Random fragmentation & coalescence

Bernhard Mehlig, Department of Physics, Göteborg University, Sweden

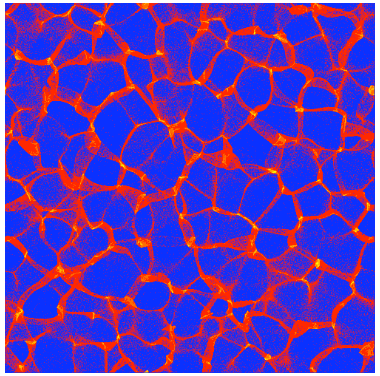
a) Clustering in turbulent flows

b) Caustics in turbulent flows

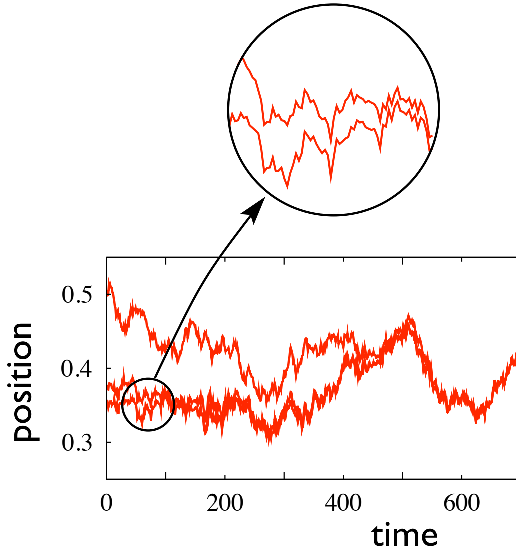
c) Genetics (ancestral recombination graphs)

d) Conduction in disordered solids

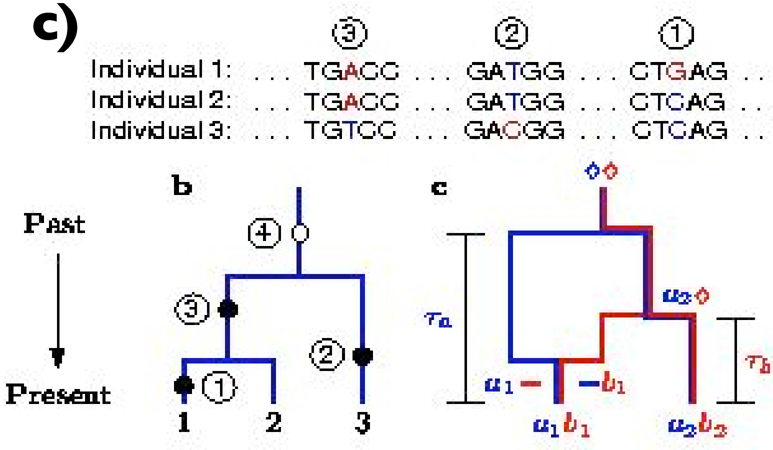
b)



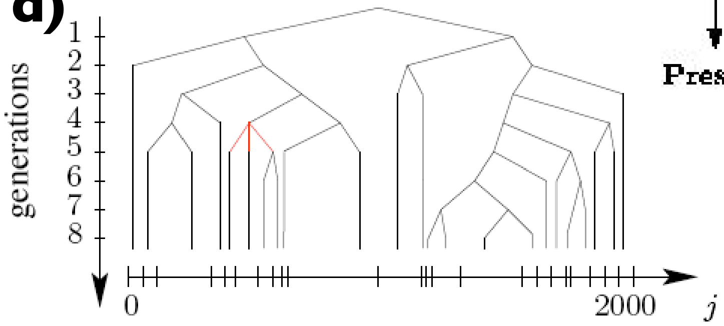
a)



c)



d)



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a) Clustering in turbulent flows

