Lifshitz Space-Times for Schrödinger Holography

Jelle Hartong

Niels Bohr Institute

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Based on collaborations with

Elias Kiritsis and Niels Obers

and with

Eric Bergshoeff and Jan Rosseel

Introduction

- Many systems in nature exhibit critical points with non-relativistic scale invariance. Such systems typically have Lifshitz symmetries:

$$
\begin{aligned}
D_z &: \quad \vec{x} \to \lambda \vec{x} \qquad t \to \lambda^z t \,, \\
H &: \quad t \to t + a \,, \\
P_i &: \quad x^i \to x^i + a^i \,, \\
J_{ij} &: \quad x^i \to R^i{}_j x^j \,.
\end{aligned}
$$

- Lifshitz algebra (only nonzero commutators shown, left out J_{ij} and $z \neq 1$):

$$
[D_z, H] = -zH \,, \qquad [D_z, P_i] = -P_i \,.
$$

- An example of a symmetry group that also displays non-relativistic scale invariance but which is larger than Lifshitz is the Schrödinger group.
- Additional symmetries are Galilean boosts G_i ($x^i \to x^i + v^i t$) and a particle number symmetry M.
- Schrödinger algebra (only nonzero commutators shown, left out J_{ij} and $z \neq 1, 2$):

$$
\begin{aligned}
[D_z, H] &= -zH\,, & [D_z, P_i] &= -P_i\,, & [D_z, M] &= (z-2)M \\
[D_z, G_i] &= (z-1)G_i\,, & [H, G_i] &= P_i\,, & [P_i, G_j] &= M\delta_{ij}
\end{aligned}
$$

- When $z = 2$ there is an additional special conformal symmetry K.

- Aim: to construct holographic techniques for (strongly coupled) systems with NR symmetries.
- Lifshitz holography initiated by: [Kachru, Liu, Mulligan, 2008].

$$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{1}{r^2}\left(dr^2 + d\vec{x}^2\right)$$

- From a different perspective, Lifshitz space-times form interesting examples of non-AdS space-times for which it appears to be possible to construct explicit holographic techniques.
- Punch line of this talk:

Field theories dual to Lifshitz space-times are Schrödinger invariant!

Outline Talk

- Newton–Cartan geometry
- Asymptotically locally Lifshitz space-times
- Two arguments why the dual field theory is Schrödinger invariant:
    - Schrödinger 1: sources (torsional NC geometry), vevs and Ward identities [JH, Kiritsis, Obers, to appear], [Bergshoeff, JH, Rosseel, to appear].
    - Schrödinger 2: bulk vs boundary Killing symmetries [JH, Kiritsis, Obers, to appear].

Newton–Cartan Geometry I

- GR is a diff invariant theory whose tangent space invariance group is the Poincaré group.
- Newtonian gravity is a diff invariant theory known as Newton–Cartan gravity whose tangent space invariance group is the Bargmann algebra [Andringa, Bergshoeff, Panda, de Roo, 2011]: J_{ij}, P_i, G_i, H, M with M central and

$$[H, G_i] = P_i \,, \qquad [P_i, G_j] = M\delta_{ij}$$

- We are not interested in the equations of motion, only in the geometrical framework. The geometry on the boundary in our holographic setup is not dynamical.

From Poincaré to GR

- Local Poincaré: P_a, M_{ab} (gauging):

$$
\begin{aligned}
A_\mu &= P_a e^a_\mu + \frac{1}{2} M_{ab} {\omega_\mu}^{ab} \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu \,, A_\nu] = P_a {R_{\mu\nu}}^a(P) + \frac{1}{2} M_{ab} {R_{\mu\nu}}^{ab}(M) \\
\delta A_\mu &= \partial_\mu \Lambda + [A_\mu \,, \Lambda] \,, \qquad \Lambda = \xi^\mu A_\mu + \frac{1}{2} M_{ab} \lambda^{ab}
\end{aligned}
$$

- GR follows from the curvature constraint:

$$
{R_{\mu\nu}}^a(P) = 0 \left\{
\begin{array}{l}
{\omega_\mu}^{ab} = \text{spin connection: expr. in terms of } e^a_\mu \\
\delta A_\mu = \mathcal{L}_\xi A_\mu + \frac{1}{2} M_{ab} \partial_\mu \lambda^{ab} + \frac{1}{2} [A_\mu \,, M_{ab}] \lambda^{ab} \\
{R_{\mu\nu}}^{ab}(M) = \text{Riemann curvature 2-form} \\
\nabla_\mu \ \ \text{defined via vielbein postulate}
\end{array}
\right.
$$

