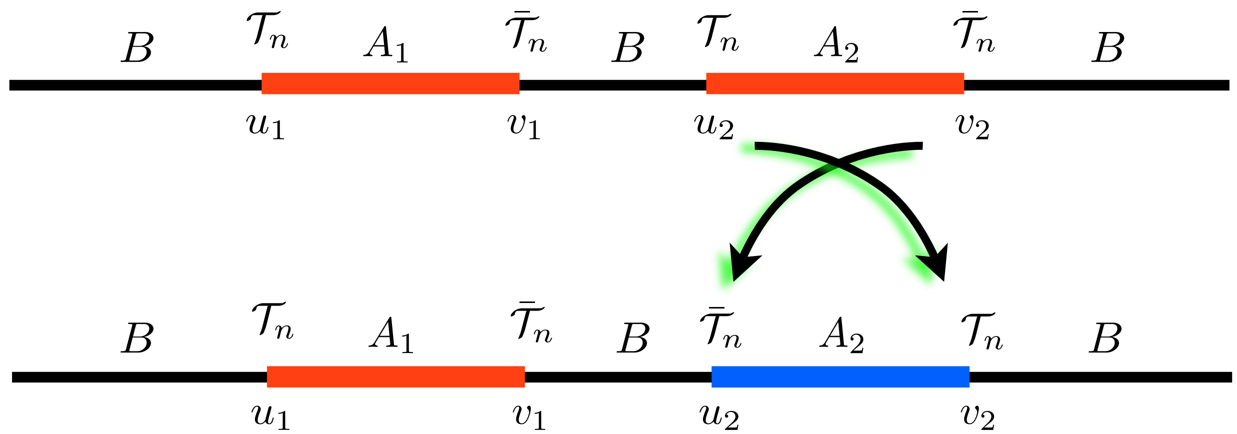

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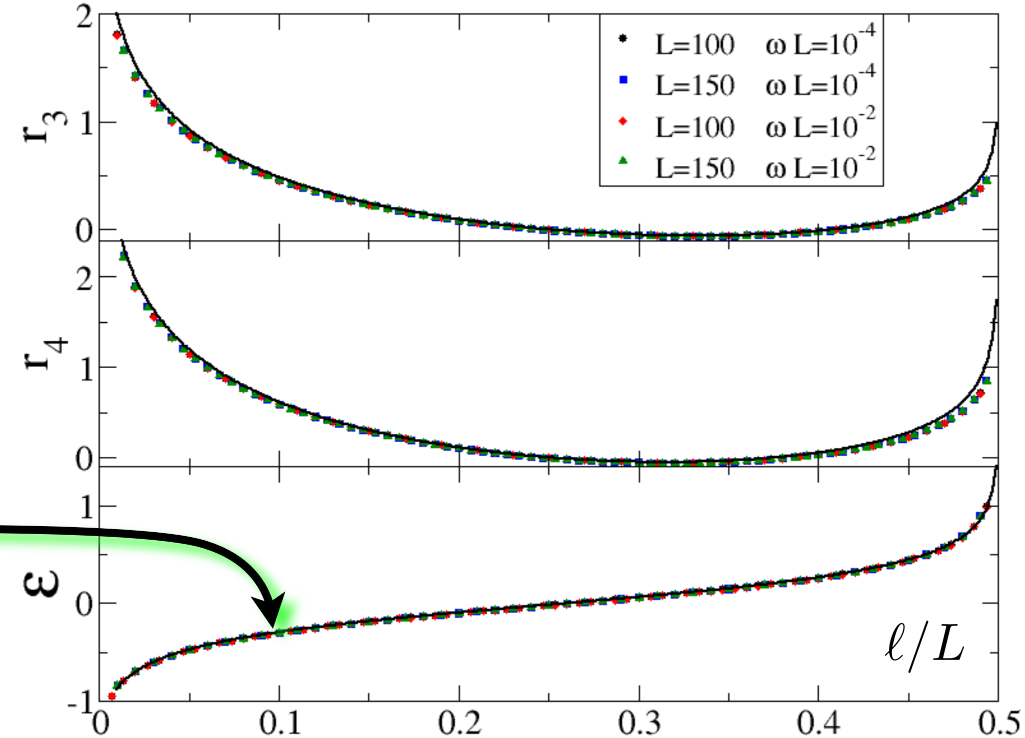
Two adjacent intervals: harmonic chain & Ising model

- Critical periodic harmonic chain

Finite system: $\ell \longrightarrow (L/\pi)\sin(\pi\ell/L)$

$$r_n = \ln \frac{\mathrm{Tr}(\rho_A^{T_{A_2=\ell}})^n}{\mathrm{Tr}(\rho_A^{T_{A_2=L/4}})^n}$$

$$\boxed{\frac{1}{4}\log\frac{\sin(\pi\ell_1/L)\sin(\pi\ell_2/L)}{\sin(\pi[\ell_1+\ell_2]/L)} + \mathrm{cnst}}$$



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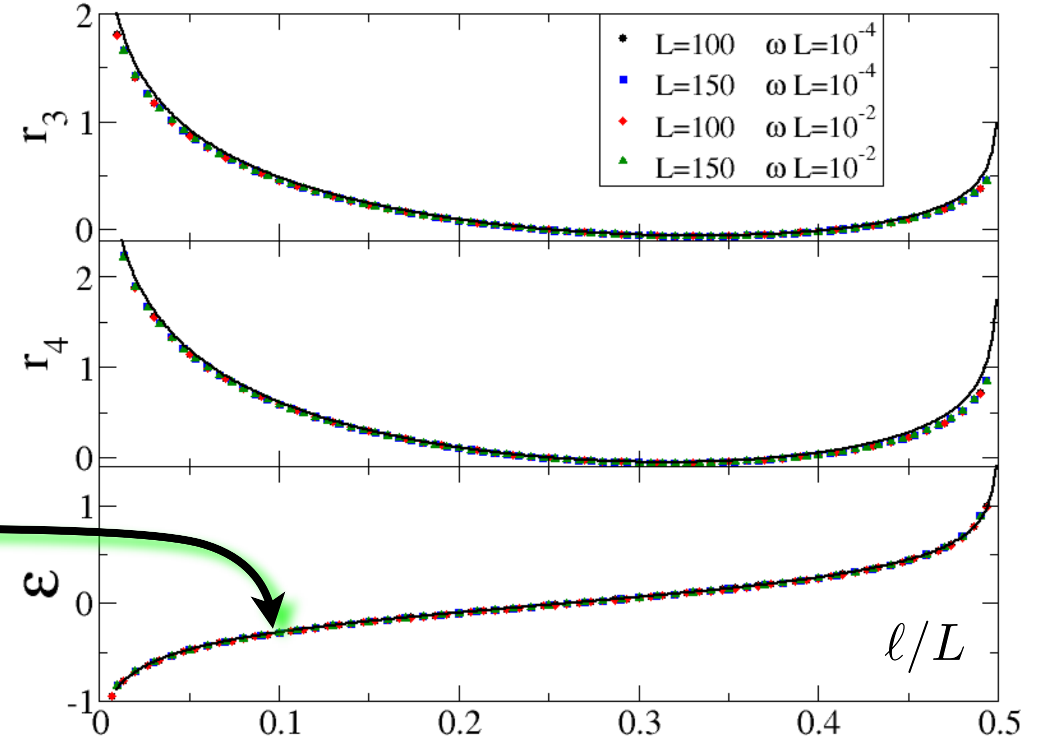
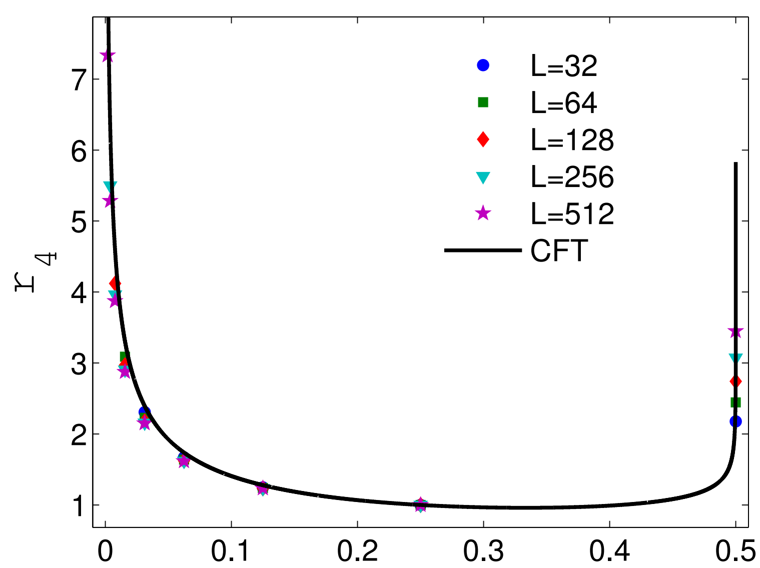
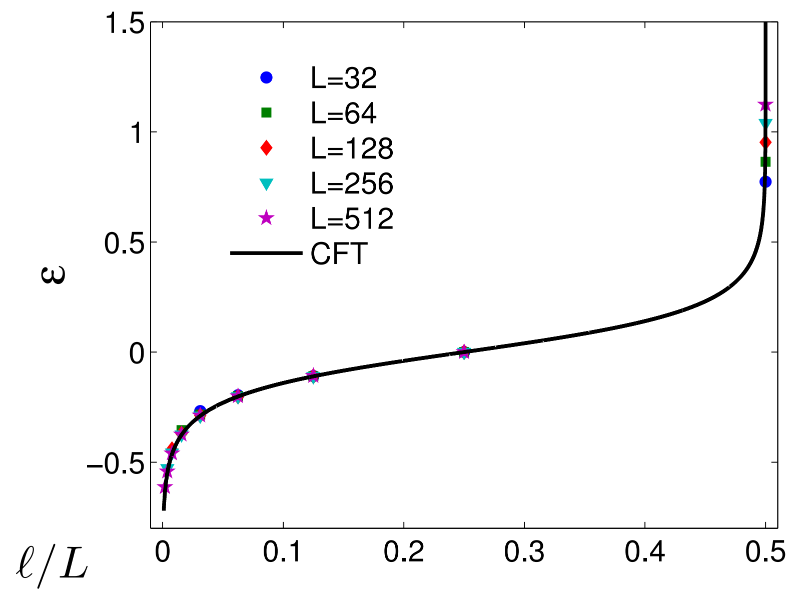


