Dynamics and observational constraints on Brans-Dicke cosmological model

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Published papers

- OH, M. Kamionka, M. Szydłowski, *Dynamics and cosmological constraints on Brans-Dicke cosmology*, `arXiv:1404.7112 [astro-ph.CO]`
- OH, M. Szydłowski, *Dynamical complexity of the Brans-Dicke cosmology*, JCAP12(2013)016, `arXiv:1310.1961 [gr-qc]`
- OH, M. Szydłowski, *Brans-Dicke theory and the emergence of ΛCDM model*, Phys. Rev. D88, 064018 (2013), `arXiv:1304.3300 [gr-qc]`

The Brans-Dicke cosmology

The action for the Brans-Dicke theory in so-called Jordan frame is in the following form

$$S = \int \mathrm{d}^4x\sqrt{-g}\left\{\phi R - \frac{\omega_{\mathrm{BD}}}{\phi}\nabla^{\alpha}\phi\nabla_{\alpha}\phi - 2\,V(\phi)\right\} + 16\pi S_m \qquad (1)$$

where the barotropic matter is described by

$$S_m = \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_m\,, \qquad (2)$$

and ω_{BD} is a dimensionless parameter of the theory.

FRW metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\Big(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\Big) \tag{3}$$

the energy conservation condition

$$3H^2 = \frac{\omega_{\mathrm{BD}}}{2}\frac{\dot{\phi}^2}{\phi^2} + \frac{V(\phi)}{\phi} - 3H\frac{\dot{\phi}}{\phi} + \frac{8\pi}{\phi}\rho_m \tag{4}$$

the acceleration equation

$$\dot{H} = -\frac{\omega_{\mathrm{BD}}}{2}\frac{\dot{\phi}^2}{\phi^2} - \frac{1}{3+2\omega_{\mathrm{BD}}}\frac{2\,V(\phi) - \phi V'(\phi)}{\phi} + 2H\frac{\dot{\phi}}{\phi} - \frac{8\pi}{\phi}\rho_m\frac{2+\omega_{\mathrm{BD}}(1+w_m)}{3+2\omega_{\mathrm{BD}}} \tag{5}$$

the dynamical equation for the BD scalar

$$\ddot{\phi} + 3H\dot{\phi} = 2\frac{2\,V(\phi) - \phi\,V'(\phi)}{3+2\omega_{\mathrm{BD}}} + 8\pi\rho_m\frac{1-3w_m}{3+2\omega_{\mathrm{BD}}} \tag{6}$$

Dynamical variables

In what follows we introduce following energy phase space variables

$$x \equiv \frac{\dot{\phi}}{H\phi}, \qquad y \equiv \sqrt{\frac{V(\phi)}{3\phi}}\frac{1}{H}, \qquad \lambda \equiv -\phi\frac{V'(\phi)}{V(\phi)}$$

the energy conservation condition (4) can be presented as

$$\Omega_m = \frac{8\pi\rho_m}{3H^2\phi} = 1 + x - \frac{\omega_{\mathrm{BD}}}{6}x^2 - y^2\,, \tag{7}$$

and the acceleration equation (5)

$$\begin{aligned}\frac{\dot{H}}{H^2} = 2x - \frac{\omega_{\mathrm{BD}}}{2}x^2 - \frac{3}{3+2\omega_{\mathrm{BD}}}y^2(2+\lambda) \\ - 3\left(1 + x - \frac{\omega_{\mathrm{BD}}}{6}x^2 - y^2\right)\frac{2+\omega_{\mathrm{BD}}(1+w_m)}{3+2\omega_{\mathrm{BD}}}\,,\end{aligned} \tag{8}$$

Dynamical system

$$
\begin{aligned}
x' &= -3x - x^2 - x\frac{\dot{H}}{H^2} + \frac{6}{3+2\omega_{\mathrm{BD}}}y^2(2+\lambda) + \\
&+ 3\left(1 + x - \frac{\omega_{\mathrm{BD}}}{6}x^2 - y^2\right)\frac{1-3w_m}{3+2\omega_{\mathrm{BD}}}\,, \\
y' &= -y\left(\frac{1}{2}x(1+\lambda) + \frac{\dot{H}}{H^2}\right)\,, \\
\lambda' &= x\lambda\Big(1 - \lambda(\Gamma - 1)\Big)\,,
\end{aligned}
\tag{9}
$$

where $()' = \frac{\mathrm{d}}{\mathrm{d}\ln a}$ and

$$
\Gamma = \frac{V''(\phi)V(\phi)}{V'(\phi)^2}\,.
$$

From now on we will assume that $\Gamma = \Gamma(\lambda)$. The critical points of the system (9) depend on the explicit form of the $\Gamma(\lambda)$ function. One can notice that the single critical point $(x^* = 0\,, y^* = \pm 1\,, \lambda^* = -2)$ do not depend on the assumed $\Gamma(\lambda)$. Additionally, the acceleration equation (8) calculated at this point vanishes, giving rise to the deSitter expansion.

