Spin Matrix Theory

A quantum mechanical model of the AdS/CFT correspondence

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Based on paper to appear by TH and M. Orselli
+ earlier papers: hep-th/0605234, hep-th/0608115, arXiv:0707.1621, arXiv:0806.3370

Motivation:

Goal: A quantitative understanding of the AdS/CFT correspondence beyond $N=\infty$ and SUSY

Why? - Quantum black holes
- Giant Gravitons

AdS/CFT at $N=\infty$:

planar limit of $\mathcal{N}=4$ SYM

Integrable spin chain

Tree-level string theory on $AdS_5 \times S^5$

λ

0 λ small

λ finite

λ large

$N=\infty$: Integrable spin chain as connecting link

Generalize this to finite N?

Generalize spin chain to finite N?

÷ Integrability symmetry $\Rightarrow$ Need other simplifying feature

Simplification: Limit $T \to 0$, $\Omega \to \Omega_{crit}$, $\lambda \to 0$

For $N=\infty$:

planar $\mathcal{N}=4$ SYM in subsector

nearest neighbor spin chain
- rescaled coupling g
- Non-relativistic QM

Non-relativistic string theory

$\longrightarrow g$

g small

g finite

g large

Idea: Use simplifying limit to find generalization of nearest-neighbor spin chain to finite N

$\Rightarrow$ Spin Matrix theory

Plan for talk:

- Define Spin Matrix theory (SMT)
- SMT describes $\mathcal{N}=4$ SYM at critical points
- Phases of SMT (QM model of AdS/CFT)
- Discussion

Spin Matrix theory (SMT):

New quantum mechanical theory

Defined given: Adjoint repr. of $U(N)$ → $N \times N$ matrices

Representation R_s of semi-simple (super-) Lie group G_s

→ Called the spin repr. / spin group

- Harmonic oscillators $(a_s^{\dagger})^i{}_j$ $\quad s \in R_s$

  (i,j) : $N \times N$ matrix indices

$$\left[(a^s)^i{}_j , (a^{\dagger}_{s'})^k{}_l \right] = \delta^s_{s'} \delta^i_l \delta^k_j \qquad (a^s)^i{}_j |0\rangle = 0$$

- Singlet condition: $\Phi^i{}_j |\phi\rangle = 0$; $\Phi^i{}_j = \sum_{s \in R_s} \sum_{k=1}^{N} \left((a_s^{\dagger})^i{}_k (a^s)^k{}_j - (a_s^{\dagger})^k{}_j (a^s)^i{}_k \right)$

⇒ Defines Hilbert space of SMT

Hilbert space spanned by

$$\mathrm{Tr}\left(a^\dagger_{s_1} a^\dagger_{s_2} \cdots a^\dagger_{s_k}\right) \mathrm{Tr}\left(a^\dagger_{s_{k+1}} \cdots\right) \cdots \mathrm{Tr}\left(\cdots a^\dagger_{s_L}\right) |0\rangle$$

Hamiltonian of SMT

- Annihilate 2, Create 2
- Commute with G_s generators
- Spin & Matrix parts factorized

$$H_{int} = \frac{1}{N} U^{s'r'}_{sr} \sum_{\sigma \in S(4)} T_\sigma \left(a^\dagger_{s'}\right)^{i_{\sigma(1)}}{}_{i_3} \left(a^\dagger_{r'}\right)^{i_{\sigma(2)}}{}_{i_4} \left(a^s\right)^{i_{\sigma(3)}}{}_{i_1} \left(a^r\right)^{i_{\sigma(4)}}{}_{i_2}$$

We make choice

$$\sum_{\sigma \in S(4)} T_\sigma \sigma = (14) + (23) - (12) - (34)$$

Motivation: $\mathcal{N}=4$ SYM

Spin part:

irreducible representations

$$U : R_s \otimes R_s \longrightarrow R_s \otimes R_s \; ; \quad R_s \otimes R_s = \sum_J V_J$$

U commutes with G_s generators $\Longrightarrow$ $U^{s'r'}_{sr} = \sum_J C_J \left(P_J\right)^{s'r'}_{sr}$

projector to V_J

H_{int} determined by constants C_J

Complete Hamiltonian:

$$H = \mu_0 L + g H_{int} - \sum_p \mu_p K_p$$

$L = tr(a^+_s a^s)$
"Length" operator

K_p: Cartan gen's of G_s

Choice (rescaling): $\mu_0 = 1$

Partition fct:

$$Z(\beta, \mu_p) = Tr\left(e^{-\beta\left(L + g H_{int} - \sum_p \mu_p K_p\right)}\right)$$

g: coupling const. of SMT

Nearest neighbor spin chains from SMT :