Newton–Cartan Geometry II

- Gauging Bargmann [Andringa, Bergshoeff, Panda, de Roo, 2011] H, P_a, M, J_{ab}, G_a (a is a spatial index):

$$
\begin{aligned}
A_\mu &= H\tau_\mu + P_a e^a_\mu + M m_\mu + \frac{1}{2} J_{ab}\omega_\mu{}^{ab} + G_a \omega_\mu{}^a \\
F_{\mu\nu} &= H R_{\mu\nu}(H) + P_a R_{\mu\nu}{}^a(P) + M R_{\mu\nu}(M) + \dots \\
\delta A_\mu &= \partial_\mu \Lambda + [A_\mu\,, \Lambda]\,, \qquad \Lambda = \xi^\mu A_\mu + M\sigma + \frac{1}{2} J_{ab}\lambda^{ab} + G_a \lambda^a
\end{aligned}
$$

- Curv. constraints: $R_{\mu\nu}(H) = R_{\mu\nu}{}^a(P) = R_{\mu\nu}(M) = 0$. Independent fields: τ_μ, e^a_μ, m_μ transforming as:

$$
\begin{aligned}
\delta\tau_\mu &= \mathcal{L}_\xi \tau_\mu \\
\delta e^a_\mu &= \mathcal{L}_\xi e^a_\mu + \lambda^a \tau_\mu + \lambda^a{}_b e^b_\mu \\
\delta m_\mu &= \mathcal{L}_\xi m_\mu + \partial_\mu \sigma + \lambda_a e^a_\mu
\end{aligned}
$$

Asymptotically Locally Lifshitz Space-Times

- For a bulk theory of the form

$$S = \int d^4x\sqrt{-g}\left(R - \frac{1}{4}Z(\Phi)F^2 - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}W(\Phi)B^2 - V(\Phi)\right)$$

$$ds^2 = \frac{dr^2}{R(\Phi)r^2} - E^0E^0 + \delta_{ab}E^aE^b\,, \qquad B_M = A_M - \partial_M\Xi$$

the AlLif boundary conditions are [Ross, 2011], [Christensen, JH, Obers, Rollier, 2013], [JH, Kiritsis, Obers, to appear]:

$$\begin{aligned} E^0_\mu &\propto r^{-z}\tau_\mu + \ldots & E^a_\mu &\propto r^{-1}e^a_\mu + \ldots \\ A_\mu - \alpha(\Phi)E^0_\mu &\propto r^{z-2}\tilde{m}_\mu + \ldots & A_r &= (z-2)r^{z-3}\chi + \ldots \\ \Xi &= r^{z-2}\chi + \ldots & \Phi &= r^\Delta\phi + \ldots \end{aligned}$$

Transformations of the sources

- The local bulk symmetries are: local Lorentz transformations, gauge transformations acting on A_M and Ξ and diffs preserving the metric gauge.
- The way these symmetries act on the sources τ_μ, e^a_μ, $\tilde{m}_\mu$, χ is the same as the action of the Bargmann algebra plus local dilatations, i.e. the Sch algebra.
- There is thus a Schrödinger Lie algebra valued connection given by ($\tilde{m}_\mu = m_\mu - (z-2)\chi b_\mu$):

$$A_\mu = H\tau_\mu + P_a e^a_\mu + M m_\mu + \frac{1}{2} J_{ab} {\omega_\mu}^{ab} + G_a {\omega_\mu}^a + D b_\mu$$

with appropriate curvature constraints whose transformations reproduce those of the sources.

Torsional Newton–Cartan (TNC) Geometry

- Inverse vielbeins v^μ and e^μ_a:

$$v^\mu \tau_\mu = -1\,, \qquad v^\mu e^a_\mu = 0\,, \qquad e^\mu_a \tau_\mu = 0\,, \qquad e^\mu_a e^b_\mu = \delta^b_a$$

- From the vielbeins, their inverses and $M_\mu = \tilde{m}_\mu - \partial_\mu \chi$ we can build the following invariants: τ_μ and

$$\begin{aligned} h^{\mu\nu} &= \delta^{ab} e^\mu_a e^\nu_b\,, & \hat{v}^\mu &= v^\mu - h^{\mu\nu} M_\nu \\ \bar{h}_{\mu\nu} &= \delta_{ab} e^a_\mu e^b_\nu - \tau_\mu M_\nu - \tau_\nu M_\mu\,, & \Phi_N &= -v^\mu M_\mu + \tfrac{1}{2} h^{\mu\nu} M_\mu M_\nu \end{aligned}$$