- Ising model:

Monte-Carlo analysis [Alba, (2013)]

Tree Tensor Network [Calabrese, Tagliacozzo, E.T., (2013)]





Two disjoint intervals: periodic harmonic chains

- Previous numerical results for $\mathcal{E}$: Ising (DMRG) and harmonic chains

[Wichterich, Molina-Vilaplana, Bose, (2009)]
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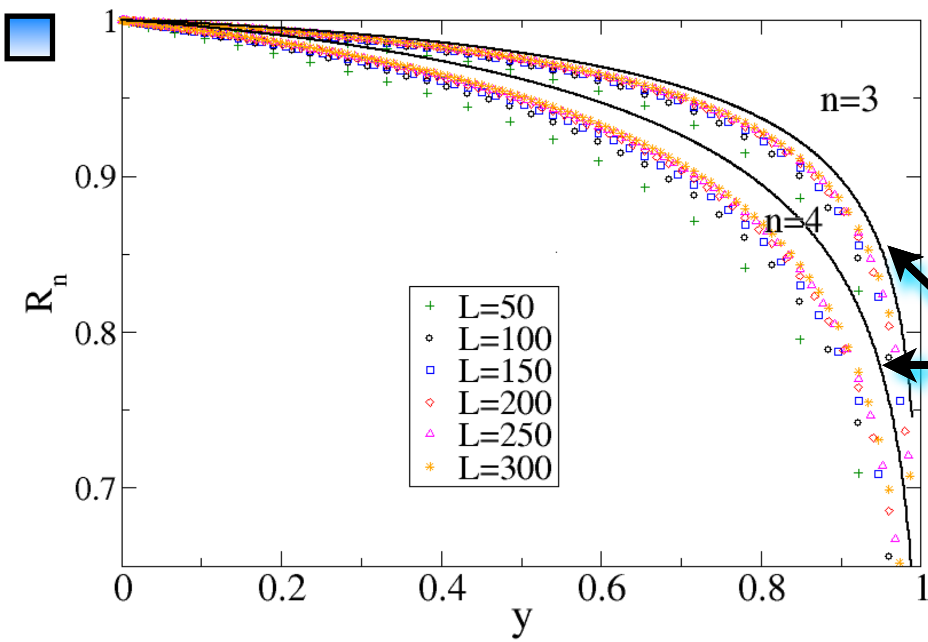
- 

CFT curves

- Two disjoint intervals

[Calabrese, Cardy, E.T., (2012)]

$$\boxed{R_n = \frac{\mathrm{Tr}(\rho_A^{T_2})^n}{\mathrm{Tr}\,\rho_A^n}}$$

$$R_n = \left[\frac{(1-y)^{\frac{2}{3}(n-\frac{1}{n})} \prod_{k=1}^{n-1} F_{\frac{k}{n}}(y)\, F_{\frac{k}{n}}(1-y)}{\prod_{k=1}^{n-1} \mathrm{Re}\big(F_{\frac{k}{n}}(\frac{y}{y-1})\, \bar{F}_{\frac{k}{n}}(\frac{1}{1-y})\big)} \right]^{\frac{1}{2}}$$

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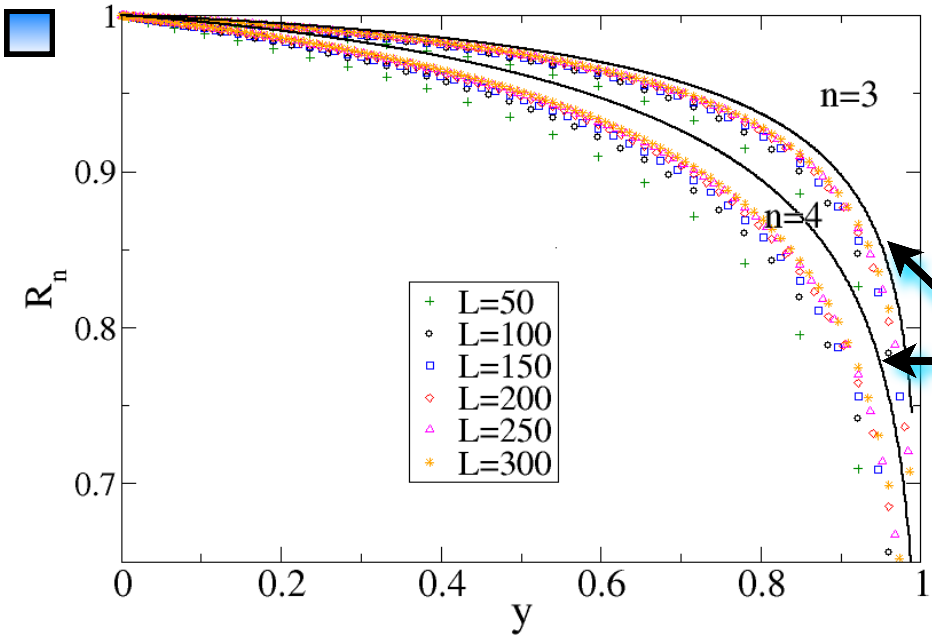

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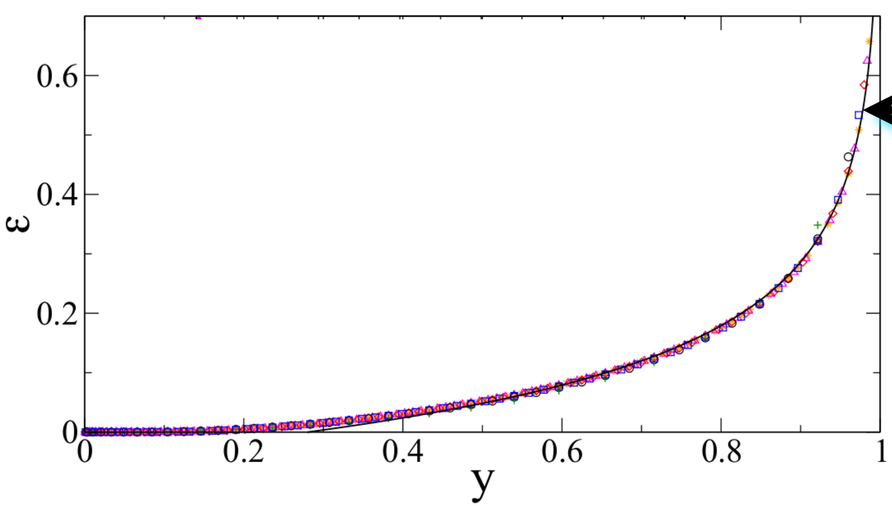

- Analytic continuation for $y \sim 1$

$$\mathcal{E} = -\frac{1}{4}\log(1-y) + \log K(y) + \mathrm{cnst}$$

- Analytic continuation $n_e \to 1$ for $0 < y < 1$ is missing
- It goes to zero faster than any power

Two disjoint intervals: Ising model

[Calabrese, Tagliacozzo, E.T., (2013)]

- CFT

$$\boxed{\mathcal{G}_n(y) = (1-y)^{(n-1/n)/6}\,\frac{\sum_{\mathbf{e}} |\Theta[\mathbf{e}](\mathbf{0}|\tau(\frac{y}{y-1}))|}{2^{n-1}\prod_{k=1}^{n-1} |F_{k/n}(\frac{y}{y-1})|^{1/2}}}$$

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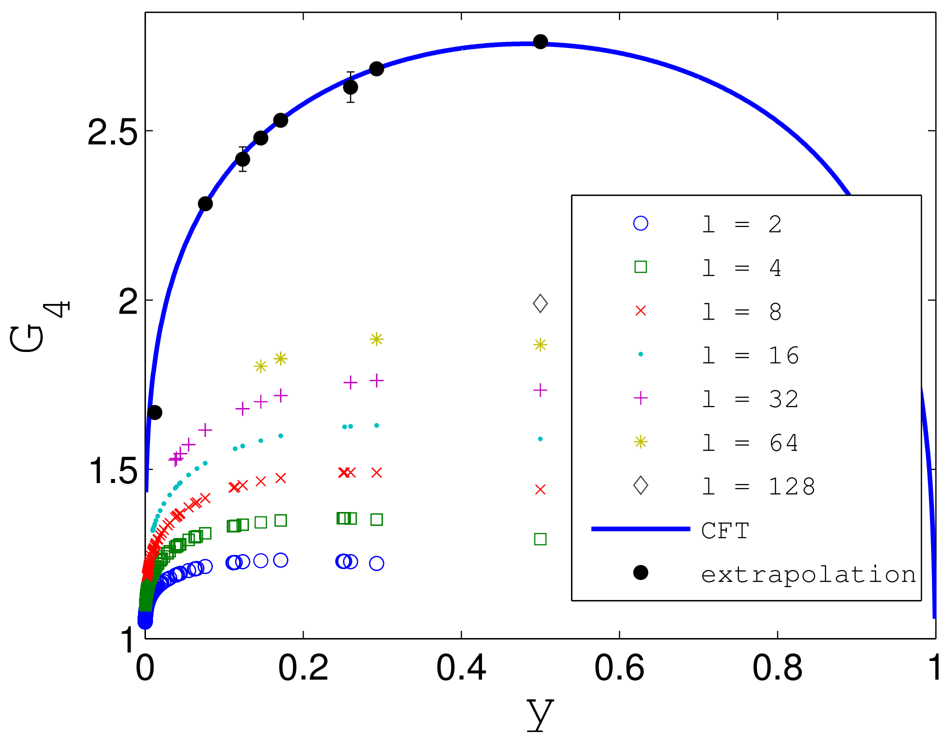
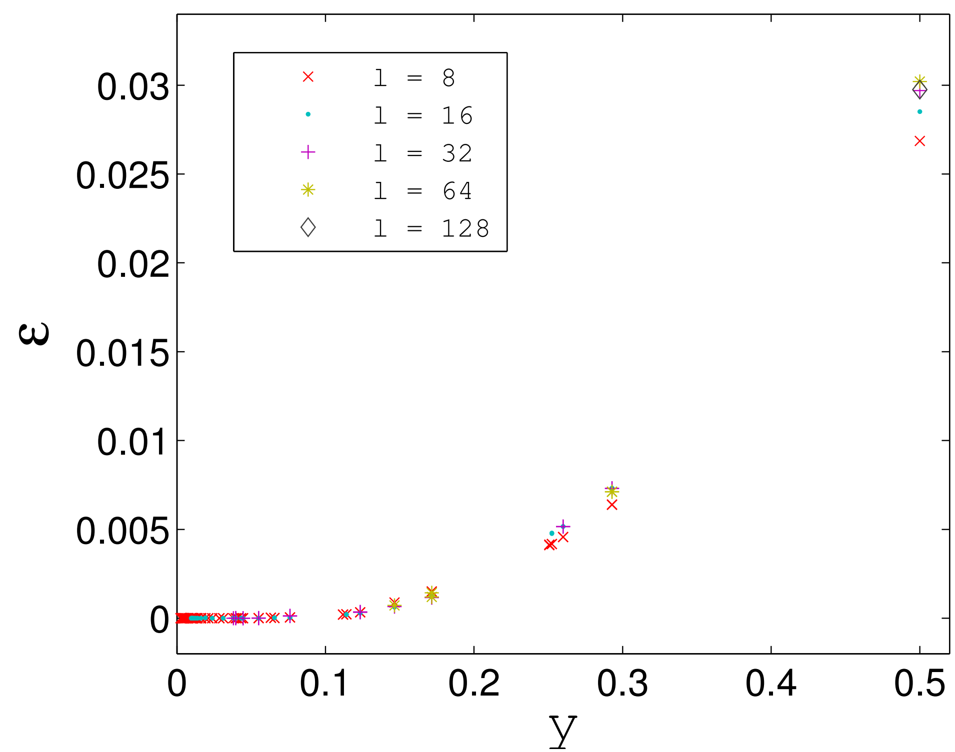
- Tree tensor network:





One interval at finite temperature: a naive approach

[Calabrese, Cardy, E.T., (2014)]

- Logarithmic negativity $\mathcal{E}$ of one interval at finite $T = 1/\beta$

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- A naive approach: compute $\langle \mathcal{T}_n^2(u) \bar{\mathcal{T}}_n^2(v) \rangle_\beta$ through the conformal map relating the cylinder to the complex plane

$$\mathcal{E}_{\text{naive}} = \frac{c}{2} \ln \left(\frac{\beta}{\pi a} \sinh \frac{\pi \ell}{\beta} \right) + 2 \ln c_{1/2}$$

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Problems:

➡ The Rényi entropy $n = 1/2$ is not an entanglement measure at finite T

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Problems:

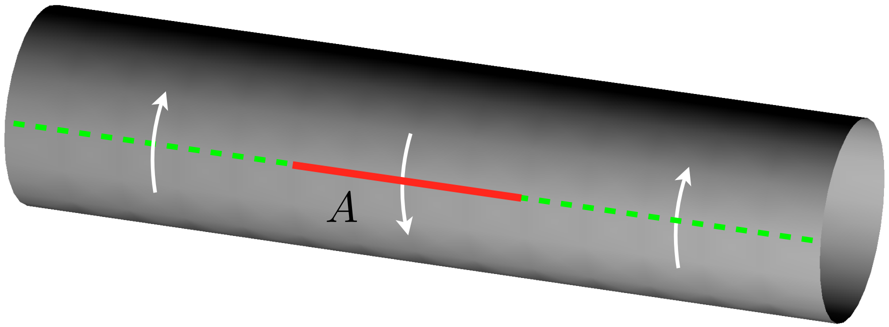
→ The Rényi entropy $n = 1/2$ is not an entanglement measure at finite T

→ $\mathcal{E}_{\text{naive}}$ is an increasing function of T, linearly divergent at high T

Entanglement should decrease as the system becomes classical

One interval at finite temperature in the infinite line

(connection to the $(j+1)$-th cylinder following the arrows)



$$\text{Single copy of } \rho_\beta^{T_A} \quad \Longrightarrow \quad \mathrm{Tr}\big(\rho_\beta^{T_A}\big)^n$$

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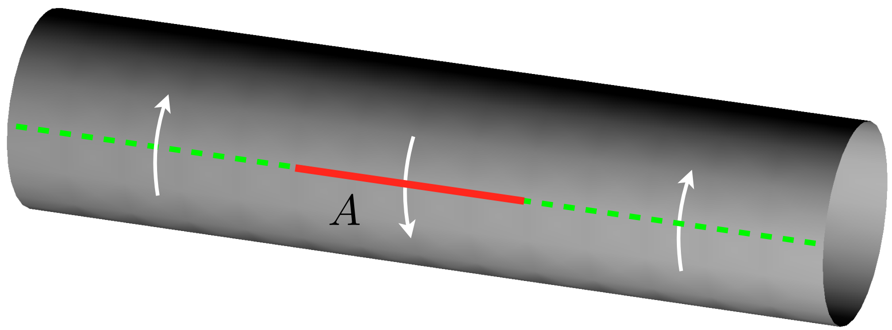

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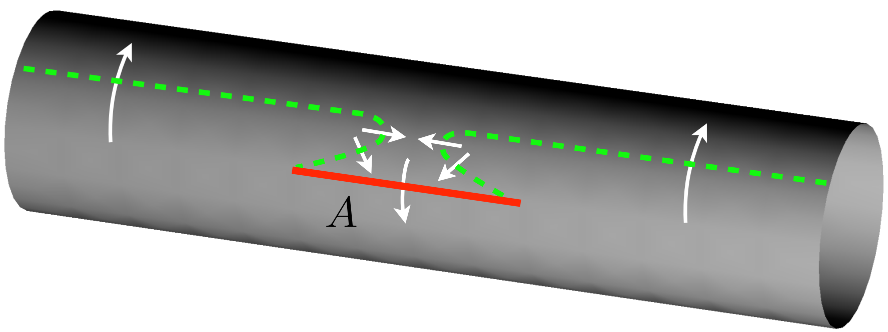
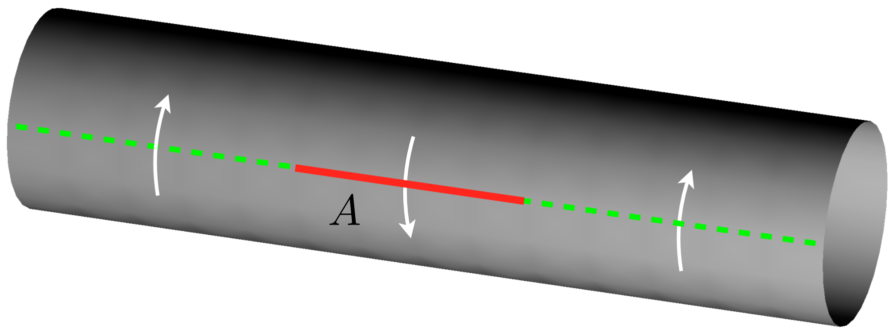


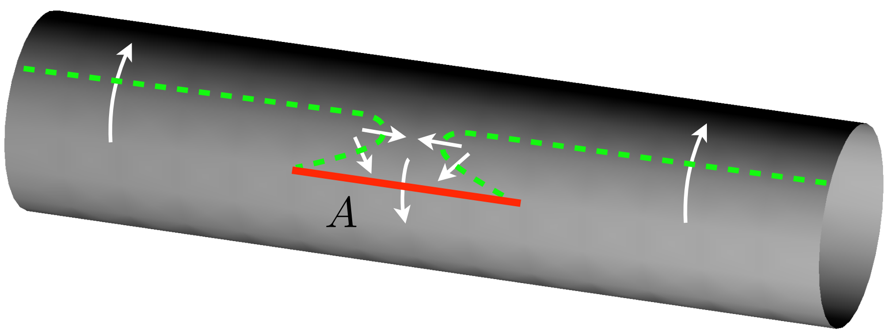
Deformation of the cut along B

One interval at finite temperature in the infinite line

(connection to the $(j+1)$-th cylinder following the arrows)



Single copy of $\rho_\beta^{T_A} \implies \mathrm{Tr}\big(\rho_\beta^{T_A}\big)^n$



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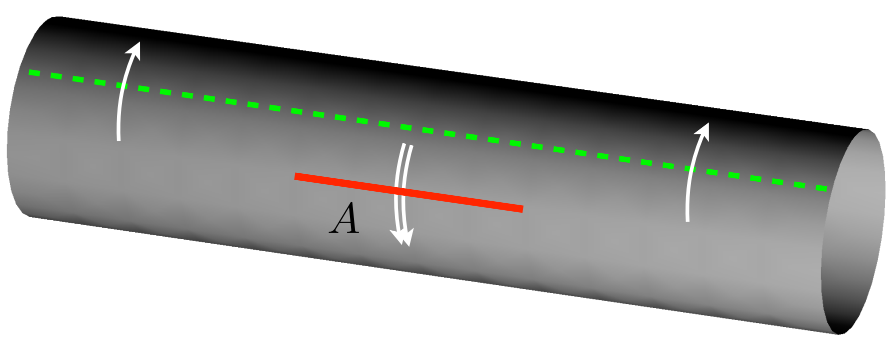
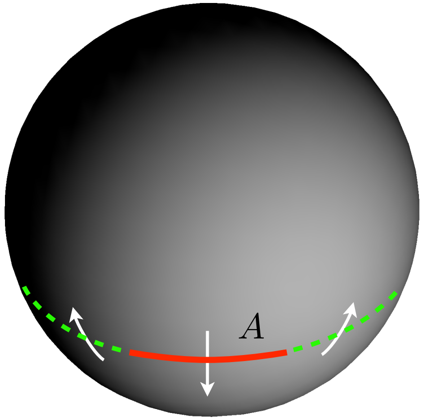


A cut remains connecting consecutive copies
$\implies$ No factorization for even n

(The double arrow indicates the connection to the $(j+2)$-th copy)

Deforming the cut at zero temperature



Single copy of $\left(|\psi\rangle\langle\psi|\right)^{T_A} \quad \Longrightarrow \quad \mathrm{Tr}\left[\left(|\psi\rangle\langle\psi|\right)^{T_A}\right]^n$

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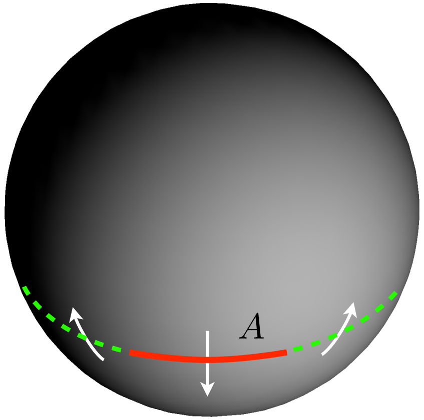