Regime with N large compared to $\langle L \rangle$

$\leftrightarrow$ Low temperature T for a given (large) N

$\Rightarrow$ Multi-trace states linearly independent

Interpret single-trace state $|s_1 s_2 \cdots s_L\rangle = \mathrm{Tr}(a^\dagger_{s_1} a^\dagger_{s_2} \cdots a^\dagger_{s_L})|0\rangle$
as spin chain state (Length L)

Action of H_{int}: $H_{int}|s_1 s_2 \cdots s_L\rangle = 2\sum_{k=1}^{L} U^{mn}_{s_k s_{k+1}} |s_1 \cdots s_{k-1}\, m\, n\, s_{k+2} \cdots s_L\rangle$
$+ O(\frac{L}{N})$

Nearest-neighbor spin chain Hamiltonian $2U^{s'r'}_{sr}$

Perturbative $\frac{1}{N}$ effects: Splitting/joining of spin chains
Weakly interacting gas of spin chains

Increase T $\rightarrow$ Finite N effects $\rightarrow$ $\left\{\begin{array}{l}\text{spin chain interpretation} \\ \text{breaks down} \\ \rightarrow \text{New phase}\end{array}\right.$

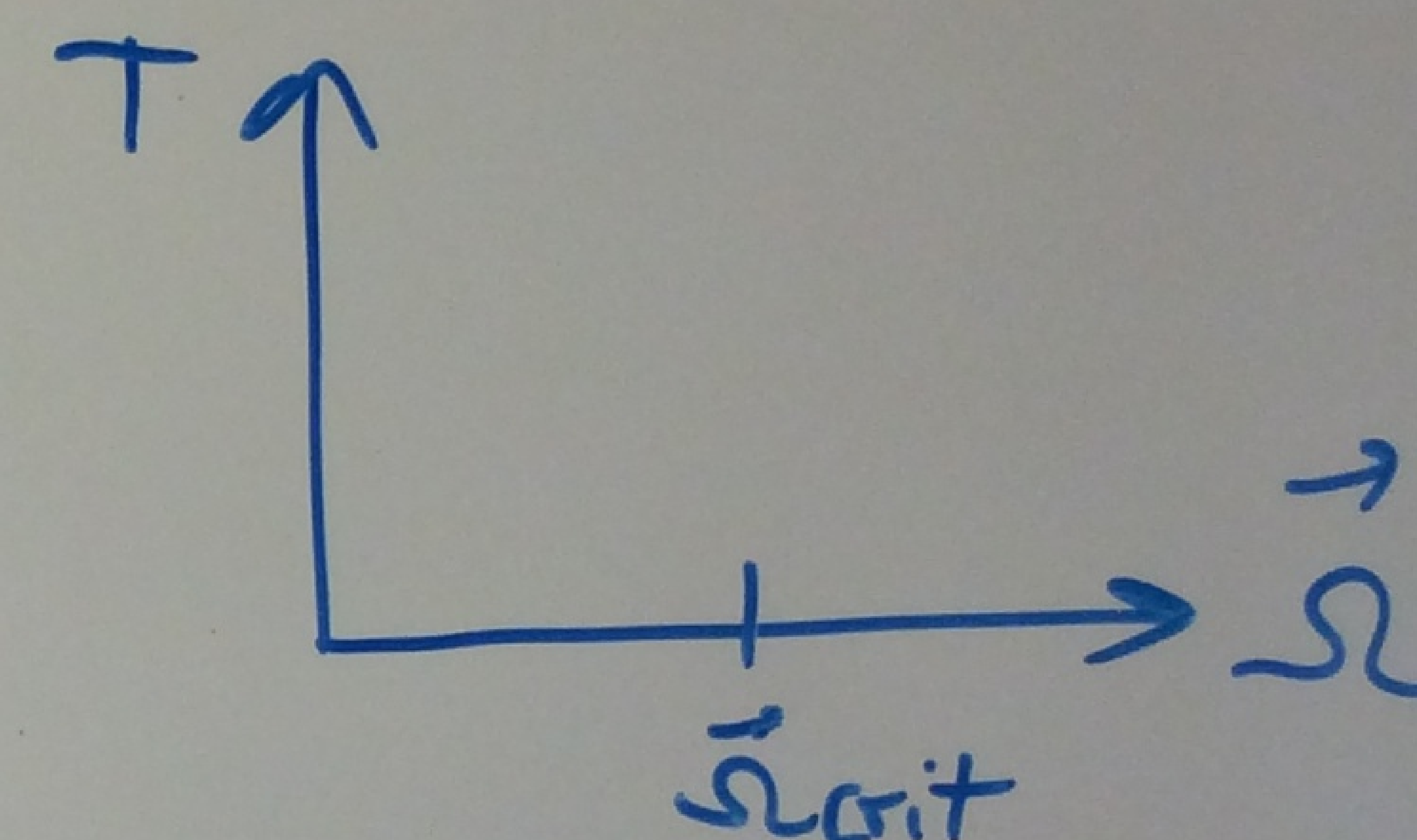
SMT from $\mathcal{N}=4$ SYM near critical points:

$\mathcal{N}=4$ SYM partition fct. $$Z(\beta,\vec{\Omega}) = \mathrm{Tr}\left(e^{-\beta D + \beta \vec{\Omega}\cdot\vec{J}}\right)$$

Dilation op. $D = D_0 + \lambda D_2 + \lambda^2 D_4 + \cdots$; $\vec{J} = (S_1, S_2, R_1, R_2, R_3)$

Limit: $(T,\vec{\Omega}) \longrightarrow (0,\vec{\Omega}_{crit})$; $\lambda \to 0$

keep $\beta\lambda$, $\beta(\vec{\Omega}-\vec{\Omega}_{crit})$ fixed



$$\beta D - \beta\vec{\Omega}\cdot\vec{J} = \beta(D-D_0) + \beta(D_0 - \vec{\Omega}_{crit}\cdot\vec{J}) - \beta(\vec{\Omega}-\vec{\Omega}_{crit})\cdot\vec{J}$$

$\beta\lambda D_2$ gives $\tilde{\beta}\, g\, H_{int}$; higher loops $\to 0$

Only states with $D_0 = \vec{\Omega}_{crit}\cdot\vec{J}$ survives

Gives $\tilde{\beta}\left(L - \sum_p n_p K_p\right)$

⇒ Resulting SMT describes $\mathcal{N}=4$ SYM near the critical point

Compared to $\mathcal{N}=4$ SYM we have

- Reduced set of states as Hilbert space
- Only one-loop interaction survives

A low energy, non-relativistic limit

Magnon dispersion relation: $D - D_0 = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}} - 1$

QFT ⟶ QM

Critical point $(T, \omega_1, \omega_2, \Omega_1, \Omega_2, \Omega_3)$	Spin group G_s	Cartan diagram for algebra	Representation R_s
$(0,0,0,1,1,0)$	$SU(2)$	$\bigcirc$	$[1]$
$(0,\frac{1}{2},0,1,1,\frac{1}{2})$	$SU(1\|2)$	$\bigcirc - \bigotimes$	$[1,0]$
$(0,0,0,1,1,1)$	$SU(2\|3)$	$\bigcirc - \bigotimes - \bigcirc - \bigcirc$	$[0,0,0,1]$
$(0,1,0,1,0,0)$	$SU(1,1)$	$\bigcirc$	$[-1]$
$(0,1,0,1,\frac{1}{2},\frac{1}{2})$	$SU(1,1\|1)$	$\bigotimes - \bigotimes$	$[0,1]$
$(0,1,0,1,1,0)$	$SU(1,1\|2)$	$\bigotimes - \bigcirc - \bigotimes$	$[0,1,0]$
$(0,1,1,0,0,0)$	$SU(1,2)$	$\bigcirc - \bigcirc$	$[0,-3]$
$(0,1,1,\frac{1}{2},\frac{1}{2},0)$	$SU(1,2\|1)$	$\bigcirc - \bigotimes - \bigotimes$	$[0,0,2]$
$(0,1,1,1,0,0)$	$SU(1,2\|2)$	$\bigcirc - \bigotimes - \bigcirc - \bigotimes$	$[0,0,0,1]$
$(0,1,1,1,1,1)$	$SU(1,2\|3)$	$\bigcirc - \bigotimes - \bigcirc - \bigcirc - \bigotimes$	$[0,0,0,1,0]$

Table 1: Critical points of $\mathcal{N}=4$ SYM theory that can be described by Spin Matrix theory. Listed are the spin groups, the Cartan diagram for the corresponding algebra and the representations (in terms of Dynkin labels) that defines the Spin Matrix Theory for a given critical point.