de Sitter state for $V(\phi) = V_0\,\phi^2$

Critical point: $x_3^* = 0$, $y_3^* = \pm 1$ with effective equation of state parameter

$$w_{\text{eff}}\big|_3^* = -1$$

linearized solutions are

$$x_3(a) = \frac{1}{w_m}\frac{1+2\omega_{\text{BD}}w_m}{3+2\omega_{\text{BD}}}\left[\Delta x - 2y_3^*\frac{1-3w_m}{1+2\omega_{\text{BD}}w_m}\Delta y\right]\left(\frac{a}{a_3^{(i)}}\right)^{\lambda_1} - \\ -\frac{1}{w_m}\frac{1-3w_m}{3+2\omega_{\text{BD}}}\left[\Delta x - 2y^*\Delta y\right]\left(\frac{a}{a_3^{(i)}}\right)^{\lambda_2},$$

$$y_3(a) = y_3^* + \frac{1}{2y_3^* w_m}\frac{1+2\omega_{\text{BD}}w_m}{3+2\omega_{\text{BD}}}\left\{\left[\Delta x - 2y_3^*\frac{1-3w_m}{1+2\omega_{\text{BD}}w_m}\Delta y\right]\left(\frac{a}{a_3^{(i)}}\right)^{\lambda_1} - \\ -\left[\Delta x - 2y^*\Delta y\right]\left(\frac{a}{a_3^{(i)}}\right)^{\lambda_2}\right\}$$

where $\lambda_1 = -3$ and $\lambda_2 = -3(1+w_m)$ are the eigenvalues of the linearization matrix and $\Delta x = x_3^{(i)} - x_3^*$, $\Delta y = y_3^{(i)} - y_3^*$ are the initial conditions.

de Sitter state for $V(\phi) = V_0\,\phi^2$

Using the linearized solutions we obtain the following form of the Hubble function in the vicinity of the critical point under considerations

$$\left(\frac{H(a)}{H(a_0)}\right)^2 \approx 1 - \Omega_{DM,0} - \Omega_{M,0} + \Omega_{DM,0}\left(\frac{a}{a_0}\right)^{-3} + \Omega_{M,0}\left(\frac{a}{a_0}\right)^{-3(1+w_m)},$$

where

$$\Omega_{DM,0} = -\frac{4}{3w_m}\frac{1+2\omega_{\text{BD}}w_m}{3+2\omega_{\text{BD}}}\left\{\Delta x - 2y_3^*\frac{1-3w_m}{1+2\omega_{\text{BD}}w_m}\Delta y\right\}\left(\frac{a_0}{a_3^{(i)}}\right)^{-3},$$

$$\Omega_{M,0} = \frac{2}{3w_m(1+w_m)}\frac{2+3\omega_{\text{BD}}w_m(1+w_m)}{3+2\omega_{\text{BD}}}\left\{\Delta x - 2y_3^*\Delta y\right\}\left(\frac{a_0}{a_3^{(i)}}\right)^{-3(1+w_m)}.$$

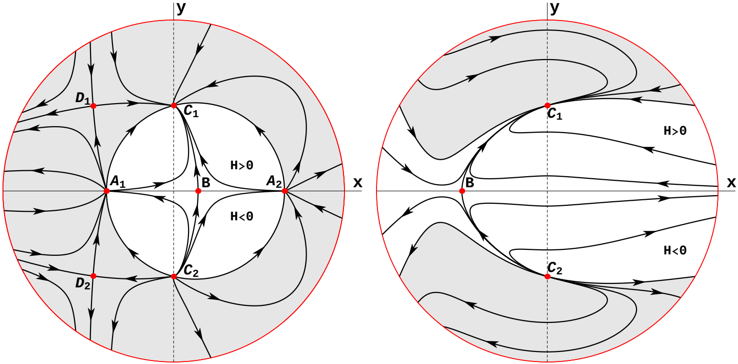


Figure : Diagrams of the evolutional paths in phase space compactified with circle at infinity for the model filled with dust matter $w_m = 0$ and : $\omega_{\mathrm{BD}} > 0$ (left) , $\omega_{\mathrm{BD}} < -5/3$ (right). Diagrams plotted for fixed values ($\omega_{\mathrm{BD}} = 5$ and $\omega_{\mathrm{BD}} = -2$), all phase space diagram in given range are topologically equivalent. The circle at infinity consists of bounces during the evolution of the universe.