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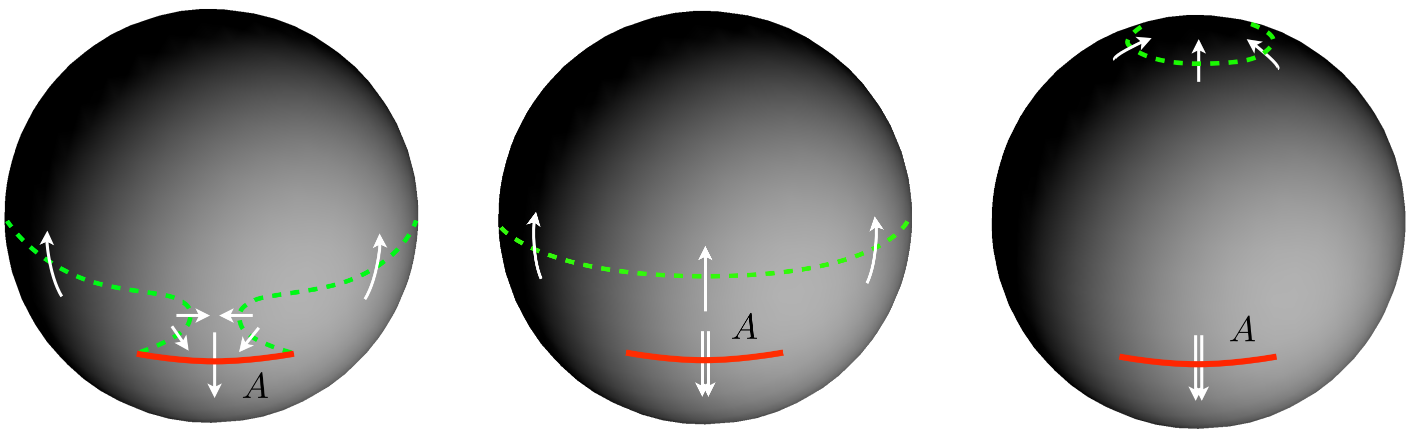

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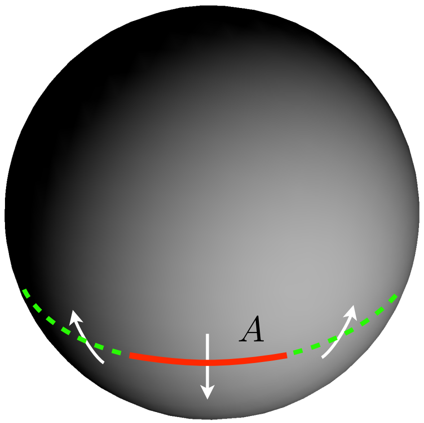

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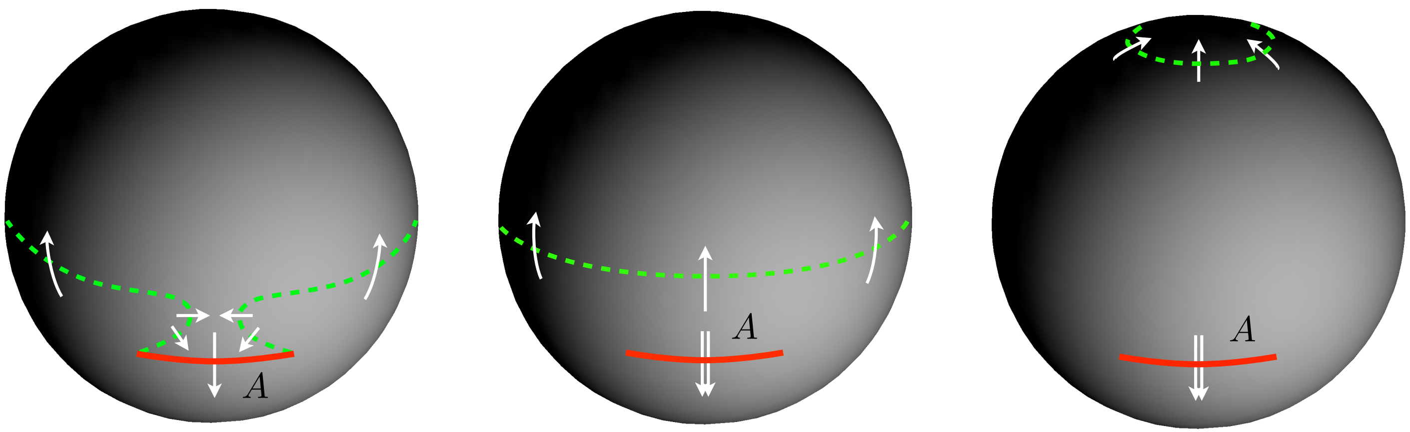

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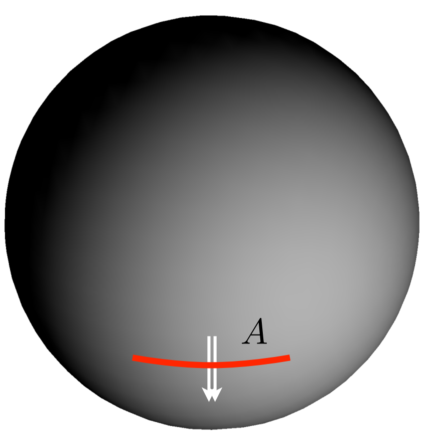
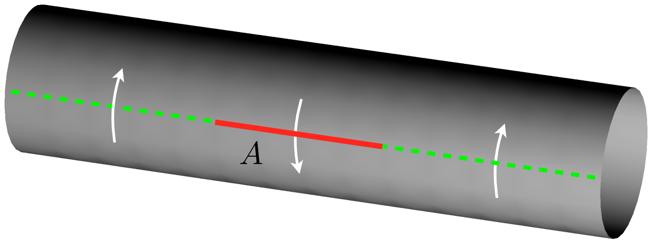


The cut connecting consecutive copies shrinks to a point
Only the connection to the $j \pm 2$ copies along A remains
$\Longrightarrow$ Factorization for even n

One interval at finite temperature in the infinite line



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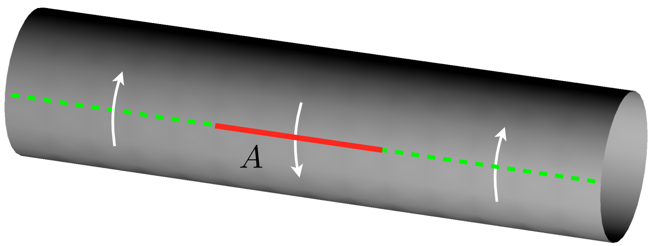

Two auxiliary twist fields at $\mathrm{Re}(w) = \pm L$, then $L \to \infty$

$$\mathcal{E}_A = \lim_{L\to\infty} \lim_{n_e\to 1} \ln \langle \mathcal{T}_{n_e}(-L) \bar{\mathcal{T}}^2_{n_e}(-\ell) \mathcal{T}^2_{n_e}(0) \bar{\mathcal{T}}_{n_e}(L) \rangle_\beta$$

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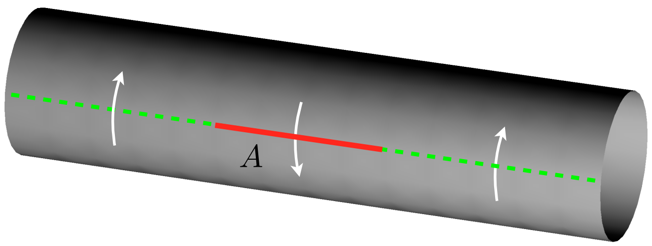

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- Conformal map the cylinder into the plane $z = e^{2\pi w/\beta}$

$$\langle \mathcal{T}_n(z_1) \bar{\mathcal{T}}^2_n(z_2) \mathcal{T}^2_n(z_3) \bar{\mathcal{T}}_n(z_4) \rangle = \frac{c_n \, c_n^{(2)}}{z_{14}^{2\Delta_n} \, z_{23}^{2\Delta_n^{(2)}}} \frac{\mathcal{F}_n(x)}{x^{\Delta_n^{(2)}}} \qquad \mathcal{F}_n(1) = 1 \qquad \mathcal{F}_n(0) = \frac{C^2_{\mathcal{T}_n \bar{\mathcal{T}}^2_n \bar{\mathcal{T}}_n}}{c_n^{(2)}}$$

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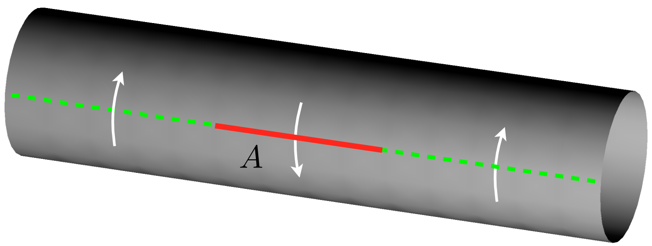

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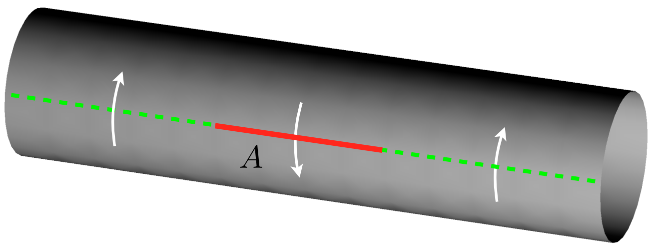
$x \to e^{-2\pi\ell/\beta}$ when $L \to \infty$

$$\boxed{f(x) \equiv \lim_{n_e\to 1} \ln[\mathcal{F}_{n_e}(x)]}$$

$$\boxed{\mathcal{E}_A = \frac{c}{2}\ln\left[\frac{\beta}{\pi a}\sinh\left(\frac{\pi\ell}{\beta}\right)\right] - \frac{\pi c\ell}{2\beta} + f(e^{-2\pi\ell/\beta}) + 2\ln c_{1/2}}$$