SU(2) Spin Matrix theory:

Describes $\mathcal{N}=4$ SYM near crit. pt. $(T,\vec{\Omega}) = (0,0,0,1,1,0)$

Spin $\frac{1}{2}$ repr. of SU(2) $\qquad \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$

Hamitonian: $C_0 = \frac{1}{8\pi^2}$, $C_1 = 0 \Rightarrow U^{s'r'}_{sr} = \frac{1}{16\pi^2}\left(\delta^{s'}_s \delta^{r'}_r - \delta^{r'}_s \delta^{s'}_r\right)$

$$\Rightarrow H_{int} = -\frac{1}{8\pi^2 N} \mathrm{Tr}\left([a^\dagger_\uparrow, a^\dagger_\downarrow][a^\uparrow, a^\downarrow] \right)$$

Full Hamiltonian $\quad H = L + g H_{int} - \mu S_z \quad$ (we choose $\mu = 0$)

Part. fct. $\quad Z(\beta) = \mathrm{Tr}\left(e^{-\beta(L + g H_{int})}\right)$

For given large N:

$$\begin{cases} \text{low } T \Leftrightarrow N = \infty + \frac{1}{N} \text{ perturbations} \\ \text{high } T \Leftrightarrow \text{finite } N \text{ effects} \\ \qquad\qquad\quad \text{Non-perturbative in } \frac{1}{N} \end{cases}$$

SU(2) SMT at low T:

For $N=\infty$: Solved for all g in terms of ferromagn. $XXX_{\frac{1}{2}}$ Heisenberg spin chain

For $N<\infty$: Phase transition at $T_c(g,N) < T_H(g)$

From $\log Z \sim O(1)$ to $\log Z \sim O(N^2)$

What is the high-T phase above T_c?

SU(2) SMT at high T:

We need to understand finite-N effects

Consider generic SMT for $g=0$

Exact formula for part. fct. is

Dutta, Gopakumar

$$Z(\beta,\mu_p)_{g=0} = 1 + \sum_{n=1}^{\infty} \sum_{k} \sum_{r} \prod_{j=1}^{n} \frac{z(j\beta,\mu_p)^{k_j}}{k_j!\, j^{k_j}} |\chi(r,k)|^2$$

Single spin part. fct. $z(\beta,\mu_p) = \sum_{s\in R_s} \langle s| e^{-\beta + \sum_p \beta\mu_p K_p} |s\rangle$

(we assume R_s is bosonic)

n: symmetric group S_n

$k = (k_1, k_2, \dots, k_n)$ conjugacy classes of S_n

$r = [r_1, r_2, \dots, r_n]$ irreducible repr. of S_n

$\chi(r,k)$: character of S_n given repr. r and conj. class k

Apply to $z(n\beta) = q\,x^n$, $x \equiv e^{-\beta}$

$q = 1$: Free $U(1)$ SMT for $\mu_p = 0$

$q \geq 2$: Free $SU(q)$ SMT for $\mu_p = 0$

Part. fct.:

$$Z_{q,N}(\beta)_{g=0} = 1 + \sum_{n=1}^{\infty} x^n \sum_{k} \sum_{r} \frac{q^{k_j}}{k_j!\, j^{k_j}} |\chi(r,k)|^2$$

Goal: To understand this for $x \to 1$ $(T \to \infty)$

For $g = 1$: Easy to see: $$Z_{1,N}(\beta)_{g=0} = \prod_{n=1}^{N} \frac{1}{1-x^n}$$

(like N bosons in harmonic osc. potential)

For large T: $$Z_{1,N}(\beta)_{g=0} \simeq \frac{1}{N!}(1-x)^{-N}$$

(N indistinguishable 1D harmonic oscillators)

$$\Rightarrow \log Z_{1,N}(\beta)_{g=0} \simeq N \log N - N \log T$$

$$\simeq -N \log T \qquad (\text{for } T \gg N)$$

(N 1D harmonic oscillators)

Classical limit in stat. mech.:

Bose-Einstein $\rightarrow$ Maxwell-Boltzmann for large T

Fermi-Dirac $\rightarrow$ Maxwell-Boltzmann for large T

For $q \geq 2$ no known resummed expression

(Except for $N=\infty$: $Z_{q,N=\infty}(\beta)_{g=0} = \prod_{n=1}^{\infty} \frac{1}{1-qx^n}$)

We devised method to compute $Z_{q,N}(\beta)_{g=0}$ for particular q, N

- Use exact formula to compute finite # of x^n coefficients
- Resum by fitting to form $\frac{P(x)}{Q(x)}$
- One can test resummed form by computing more coefficients

Note: $(q,N)=(2,2)$ and $(2,3)$ was computed independently by Kimura, Ramgoolam, Turton (different method)