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Duncan, Mehlig, Östlund & Wilkinson, Phys. Rev. Lett. 95 (2005) 165503
Arvedson, Wilkinson, Mehlig & Nakamura, Phys. Rev. Lett. 96 (2006) 030601

b) Caustics and collisions in turbulent flows

Wilkinson & Mehlig, Europhys. Lett. 71 (2005) 186
Wilkinson, Mehlig & Bezuglyy, Phys. Rev. Lett. 97 (2006) 048501
Andersson, Gustavsson, Mehlig & Wilkinson, Europhys. Lett. 80 (2007) 69001
Gustavsson, Mehlig & Wilkinson, New Journal of Physics (2008), in press

c) Genetics (ancestral recombination graphs)

A. Eriksson & B. Mehlig, Genetics 169 (2005) 1175
A. Eriksson, P. Fernström, B. Mehlig & S. Sagitov, Genetics in press (2008)

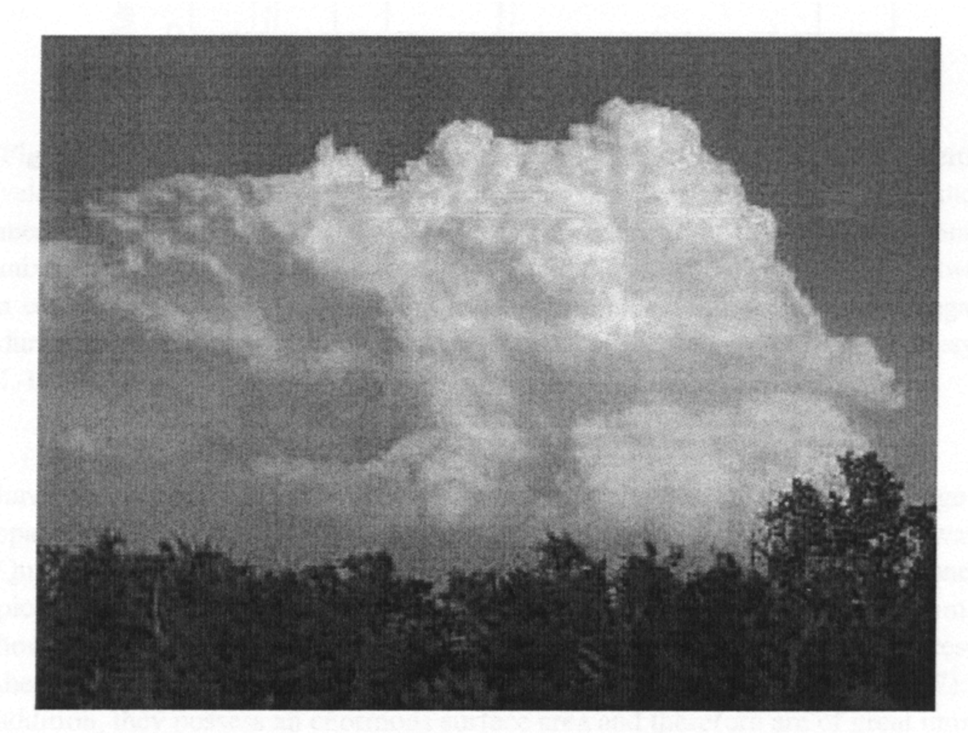
d) Conduction in disordered solids

Mehlig & Wilkinson, Prog. Theor. Phys. Suppl. 166 (2007) 136
Wilkinson, Mehlig & Bezuglyy, Phil. Mag. Lett. 88 (2008) 327

a) Clustering in random flows

Rain droplets in a turbulent cloud: drag force given by Stokes law

$$\ddot{\boldsymbol{r}} = \gamma\big(\boldsymbol{u}(\boldsymbol{r},t) - \dot{\boldsymbol{r}}\big)$$



Shaw, Annu. Rev. Fluid. Mech. 35 (2003) 183

$\boldsymbol{u}(\boldsymbol{r},t)$ turbulent air flow (strength u_0),

$\gamma = 9\rho_{\rm g}\nu/(2\rho_{\rm p}a^2)$ Stokes constant,

$\rho_{\rm g}$ and $\rho_{\rm p}$ densities of air and water,

ν kinematic viscosity,

and a particle size.