What is the value of ω_{BD} ?

- $f(R)$ theories of gravity

$$S = \int \mathrm{d}^4 x \sqrt{-g} f(R) \quad \rightarrow \quad S = \int \mathrm{d}^4 x \Big\{ \phi R - 2V(\phi) \Big\} \qquad \omega_{\mathrm{BD}} = 0$$

- low-energy limit of the bosonic string theory

$$S = \int \mathrm{d}^4 x \sqrt{-g} e^{-2\Phi} \Big\{ R + 4\nabla_\alpha \Phi \nabla^\alpha \Phi - \Lambda \Big\} \quad \rightarrow$$

$$S = \int \mathrm{d}^4 x \sqrt{-g} \Big\{ \phi R + \frac{1}{\phi} \nabla_\alpha \phi \nabla^\alpha \phi - \Lambda \phi \Big\} \qquad \omega_{\mathrm{BD}} = -1$$

An arbitrary potential function

The critical point ($x^* = 0$, $y^* = 1$, $\lambda^* = -2$) corresponds to the de Sitter expansion.
The eigenvalues of the linearization matrix are

$$l_1 = -3(1 + w_m)\,,$$
$$l_{2,3} = -\frac{3}{2}\left(1 \pm \sqrt{\frac{3 + 2\omega_{\mathrm{BD}} + \delta}{3 + 2\omega_{\mathrm{BD}}}}\right)\,,$$

where δ parameter is defined as

$$\delta = \frac{8}{3}\lambda^*\big(1 - \lambda^*(\Gamma(\lambda^*) - 1)\big) = \frac{16}{3}\big(1 - 2\,\Gamma^*\big)\,,$$

and depends on the second derivative of the potential function at the de Sitter state.

Model 1

In the first case, characterized by the purely real eigenvalues, we make the following substitution

$$\frac{\delta}{3+2\omega_{\mathrm{BD}}} = \frac{4}{9}n(n-3)\,.$$

The Hubble function is

$$\left(\frac{H(a)}{H(a_0)}\right)^2 = \Omega_{\Lambda,0} + \Omega_{M,0}\left(\frac{a}{a_0}\right)^{-3} + \Omega_{n,0}\left(\frac{a}{a_0}\right)^{-n} + \Omega_{3n,0}\left(\frac{a}{a_0}\right)^{-3+n}\,,$$

where

$$\Omega_{M,0} = \left(1 - \frac{16}{3\delta}\right)\Omega_{bm,0}\,,$$

and $\Omega_{n,0}$, $\Omega_{3n,0}$ are functions of the initial conditions Δx, Δy, $\Delta\lambda$ and

$$\Omega_{\Lambda,0} = 1 - \Omega_{M,0} - \Omega_{n,0} - \Omega_{3n,0}\,.$$

Model 2

For the second type of behavior in the vicinity of the de Sitter state we make the following substitution

$$\frac{\delta}{3+2\omega_{\text{BD}}} = -\frac{1}{9}(9+4n^2)\,,$$

The Hubble function is

$$\left(\frac{H(a)}{H(a_0)}\right)^2 = \Omega_{\Lambda,0} + \Omega_{M,0}\left(\frac{a}{a_0}\right)^{-3} + \\ + \left(\frac{a}{a_0}\right)^{-3/2}\left(\Omega_{cos,0}\cos\left(n\ln\left(\frac{a}{a_0}\right)\right) + \Omega_{sin,0}\sin\left(n\ln\left(\frac{a}{a_0}\right)\right)\right),$$

where

$$\Omega_{M,0} = \left(1-\frac{16}{3\delta}\right)\Omega_{bm,0}\,,$$

and $\Omega_{cos,0}$, $\Omega_{sin,0}$ are functions of the initial conditions Δx, Δy, $\Delta\lambda$ and

$$\Omega_{\Lambda,0} = 1 - \Omega_{M,0} - \Omega_{cos,0}$$

ΛCDM nested within

Carefully choosing the initial conditions for the linearized solutions

$$\Delta x = \frac{4}{\delta}\Omega_{bm,i}\,, \qquad \Delta\lambda = -\frac{1}{2}\Omega_{bm,i}\,,$$

where up to linear terms in initial conditions $\Omega_{bm,i} = \Delta x - 2\Delta y$, we have $\Omega_{n,0} = \Omega_{3n,0} = 0$ and $\Omega_{cos,0} = \Omega_{sin,0} = 0$ and the resulting form of the Hubble function is

$$\left(\frac{H(a)}{H(a_0)}\right)^2 \approx 1 - \Omega_{M,0} + \Omega_{M,0}\left(\frac{a}{a_0}\right)^{-3}\,,$$