Using Eq. (5.3) we have computed the partition function of free $SU(q)$ Spin Matrix theory for general q and for $2 \leq N \leq 5$ up to order x^{40}. Employing the assumption that the partition functions are of the form $P(x)/Q(x)$ we have subsequently resummed the series for specific values of (q, N). With $N = 2$ we have resummed the series for $2 \leq q \leq 5$, with $N = 3$ for $2 \leq q \leq 5$, with $N = 4$ for $q = 2$ and with $N = 5$ for $q = 2$. The resummed expressions for the partition functions are

$$Z_{q,2}(\beta)|_{g=0} = \frac{P_{2,2q-4}(x)}{(1-x)^{2q-2}(1-x^2)^{2q-1}} \quad \text{for} \quad 2 \leq q \leq 5 \tag{B.1}$$

$$Z_{q,3}(\beta)|_{g=0} = \frac{P_{3,10q-16}(x)}{(1-x)^{2q-2}(1-x^2)^{4q-4}(1-x^3)^{3q-2}} \quad \text{for} \quad 2 \leq q \leq 5 \tag{B.2}$$

$$Z_{2,4}(\beta)|_{g=0} = \frac{P_{4,14}(x)}{(1-x)^3(1-x^2)^4(1-x^3)^5(1-x^4)^5} \tag{B.3}$$

$$Z_{2,5}(\beta)|_{g=0} = \frac{P_{5,39}(x)}{(1-x^2)^6(1-x^3)^8(1-x^4)^6(1-x^5)^6} \tag{B.4}$$

with the polynomials

$$\begin{gathered} P_{2,0} = 1\,, \quad P_{2,2} = 1 - x + x^2\,, \quad P_{2,4} = 1 - 2x + 4x^2 - 2x^3 + x^4 \\ P_{2,6} = 1 - 3x + 9x^2 - 9x^3 + 9x^4 - 3x^5 + x^6 \end{gathered} \tag{B.5}$$

$$P_{3,4} = 1 - x^2 + x^4$$

$$\begin{aligned}
P_{3,14} = {} & 1 - x - 2x^2 + 6x^3 + 6x^4 - 9x^5 + x^6 + 17x^7 + x^8 - 9x^9 \\
& +6x^{10} + 6x^{11} - 2x^{12} - x^{13} + x^{14}
\end{aligned}$$

$$\begin{aligned}
P_{3,24} = {} & 1 - 2x - x^2 + 18x^3 + 6x^4 - 30x^5 + 75x^6 + 150x^7 - 30x^8 + 30x^9 \\
& +410x^{10} + 238x^{11} - 76x^{12} + 238x^{13} + 401x^{14} + 30x^{15} - 30x^{16} \\
& +150x^{17} + 75x^{18} - 30x^{19} + 6x^{20} + 18x^{21} - x^{22} - 2x^{23} + x^{24}
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
P_{3,34} = {} & 1 - 3x + 2x^2 + 34x^3 - 4x^4 - 18x^4 + 421x^6 + 624x^7 + 251x^8 \\
& +2107x^9 + 5377x^{10} + 4766x^{11} + 6384x^{12} + 16031x^{13} + 19327x^{14} \\
& +14592x^{15} + 21381x^{16} + 29839x^{17} + 21381x^{18} + 14592x^{19} + 19327x^{20} \\
& +16031x^{21} + 6384x^{22} + 4766x^{23} + 5377x^{24} + 2107x^{25} + 251x^{26} \\
& +624x^{27} + 421x^{28} - 18x^{29} - 4x^{30} + 34x^{31} + 2x^{32} - 3x^{33} + x^{34}
\end{aligned}$$

for $N = 3$ and

$$P_{4,14} = 1 - x - x^2 + 2x^4 + 2x^5 - 4x^7 + 2x^9 + 2x^{10} - x^{12} - x^{13} + x^{14}$$

$$\begin{aligned}
P_{5,39} = {} & 1 + 2x - 6x^3 - 9x^4 + 2x^5 + 25x^6 + 38x^7 + 17x^8 - 34x^9 - 68x^{10} \\
& -34x^{11} + 73x^{12} + 176x^{13} + 171x^{14} + 34x^{15} - 127x^{16} - 156x^{17} - 2x^{18} \\
& +218x^{19} + 322x^{20} + 218x^{21} - 2x^{22} - 156x^{23} - 127x^{24} + 34x^{25} \\
& +171x^{26} + 176x^{27} + 73x^{28} - 34x^{29} - 68x^{30} - 34x^{31} + 17x^{32} + 38x^{33} \\
& +26x^{34} + 2x^{35} - 9x^{36} - 6x^{37} + 2x^{38} + x^{39}
\end{aligned} \tag{B.7}$$

for $N = 4$ and $N = 5$.