Model

Assumptions

1. spherical particles of mass m and radius a move indepedently (until they come into contact),
2. particles do not affect the flow,
3. drag force given by Stokes law $\ddot{\boldsymbol{r}} = \gamma\big(\boldsymbol{u}(\boldsymbol{r},t) - \dot{\boldsymbol{r}}\big)$,
4. velocity field $\boldsymbol{u}(\boldsymbol{r},t)$ isotropic, homogenous, stationary, and incompressible Gaussian random function.

Dimensionless parameters

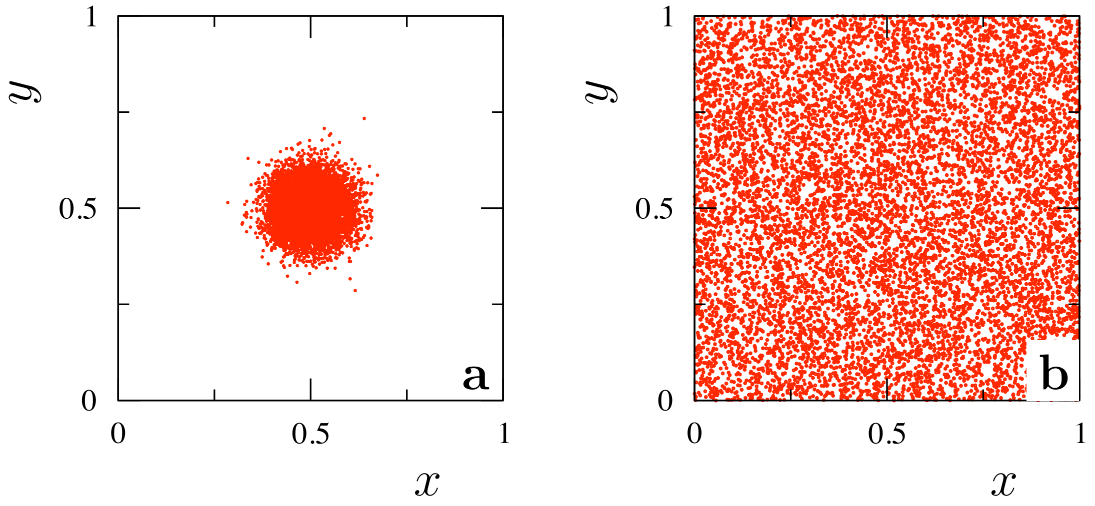
$$\mathrm{St} = \frac{1}{\gamma\tau}\,, \quad \mathrm{Ku} = \frac{u_0\tau}{\ell}\,, \quad n\ell^d\,, \quad a/\ell$$

St Stokes number, Ku Kubo number, n particle density,

ℓ and τ Kolmogorov length and time (correlation length and time).

Mixing by random stirring

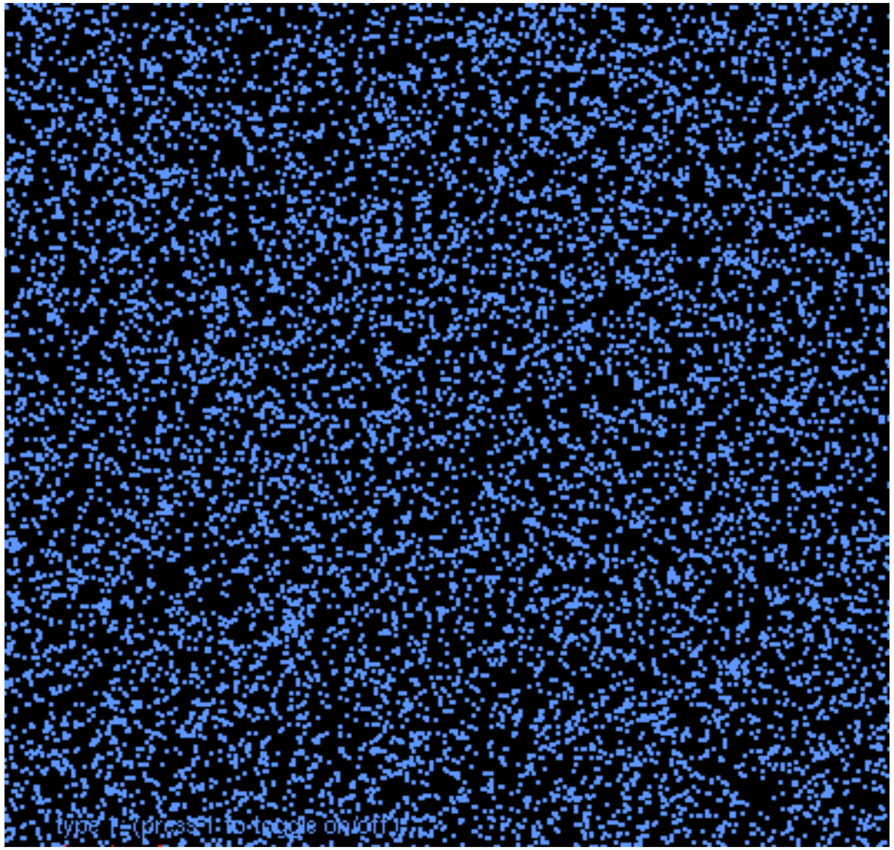
Computer simulation of 10^4 particles (red) in two-dimensional random flow (periodic boundary conditions in space)