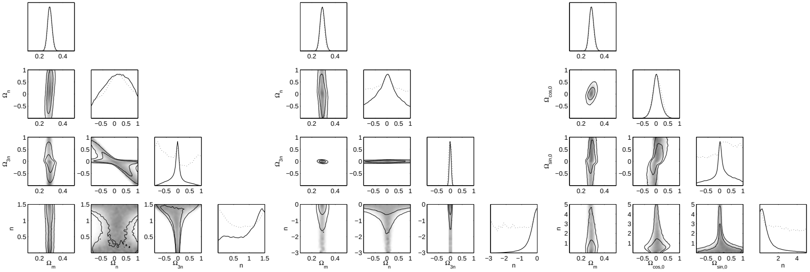
where

$$\Omega_{M,0} = \left(1 - \frac{16}{3\delta}\right)\Omega_{bm,0}\,.$$

This Hubble function describes the ΛCDM model with direct interpretation of the second term in the brackets as proportional to density parameter of the dark matter in the model

$$\Omega_{dm,0} = -\frac{16}{3\delta}\Omega_{bm,0}\,.$$

Confidence levels

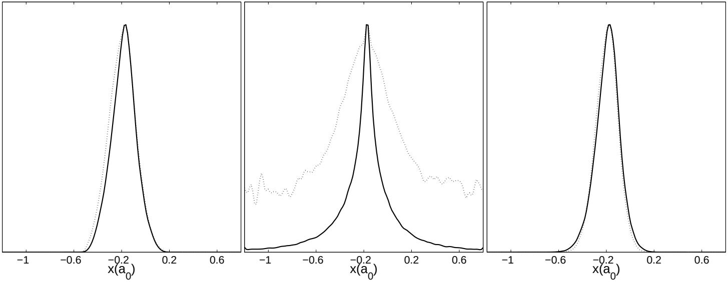


Derived quantities

$$x(a_0) = \frac{\dot{\phi}}{H\phi}\Big|_0 = -\frac{\dot{G}}{HG}\Big|_0$$

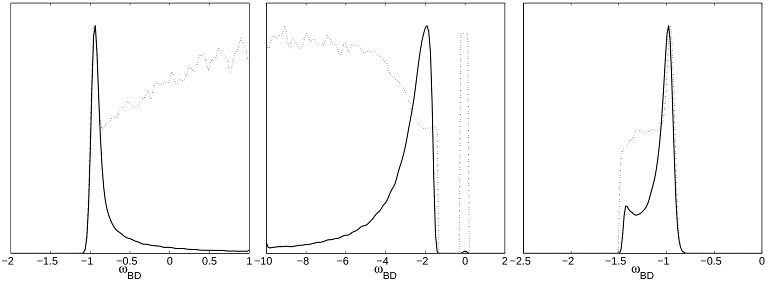
$$x(a_0) = \frac{3}{4}\left(\Omega_{bm,0} - \Omega_{M,0}\right) - \frac{n}{n+1}\Omega_{n,0} - \frac{n-3}{n-4}\Omega_{3n,0}\,,$$

$$x(a_0) = \frac{3}{4}\left(\Omega_{bm,0} - \Omega_{M,0}\right) + \frac{2}{4n^2+25}\left(5\,\Omega_{cos,0} + 2n\,\Omega_{sin,0}\right) - \Omega_{cos,0}\,.$$



Derived quantities

$$\omega_{\rm BD} = -\frac{3}{2} + \frac{6}{n(n-3)}\frac{\Omega_{bm,0}}{\Omega_{bm,0}-\Omega_{M,0}}\,, \qquad \omega_{\rm BD} = -\frac{3}{2} - \frac{24}{9+4n^2}\frac{\Omega_{bm,0}}{\Omega_{bm,0}-\Omega_{M,0}}\,.$$



Conclusions

- We have translated the geometrical approach to the dark energy and dark matter problems in to the substantial approach which in order can be used to test and select the cosmological models by astronomical data.
- In the vicinity of the critical point corresponding to the deSitter state, for carefully chosen initial conditions, we have obtained the corresponding form of the Hubble function which is indistinguishable from the standard cosmological ΛCDM model.
- We shown that in the models with the de Sitter state in the form of a stable node or a sink type critical point vales of the $\omega_{\rm BD}$ parameter close to the value suggested by the low-energy limit of the bosonic string theory are favored.