What do we learn from this?

For all (q,N) computed we find for $x \to 1$:

$$Z_{q,N}(\beta)_{g=0} \simeq \frac{a_{q,N}}{(1-x)^{(q-1)N^2+1}}$$

But no obvious interpretation of $a_{q,N}$ coefficient in terms of statistics

$a_{q,N}$	$q=2$	$q=3$	$q=4$	$q=5$
$N=2$	$1/2^3$	$1/2^9$	$1/2^6$	$5/2^9$
$N=3$	$1/2^4 3^4$	$7/2^8 3^6$	$\frac{409}{2^{10}3^{10}}$	$\frac{5\cdot 14159}{2^{16}\cdot 3^{12}}$
$N=4$	$1/2^{13}3^5$			
$N=5$	$\frac{193}{2^{18}3^8 5^5}$			

In classical limit:

$$\log Z_{q,N}(\beta)_{g=0} \simeq -\left[(q-1)N^2+1\right]\log T \quad \text{for } T \to \infty$$

$\Rightarrow$ Conjecture: Free $SU(q)$ SMT $\simeq$ $(q-1)N^2+1$ one-dim. harmonic osc for large T (classical limit)

High-temp. phase of free SU(q) SMT :

$$(q-1)N^2+1 \text{ harmonic oscillators}$$

$\rightarrow$ A partially deconfined phase

"Deconf." since for $N \to \infty$ $\begin{cases} \log Z \sim O(1) & \text{for } T < T_H = \frac{1}{\log q} \\ \log Z \sim N^2 & \text{for } T > T_H = \frac{1}{\log q} \end{cases}$

"Partial" since without singlet condition the high-temp. phase would be qN^2 harmonic oscillators

Singlet condition removes N^2-1 harmonic oscillators

Interpretation of high-temp. phase?

Spin chains break up in smaller constituents?

No – This interpretation is flawed

Take Plethystic log → Gives single trace part. fct from which $Z_{q,N}(\beta)_{g=0}$ can be generated

For $(q,N) = (2,2), (2,3), (3,2)$:

→ $Z_{q,N}(\beta)_{g=0}$ can be generated from finite no. of single traces (+ algebraic constraints)

But those are exceptions → Any higher q or N

→ One needs single-traces of arbitrarily large lengths!

Not clear what the constituents of the high-temp. phase are

Classical matrix model description for any q

Classical limit points to semi-classical regime at high T

Coherent states for $SU(2)$ SMT: ($q=2$ from now)

$$|\lambda\rangle = \mathcal{N}_\lambda \exp\left(\sum_s \mathrm{Tr}(\lambda_s a_s^\dagger)\right)|0\rangle \; ; \; \langle\lambda|\lambda\rangle = 1$$

$$\lambda_s = \frac{1}{\sqrt{2}}(X_s + i P_s)$$: X_s, P_s hermitian $N\times N$ matrices

$s = \uparrow, \downarrow$

$$(a^s)^i{}_j |\lambda\rangle = (\lambda_s)^i{}_j |\lambda\rangle$$

Singlet condition: $\langle\lambda|\Phi^i{}_j|\lambda\rangle = 0$

$$\Leftrightarrow \quad \sum_s [X_s, P_s] = 0$$ Gauss constraint

Hamiltonian for SU(2) SMT: $H = L + g H_{int}$:

$$\langle \lambda | H | \lambda \rangle = \frac{1}{2} \sum_s \mathrm{Tr}\left(P_s^2 + X_s^2\right)$$

$$- \frac{g}{32\pi^2 N} \mathrm{Tr}\Big([X_\uparrow, X_\downarrow]^2 + [P_\uparrow, P_\downarrow]^2 + [X_\uparrow, P_\downarrow]^2 + [X_\downarrow, P_\uparrow]^2 + [X_\uparrow, P_\uparrow]^2 + [X_\downarrow, P_\downarrow]^2 \Big)$$

+ Gauss constraint $\sum_s [X_s, P_s] = 0$

Classical matrix model that describes SU(2) SMT in high-T semi-classical regime for any g

Previously for $g=0$ we found N^2+1 harmonic oscillators at high T $\rightarrow$ Fits with matrix model?