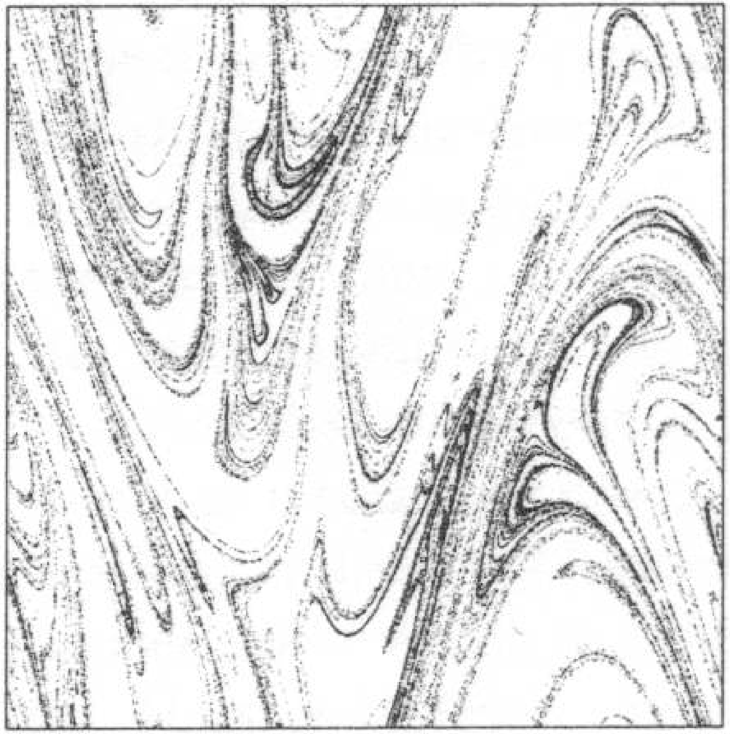
a initial distribution, **b** particle positions after random stirring.

\`Unmixing´

Computer simulation of 10^4 particles (blue) in two-dimensional smooth random incompressible flow $\boldsymbol{u}(\boldsymbol{r},t)$ (periodic boundary conditions in space)