- The affine connection is

$$\Gamma^\rho_{\mu\nu} = -\hat{v}^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} \left(\partial_\mu \bar{h}_{\nu\sigma} + \partial_\nu \bar{h}_{\mu\sigma} - \partial_\sigma \bar{h}_{\mu\nu} \right)$$

with torsion: $\Gamma^\rho_{[\mu\nu]} = -\frac{1}{2} \hat{v}^\rho (\partial_\mu \tau_\nu - \partial_\nu \tau_\mu)$

Vevs and Ward identities I

- Assuming holographic renormalizability it can be shown that the general form of the variation of the on-shell action takes the form:

$$\delta S^{\text{os}}_{\text{ren}} = \int d^3x e \left[-S^0_\mu \delta v^\mu + S^a_\mu \delta e^\mu_a + T^0 \delta\tilde{m}_0 + T^a \delta\tilde{m}_a + \langle O_\chi \rangle \delta\chi + \langle O_\phi \rangle \delta\phi - \mathcal{A}\frac{\delta r}{r}\right]$$

- The vevs and sources can be used to define:

$$\mathcal{T}^\mu{}_\nu = -\left(S^0_\nu + T^0 \partial_\nu \chi\right) v^\mu + \left(S^a_\nu + T^a \partial_\nu \chi\right) e^\mu_a \qquad \text{bdry EM tensor}$$

$$T^\mu = -T^0 v^\mu + T^a e^\mu_a \qquad \text{mass current}$$

- Vielbein components of $\mathcal{T}^\mu{}_\nu$ provide the energy density, energy flux, momentum density and stress.

Vevs and Ward identities II

- The Ward identities are (ignoring the dilaton ϕ):

$$
\begin{aligned}
0 &= -\hat{e}^a_\mu T^\mu + \tau_\mu e^{\nu a}\mathcal{T}^\mu{}_\nu && \text{boosts} \\
0 &= \hat{e}^a_\mu e^{\nu b}\mathcal{T}^\mu{}_\nu - (a \leftrightarrow b) && \text{rotations} \\
\mathcal{A} &= -z\hat{v}^\nu \tau_\mu \mathcal{T}^\mu{}_\nu + \hat{e}^a_\mu e^\nu_a \mathcal{T}^\mu{}_\nu + 2(z-1)\Phi_N \tau_\mu T^\mu && \text{dilatations} \\
\langle O_\chi \rangle &= e^{-1}\partial_\mu \left(e T^\mu\right) && \text{gauge trafos} \\
0 &= \nabla_\mu \mathcal{T}^\mu{}_\nu + 2\Gamma^\rho_{[\mu\rho]}\mathcal{T}^\mu{}_\nu - 2\Gamma^\mu_{[\nu\rho]}\mathcal{T}^\rho{}_\mu \\
&\quad -T^\mu \hat{e}^a_\mu \mathcal{D}_\nu M_a + \tau_\mu T^\mu \partial_\nu \Phi_N && \text{diffs}
\end{aligned}
$$

- We used Galilean boost invariant vielbeins τ_μ, $\hat{e}^a_\mu$, $\hat{v}^\mu$, e^μ_a and density $e = \det\left(\tau_\mu, e^a_\mu\right)$.
- ∇_μ contains the affine TNC connection and $\mathcal{D}_\mu$ contains the Bargmann boost and rotation connections.

TNC Killing vectors and flat NC space-time

- Conserved currents: $\partial_\mu \left(e K^\nu \mathcal{T}^\mu{}_\nu \right) = 0$ whenever K^μ is a TNC conformal Killing vector:

$$\begin{aligned} \mathcal{L}_K \tau_\mu &= -z\Omega\tau_\mu\,, & \mathcal{L}_K \hat{v}^\mu &= z\Omega\hat{v}^\mu\,, & \mathcal{L}_K \bar{h}_{\mu\nu} &= -2\Omega\bar{h}_{\mu\nu} \\ \mathcal{L}_K h^{\mu\nu} &= 2\Omega h^{\mu\nu}\,, & \mathcal{L}_K \Phi_N &= 2(z-1)\Omega\Phi_N\,, & \mathcal{A}\Omega &= 0 \end{aligned}$$

- Flat NC space-time:

$\tau_\mu = \delta^t_\mu$	fixing diffs
$h^{\mu\nu} = \delta^{ij}\delta^\mu_i\delta^\nu_j$	fixing diffs and flat space
$v^\mu = -\delta^\mu_t$	fixing boosts
$h_{\mu\nu} = \delta_{ij}\delta^i_\mu\delta^j_\nu$	forced by other choices
$M_\mu = \partial_\mu M$	global inertial coordinates: $\Gamma^\rho_{\mu\nu} = 0$
$\Phi_N = 0$	no Newton potential