$g=0$:

$$H = \frac{1}{2}\sum_s \mathrm{Tr}\left(P_s^2 + X_s^2\right) \; ; \quad \sum_s \left[X_s, P_s\right] = 0$$

$2N^2$ harmonic osc $\qquad\qquad$ N^2-1 conditions

Counting works: $N^2+1 = 2N^2 - (N^2-1)$

But not obvious that it is N^2+1 uncoupled harm. osc's for high T (clearly not true for finite T)

We computed $Z_{2,N}(\beta) = \int dP\, dX\, e^{-\beta H}\, \delta\left(\sum_s [X_s, P_s]\right)$

for $N=2$: $\log Z_{2,2}(\beta) \simeq -5 \log T$ for $T \to \infty$

Consistent with previous $g=0$ result!

What happens for $g \gg 1$?

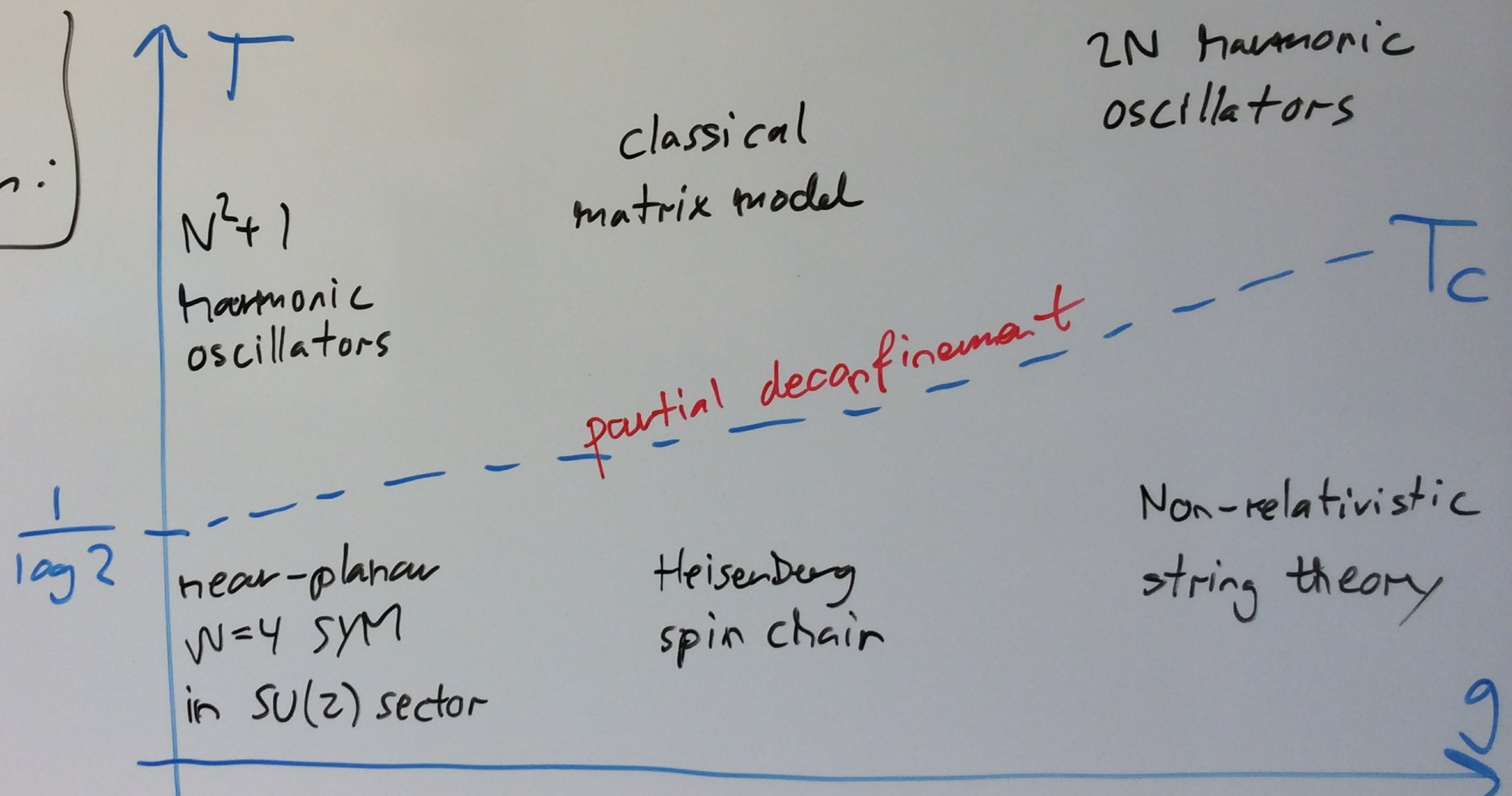
Leading order solution:

$X_\uparrow, X_\downarrow, P_\uparrow, P_\downarrow$ are diagonal

Gives potential = 0
and solves Gauss constraint

$\Rightarrow$ 2N harmonic oscillators

SU(2) SMT
phase diagram:



QM model of AdS/CFT

Discussion:

SMT describes $\mathcal{N}=4$ SYM near critical points

→ A QM model of the AdS/CFT correspondence

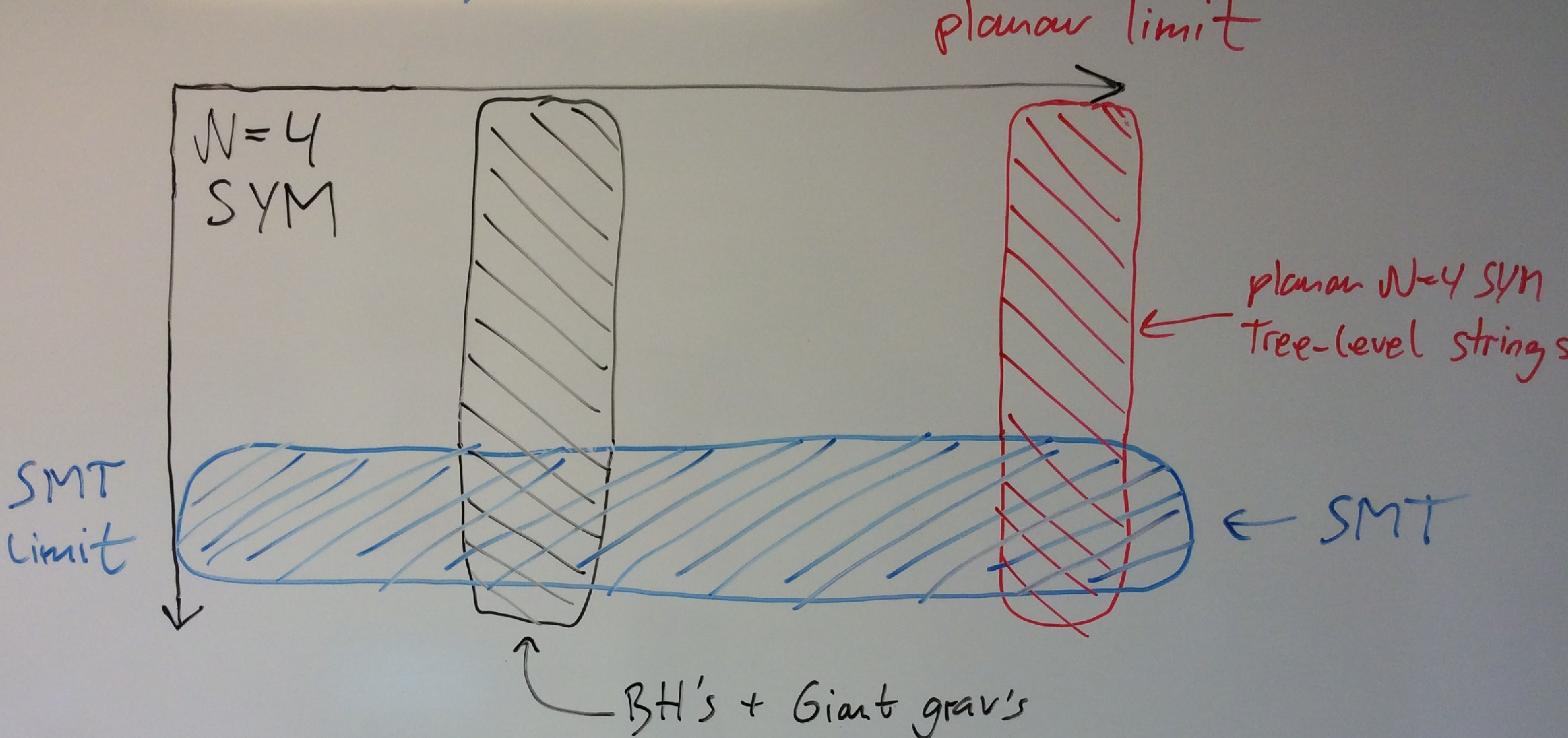
SMT generalizes spin chains as connecting link in AdS/CFT

SMT simple enough to take strong coupling limit

We explored new semi-classical regime at high T

2N harmonic oscillators	⟷	Highly excited gas of $\frac{1}{4}$ BPS Giant Gravitons

General philosophy:



SU(2) SMT for large g & T $\longrightarrow$ Giant grav's

SU(1,2|3) SMT for large g & T $\longrightarrow$ Black holes + Giant grav's