One interval at finite temperature in the infinite line



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$$\boxed{\mathcal{E}_A = \lim_{L\to\infty} \lim_{n_e\to 1} \ln \langle \mathcal{T}_{n_e}(-L)\bar{\mathcal{T}}^2_{n_e}(-\ell)\mathcal{T}^2_{n_e}(0)\bar{\mathcal{T}}_{n_e}(L)\rangle_\beta}$$

- Conformal map the cylinder into the plane $z = e^{2\pi w/\beta}$

$$\langle \mathcal{T}_n(z_1)\bar{\mathcal{T}}^2_n(z_2)\mathcal{T}^2_n(z_3)\bar{\mathcal{T}}_n(z_4)\rangle = \frac{c_n\, c_n^{(2)}}{z_{14}^{2\Delta_n}\, z_{23}^{2\Delta_n^{(2)}}}\frac{\mathcal{F}_n(x)}{x^{\Delta_n^{(2)}}} \qquad \mathcal{F}_n(1) = 1 \qquad \mathcal{F}_n(0) = \frac{C^2_{\mathcal{T}_n\bar{\mathcal{T}}^2_n\bar{\mathcal{T}}_n}}{c_n^{(2)}}$$

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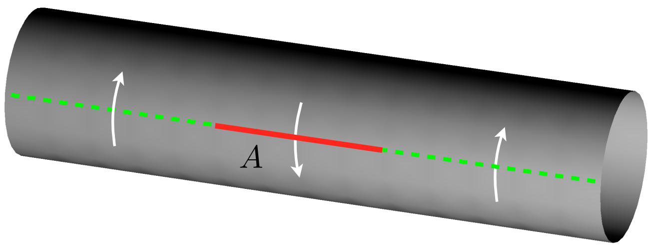
$\Rightarrow$ $\mathcal{E}_A = \mathcal{E}_B$

One interval at finite temperature in the infinite line



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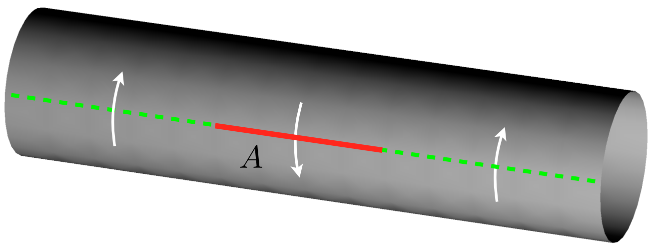

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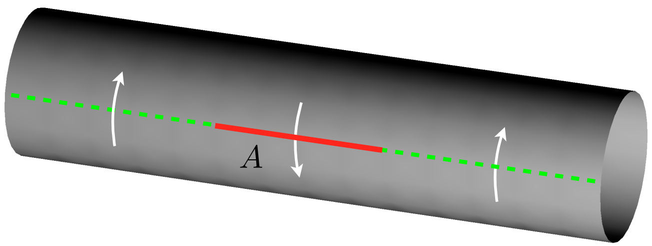

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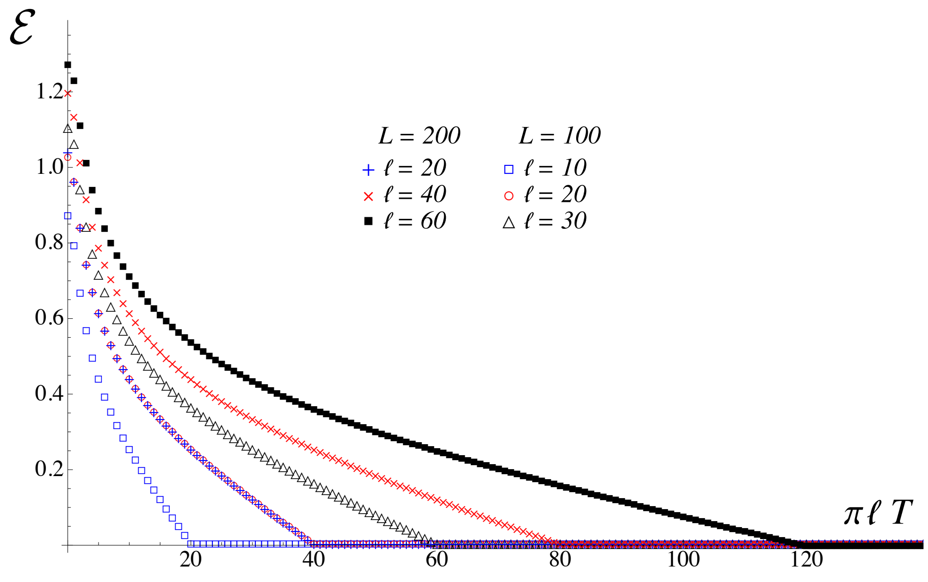
- → large T linear divergence of $\mathcal{E}_{\text{naive}}$ is canceled
- → semi infinite systems $\mathrm{Re}(w) < 0$ (BCFT) have been also studied

Harmonic chain (Dirichlet b.c.) and finite size setup



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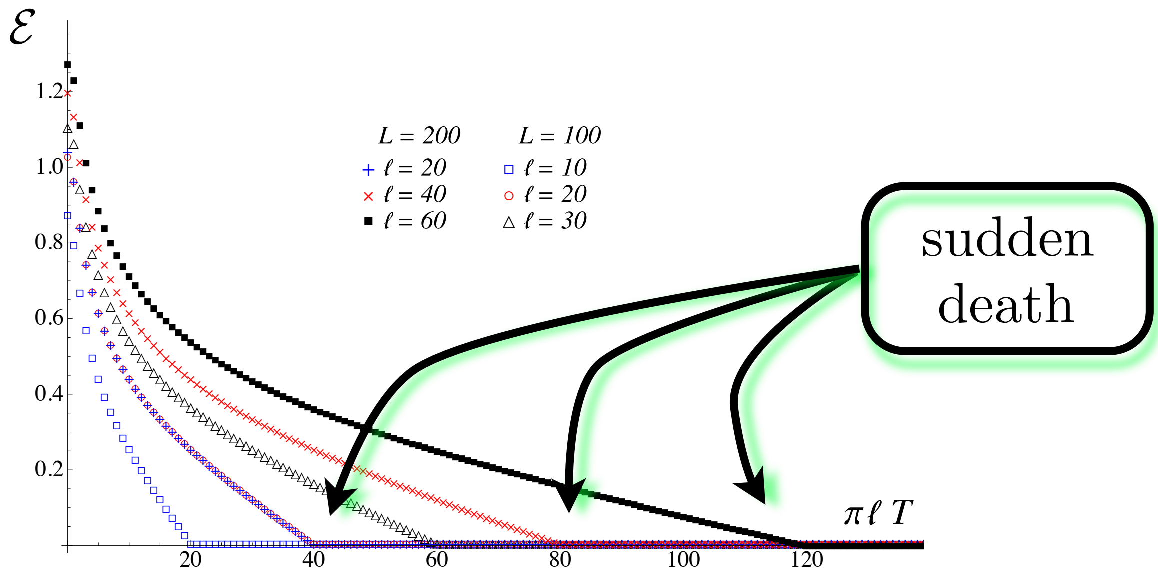

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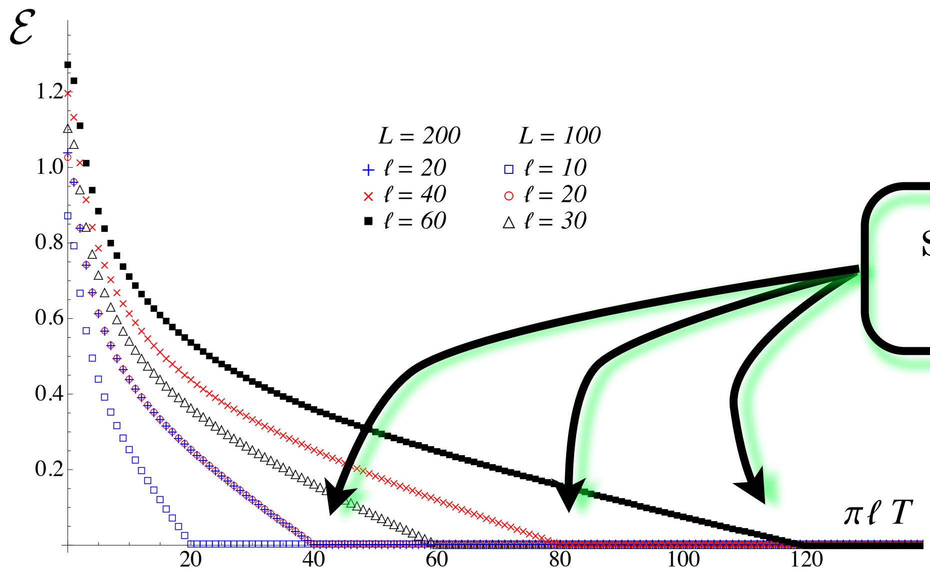