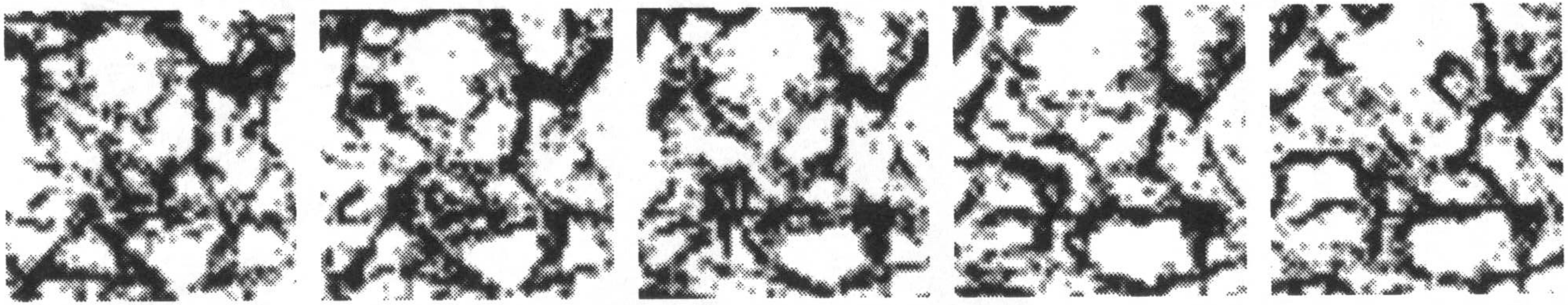


Inertial particles in turbulent flow (statistical model, two-dimensional)



Sigurgeisson & Stuart, Phys. Fluids 14 (2002) 4352
Bec, Phys. Fluids 15 (2003) L81

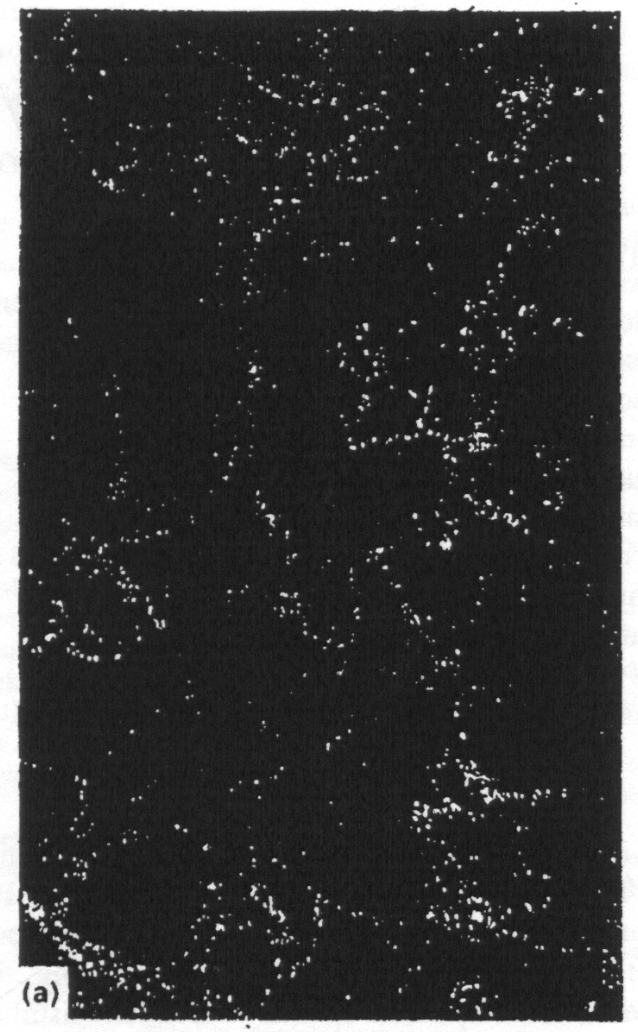
Particles falling under gravity in turbulent flow



Wang & Maxey, J. Fluid. Mech. 256 (1993) 27

Lycopodium particles in turbulent channel flow

Fessler, Kulick & Eaton, Phys. Fluids 6 (1994) 3742



Lyapunov exponents

Exponents $\lambda_1 > \lambda_2 > \lambda_3$ describe rate of contraction or expansion of small length element δr_t , area element $\delta \mathcal{A}_t$, and volume element $\delta \mathcal{V}_t$

$$
\begin{aligned}
\lambda_1 &= \lim_{t\to\infty} t^{-1} \log_{\mathrm{e}}(\delta r_t) \\
\lambda_1 + \lambda_2 &= \lim_{t\to\infty} t^{-1} \log_{\mathrm{e}}(\delta \mathcal{A}_t) \\
\lambda_1 + \lambda_2 + \lambda_3 &= \lim_{t\to\infty} t^{-1} \log_{\mathrm{e}}(\delta \mathcal{V}_t) \ .
\end{aligned}
$$

J. Sommerer & E. Ott, Science 259 (1993) 351

One-dimensional model

Determine exponent λ_1 for one-dimensional model $\ddot{x} = \gamma(u(x,t) - \dot{x})$.