Conformal Killing vectors of flat NC space-time

- The conformal Killing vectors are:

$$
\begin{aligned}
K^t &= a - z\lambda t - \alpha t^z \\
K^i &= a^i + v^i t + \lambda^i{}_j x^j - \lambda x^i - \alpha t^{z-1} x^i \\
\Omega &= \lambda + \alpha t^{z-1}
\end{aligned}
$$

provided we can solve

$$
\begin{aligned}
\mathcal{L}_K M &= v^i x^i - \frac{1}{2}(z-1)\alpha t^{z-2} x^i x^i + (z-2)\Omega M \\
0 &= \partial_t M + \frac{1}{2}\partial_i M \partial^i M \quad \text{due to } \Phi_N = 0 \text{ and } M_\mu = \partial_\mu M
\end{aligned}
$$

Two solutions: $\begin{cases} M = \text{cst} \rightarrow \text{conf. KVs: } H,\ D,\ P_i,\ J_{ij} \\ M = \frac{x^i x^i}{2t} \rightarrow \text{conf. KVs: } K,\ D,\ G_i,\ J_{ij} \end{cases}$

Field Theory on TNC Backgrounds

- Action for Schrödinger equation on a TNC background:

$$S = \int d^{d+1}x e\left[-i\phi^* \hat{v}^\mu \partial_\mu \phi + i\phi \hat{v}^\mu \partial_\mu \phi^* - h^{\mu\nu}\partial_\mu \phi \partial_\nu \phi^* + \phi\phi^* \Phi_N - V(\phi\phi^*)\right]$$

- On a flat NC background this becomes:

$$\begin{aligned} S &= \int d^{d+1}x \left[i\phi^* \left(\partial_t \phi + i\phi \partial_t M\right) - i\phi\left(\partial_t \phi^* - i\phi^* \partial_t M\right)\right. \\ &\quad \left. -\delta^{ij}\left(\partial_i \phi + i\phi \partial_i M\right)\left(\partial_j \phi^* - i\phi^* \partial_j M\right) - V(\phi\phi^*)\right] \end{aligned}$$

- Wavefunction ψ defined as $\phi = e^{-iM}\psi$.
- Space-time symmetries for $M =$ cst is the Lifshitz subalgebra of Sch given by H, D, P_i and J_{ij}.
Space-time symmetries for $M = x^i x^i / 2t$ is the Lifshitz subalgebra of Sch given by K, D, G_i and J_{ij}.

- For $M = $ cst we get the Lifshitz algebra and dual bulk:

$$H = \partial_t\,, \quad P_i = \partial_i\,, \quad D = zt\partial_t + x^i\partial_i + r\partial_r\,, \quad J_{ij} = x^i\partial_j - x^j\partial_i$$

$$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{dx^i dx^i}{r^2}$$

- For $M = \frac{x^i x^i}{2t}$ we get the Lifshitz algebra and dual bulk:

$$K = t^z\partial_t + t^{z-1}(x^i\partial_i + r\partial_r)\,, \quad G_i = t\partial_i\,, \quad D = zt\partial_t + x^i\partial_i + r\partial_r\,, \quad J_{ij}$$

$$ds^2 = \left(-\frac{1}{r^{2z}} + \frac{1}{t^2}\right) dt^2 - \frac{2dr\,dt}{rt} + \frac{dr^2}{r^2} + \frac{1}{r^2}\left(dx^i - \frac{x^i}{t}dt\right)^2$$

- Large diff relating $M = $ cst and $M = \frac{x^i x^i}{2t}$ bulk solutions:

$$\bar{t} = \frac{1}{1-z}t^{1-z}\,, \qquad \bar{x}^i = \frac{x^i}{t}\,, \qquad \bar{r} = \frac{r}{t}$$

Conclusions and Outlook

- We have defined sources for AlLif space-times and shown that they:
  - Transform under the local Schrödinger group
  - Describe a torsional NC boundary geometry
  - Lead to Sch Ward identities for the boundary EM tensor and mass current
- TNC geometries are of growing interest in CMT: [Son, 2013], [Geracie, Son, Wu, Wu, 2014], [Gromov, Abanov, 2014], [Brauner, Endlich, Monin, Penco, 2014].
- Applications to hydrodynamics. Black branes dual to systems with nonzero particle number density.