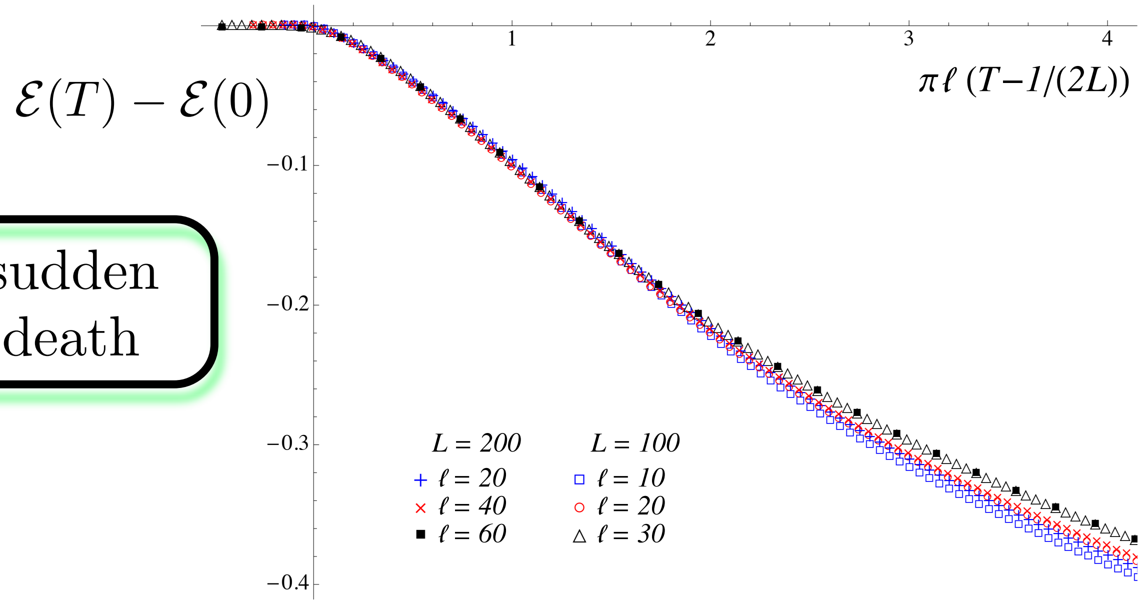

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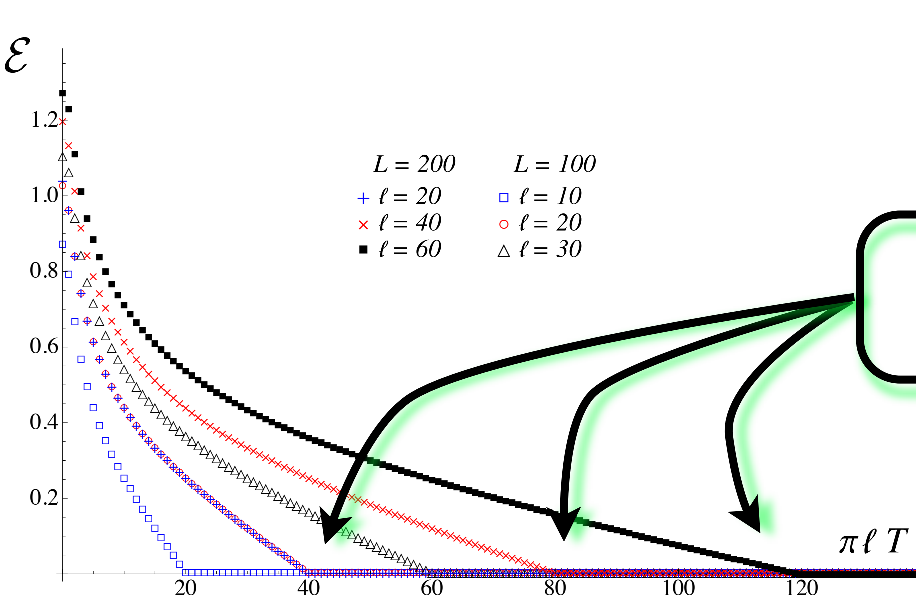
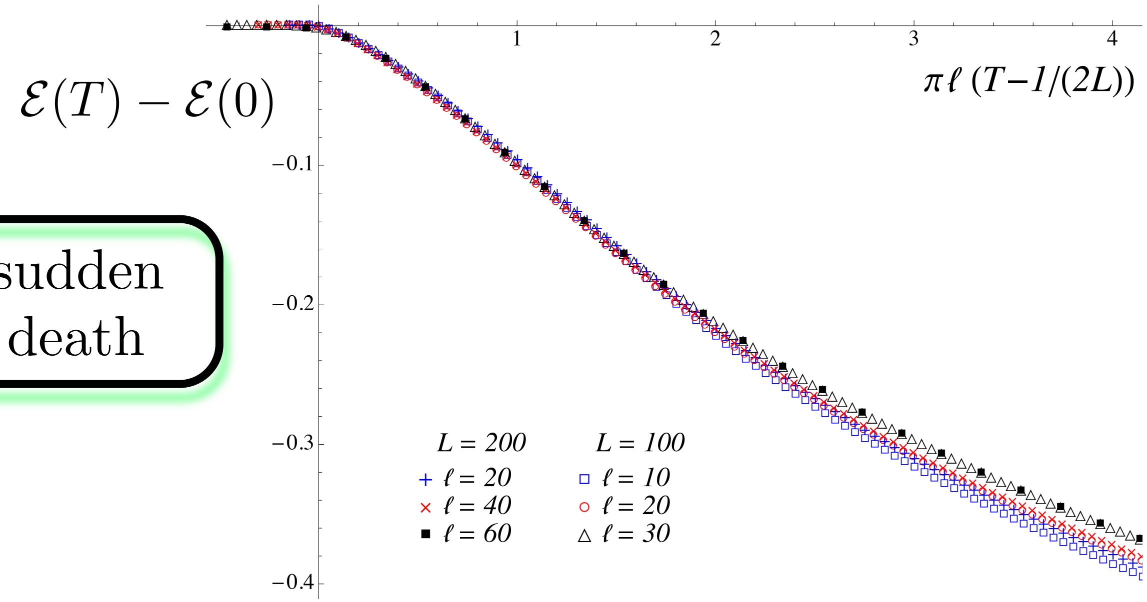
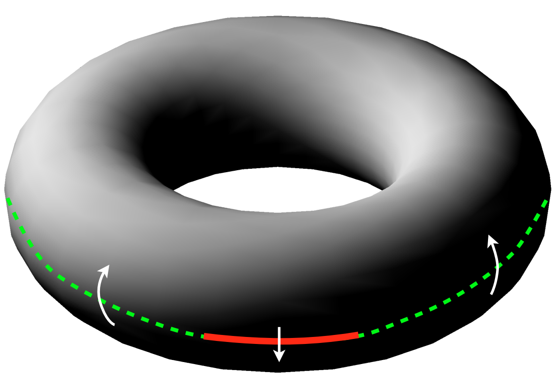
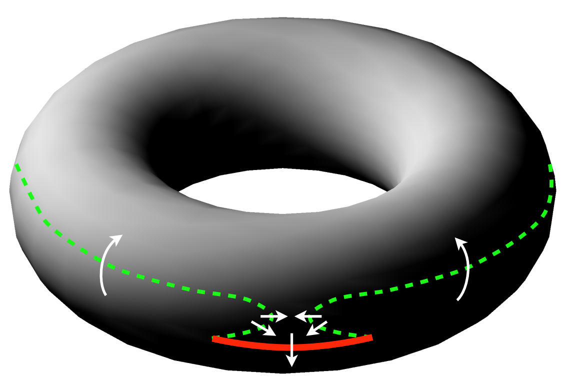




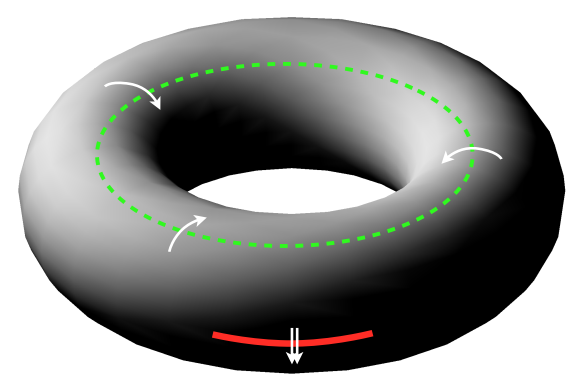
Single copy of $\rho_\beta^{T_A}$
$\Longrightarrow \;\; \mathrm{Tr}\big(\rho_\beta^{T_A}\big)^n$



Deformation of the cut along B



A cut connects consecutive copies



Global quantum quench: CFT evolution

- Global quench:
  - System prepared in the ground state $|\psi_0\rangle$ of H_0
  - At $t = 0$ sudden change of the Hamiltonian $H_0 \to H$

Unitary evolution:

$$\boxed{|\psi(t)\rangle = e^{-\mathrm{i}Ht}|\psi_0\rangle} \qquad \rho(t) = |\psi(t)\rangle\langle\psi(t)|$$

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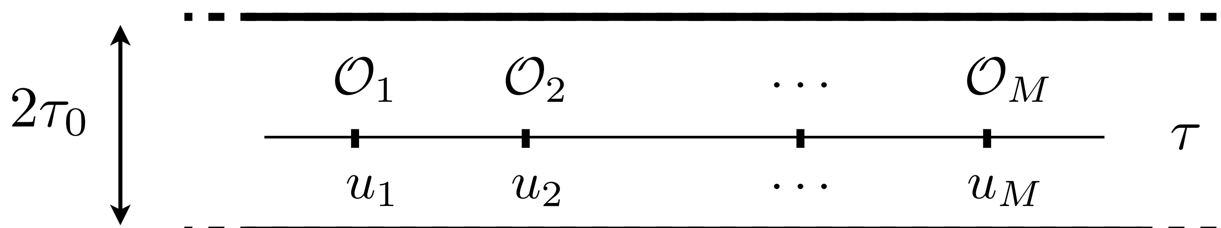
- Path integral formulation and critical H: correlation functions on the strip
  [Calabrese, Cardy, (2005), (2006), (2007)]



Analytic continuation $\tau = \tau_0 + it$, then $t \gg \tau_0$ and $|u_i - u_j| \gg \tau_0$

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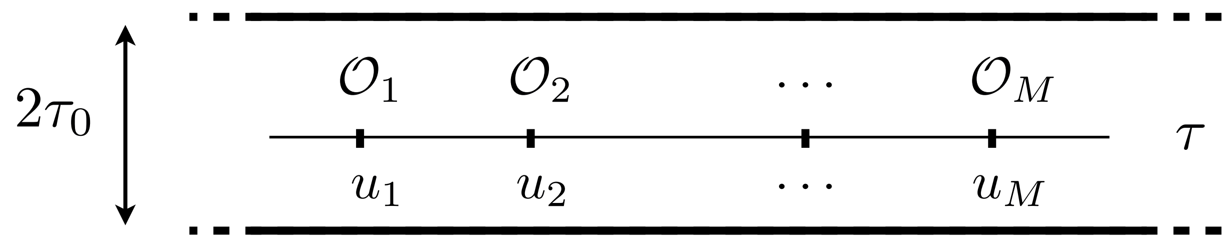

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- Rényi entropies and traces of the partial transpose:
  - $\mathrm{Tr}\rho_A^n \longrightarrow \langle \prod_{i=1}^{N} \mathcal{T}_n(u_{2i-1})\bar{\mathcal{T}}_n(u_{2i}) \rangle_{\mathrm{strip}}$ [Calabrese, Cardy, (2005)]
  - $\mathrm{Tr}(\rho_A^{T_0})^n \longrightarrow$ proper sequence of $\mathcal{T}_n$, $\bar{\mathcal{T}}_n$, $\mathcal{T}_n^2$ and $\bar{\mathcal{T}}_n^2$ within $\langle \dots \rangle_{\mathrm{strip}}$
    [Coser, E.T., Calabrese 14xx.xxxx]