Obtain series expansion for λ_1

$$\lambda_1/\gamma = -\sum_{l=1}^{\infty} c_l \mathcal{I}^l$$

where

$$\mathcal{I} = \frac{1}{2\gamma}\int_{-\infty}^{\infty} \mathrm{d}t\, \langle \partial_x u(x_t,t)\partial_x u(x_0,0)\rangle \propto \mathrm{Ku}^2\mathrm{St}\,.$$

l	c_l
1	1
2	5
3	60
4	1105
5	27120
6	828250
7	30220800
8	1282031525
9	61999046400
10	3366961243750

Mehlig & Wilkinson, Phys. Rev. Lett. 92 (2004) 250602

Corresponding results in three spatial dimensions:

Duncan, Mehlig, Östlund & Wilkinson, Phys. Rev. Lett. 95 (2005) 165503
Wilkinson, Mehlig, Östlund & Duncan, Phys. Fluids 19 (2007) 113303

As $\mathcal{I} \to 0$ obtain known results for advective limit

Falkovich, Gawedzki & Vergassola, Rev. Mod. Phys. 73 (2001) 913

One-dimensional model

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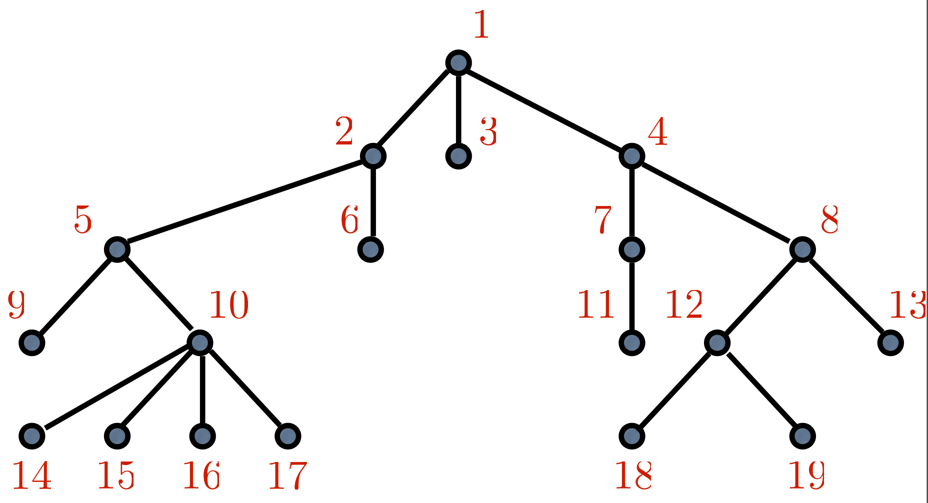
Aldous, *Brownian excursions, critical random graphs, and the multiplicative coalescent* (1996)
Spencer, *Enumerating Graphs and Brownian Motion*, Comm. Pure Appl. Math. 1 (1997) 0291
Flajolet and P. Poblete &Viola, *On the analysis of linear probing hashing*,
Algorithmica 22 (1998) 490
Janson, Random Structures & Algorithms *22 (2003) 337*