Negativity after a global quench: bipartition of the system

- Global quench of the mass in the periodic harmonic chain

$$H(\omega) = \frac{1}{2}\sum_{j=1}^{L}\left[\, p_j^2 + \omega^2 q_j^2 + (q_{j+1} - q_j)^2 \,\right] \qquad \omega_0 = 1 \;\longrightarrow\; \omega = 0$$

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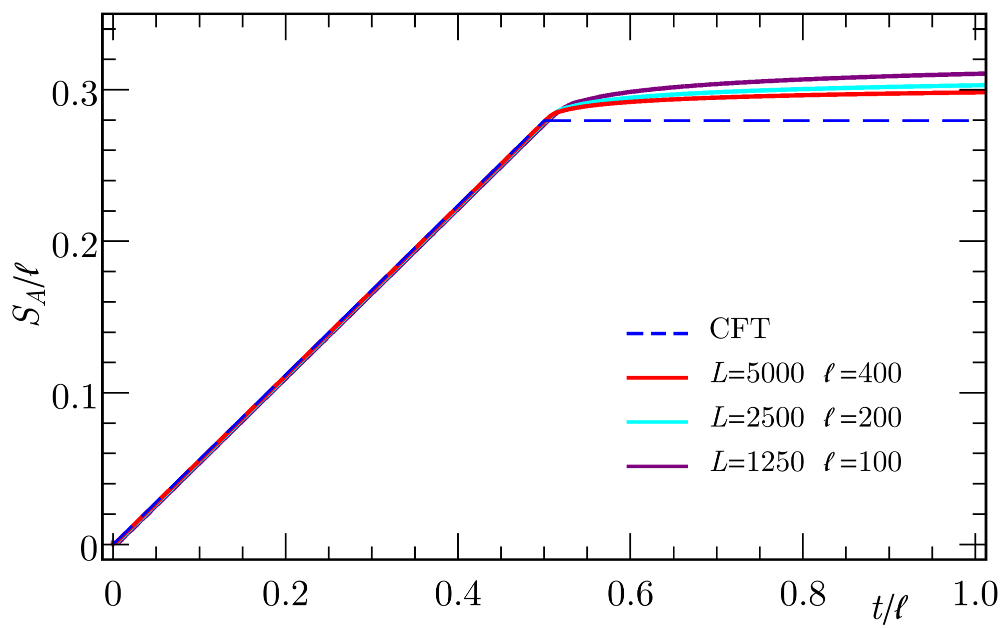

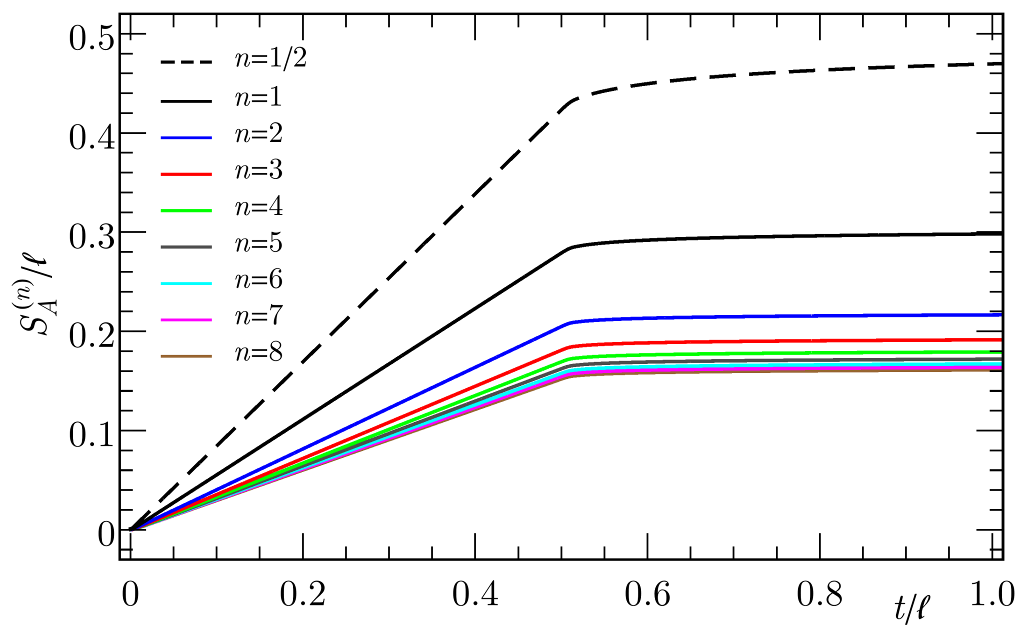

Negativity after a global quench: two adjacent intervals

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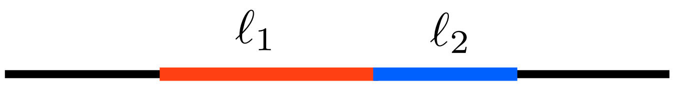
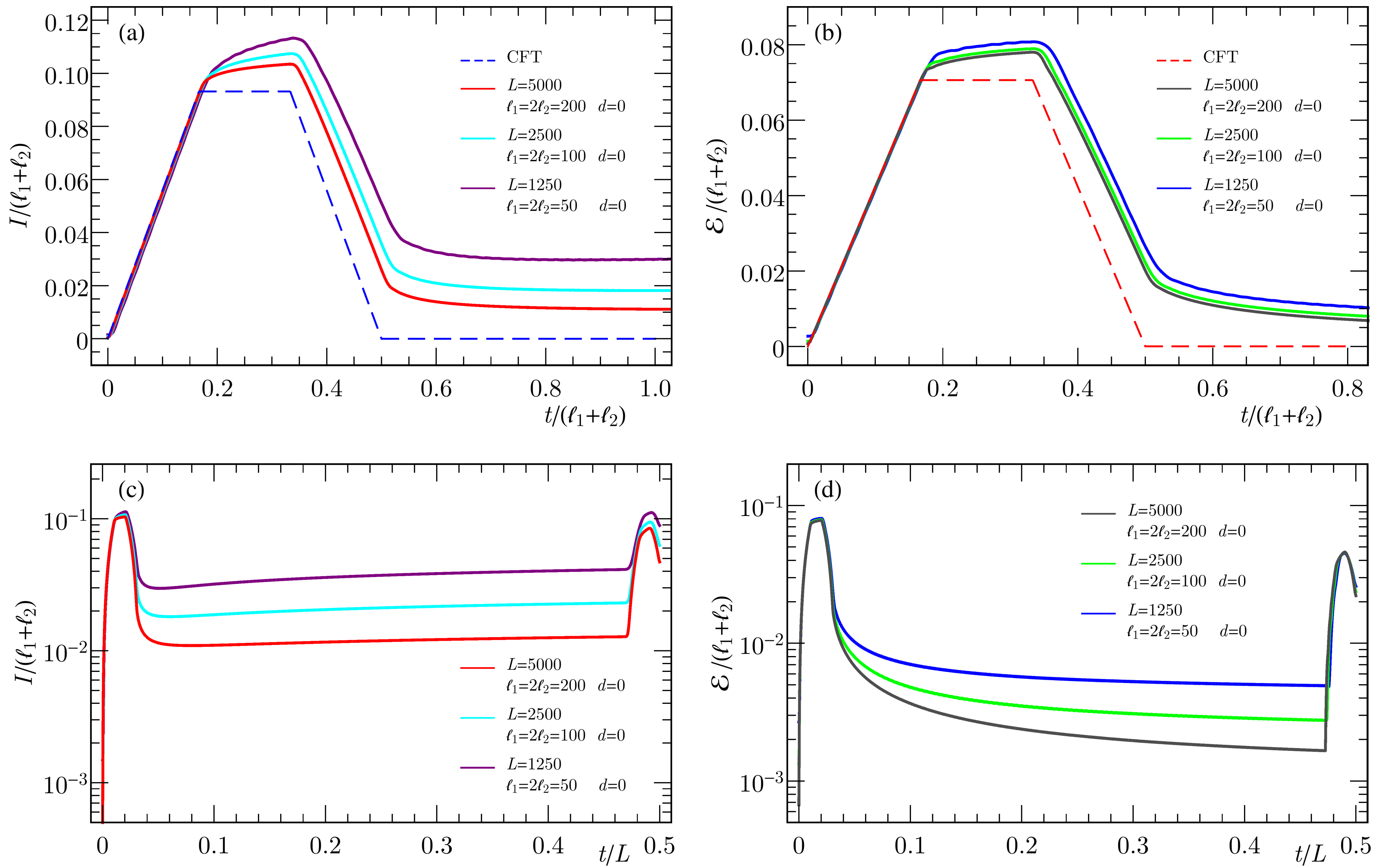
[Coser, E.T., Calabrese 14xx.xxxx]





Negativity after a global quench: two disjoint intervals

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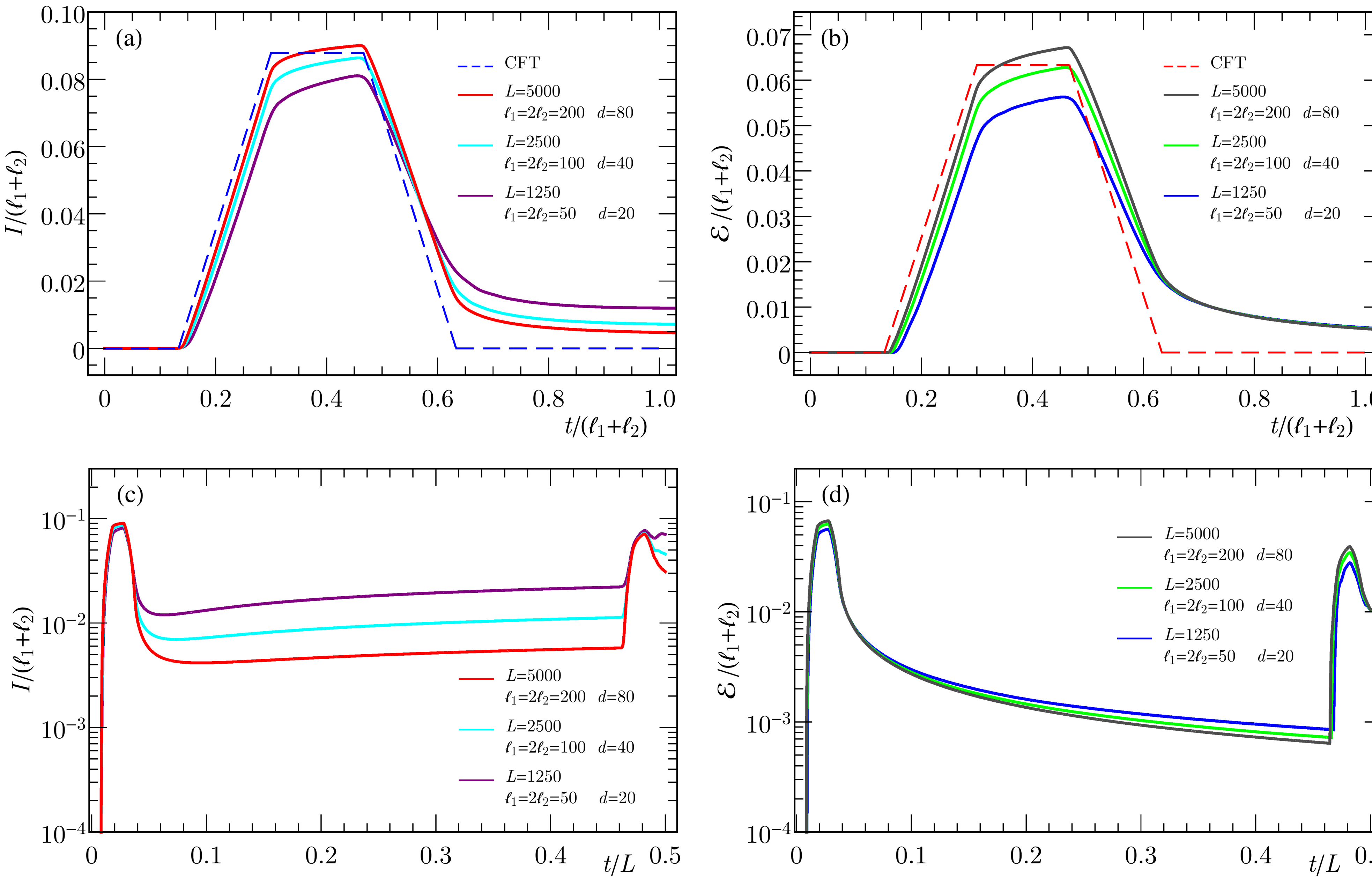
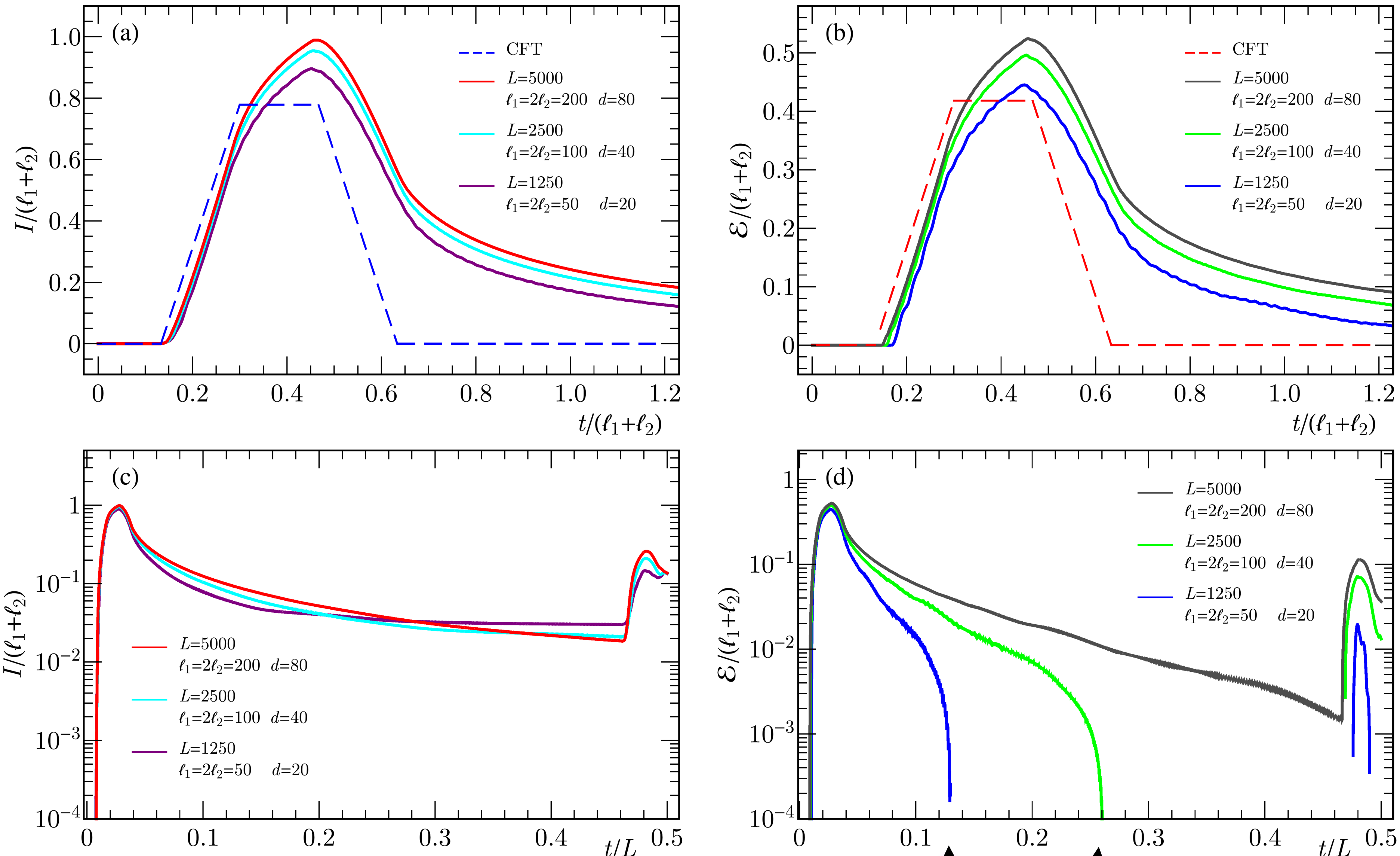


Negativity after a global quench: two disjoint intervals

$\langle \mathcal{T}_n \bar{\mathcal{T}}_n \bar{\mathcal{T}}_n \mathcal{T}_n \rangle_{\text{strip}}$

$\omega_0 = 100 \longrightarrow \omega = 0$



Sudden death of entanglement

Conclusions & open issues

- Entanglement for mixed states.

  Entanglement negativity in QFT (1+1 CFTs): $\mathrm{Tr}(\rho^{T_2})^n$ and $\mathcal{E}$

  → free boson on the line and Ising model

  → finite temperature

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  - → free compactified boson, systems with boundaries and massive case
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Thank you!

N intervals: free compactified boson & Ising model

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[Enolski, Grava, (2003)]

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  Riemann theta function with characteristic $\quad \Theta[\boldsymbol{e}](\mathbf{0}|\Omega) = \sum_{\boldsymbol{m} \in \mathbb{Z}^p} \exp\left[\mathrm{i}\pi(\boldsymbol{m}+\boldsymbol{\varepsilon})^{\mathrm{t}} \cdot \Omega \cdot (\boldsymbol{m}+\boldsymbol{\varepsilon}) + 2\pi\mathrm{i}(\boldsymbol{m}+\boldsymbol{\varepsilon})^{\mathrm{t}} \cdot \boldsymbol{\delta}\right]$

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  Riemann theta function with characteristic $\quad \Theta[\boldsymbol{e}](\mathbf{0}|\Omega) = \sum_{\boldsymbol{m} \in \mathbb{Z}^p} \exp\left[\mathrm{i}\pi (\boldsymbol{m}+\boldsymbol{\varepsilon})^{\mathrm{t}} \cdot \Omega \cdot (\boldsymbol{m}+\boldsymbol{\varepsilon}) + 2\pi\mathrm{i} (\boldsymbol{m}+\boldsymbol{\varepsilon})^{\mathrm{t}} \cdot \boldsymbol{\delta} \right]$

- Free compactified boson $(\eta \propto R^2)$ [Coser, Tagliacozzo, E.T., (2013)]

$$\boxed{\mathcal{F}_{N,n}(\boldsymbol{x}) = \frac{\Theta(\mathbf{0}|T_\eta)}{|\Theta(\mathbf{0}|\tau)|^2} \qquad T_\eta = \begin{pmatrix} \mathrm{i}\,\eta\mathcal{I} & \mathcal{R} \\ \mathcal{R} & \mathrm{i}\,\mathcal{I}/\eta \end{pmatrix} \qquad \tau = \mathcal{R} + \mathrm{i}\,\mathcal{I}}$$

period matrix

- Ising model

$$\boxed{\mathcal{F}_{N,n}^{\text{Ising}}(\boldsymbol{x}) = \frac{\sum_{\boldsymbol{e}} |\Theta[\boldsymbol{e}](\mathbf{0}|\tau)|}{2^g\, |\Theta(\mathbf{0}|\tau)|}}$$

Nasty n dependence

N intervals: free compactified boson & Ising model

- $\mathcal{R}_{N,n}$ is $\boxed{y^n = \prod_{\gamma=1}^{N} (z - x_{2\gamma-2}) \left[\prod_{\gamma=1}^{N-1} (z - x_{2\gamma-1}) \right]^{n-1}}$ $\quad g = (N-1)(n-1)$ [Enolski, Grava, (2003)]

- Partition function for a generic Riemann surface studied long ago in string theory [Zamolodchikov, (1987)] [Alvarez-Gaume, Moore, Vafa, (1986)] [Dijkgraaf, Verlinde, Verlinde, (1988)]

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- Ising model $\quad \boxed{\mathcal{F}^{\text{Ising}}_{N,n}(\boldsymbol{x}) = \frac{\sum_{\boldsymbol{e}} |\Theta[\boldsymbol{e}](\mathbf{0}|\tau)|}{2^g\,|\Theta(\mathbf{0}|\tau)|}}$

Nasty n dependence

- Two intervals case: [Caraglio, Gliozzi, (2008)] [Furukawa, Pasquier, Shiraishi, (2009)] [Calabrese, Cardy, E.T., (2009), (2011)] [Fagotti, Calabrese, (2010)] [Alba, Tagliacozzo, Calabrese, (2010), (2011)]