Wiener index

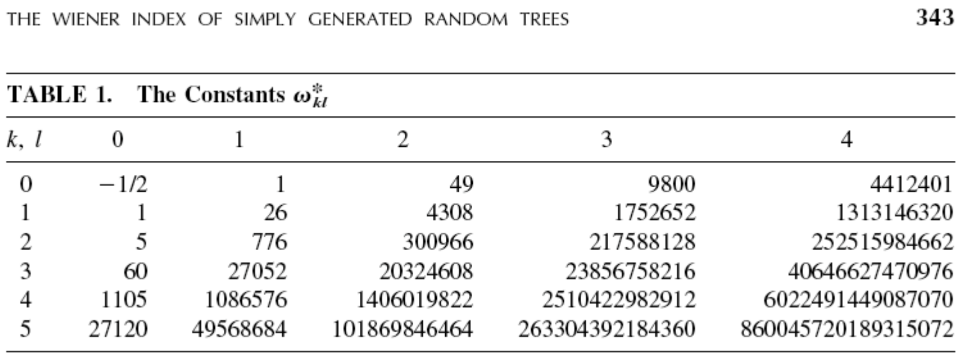
Wiener, J. Am. Chem. Soc. 69 (1947) 17

Wiener index W of a connected graph T_n of order n : sum of all distances between all pairs of vertices



$$W = \frac{1}{2}\sum_{i,j}^{n} d_{ij}\ .$$

Distribution of Wiener index for simply generated random trees. Write $W(T_n) = n^{3/2}\xi + n^{5/2}\eta$ with random variables ξ and η. Find $\langle \xi^k \eta^l \rangle = \dfrac{k!\,l!\,\sqrt{\pi}}{2^{(5k+7l-4)/2} + \Gamma\big((3k+5l-1)/2\big)}\,\omega^*_{kl}\ .$



THE WIENER INDEX OF SIMPLY GENERATED RANDOM TREES 343

TABLE 1. The Constants ω^*_{kl}

$k,\ l$	0	1	2	3	4
0	−1/2	1	49	9800	4412401
1	1	26	4308	1752652	1313146320
2	5	776	300966	217588128	252515984662
3	60	27052	20324608	23856758216	40646627470976
4	1105	1086576	1406019822	2510422982912	6022491449087070
5	27120	49568684	101869846464	263304392184360	860045720189315072

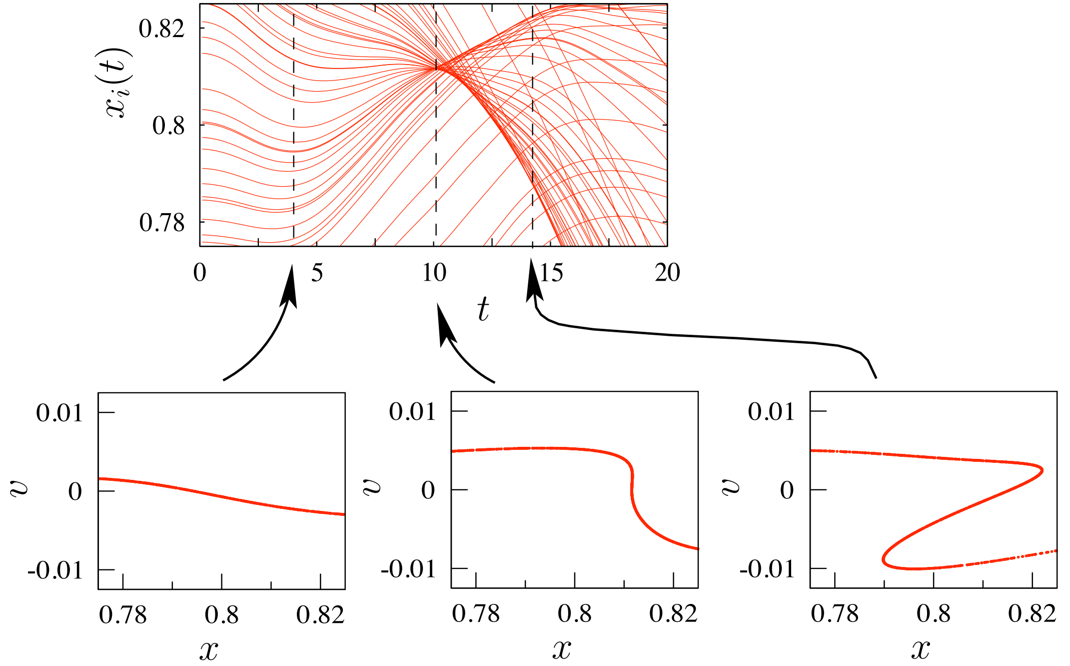
Janson, Random Structures & Algorithms *22 (2003) 337*

b) Caustics

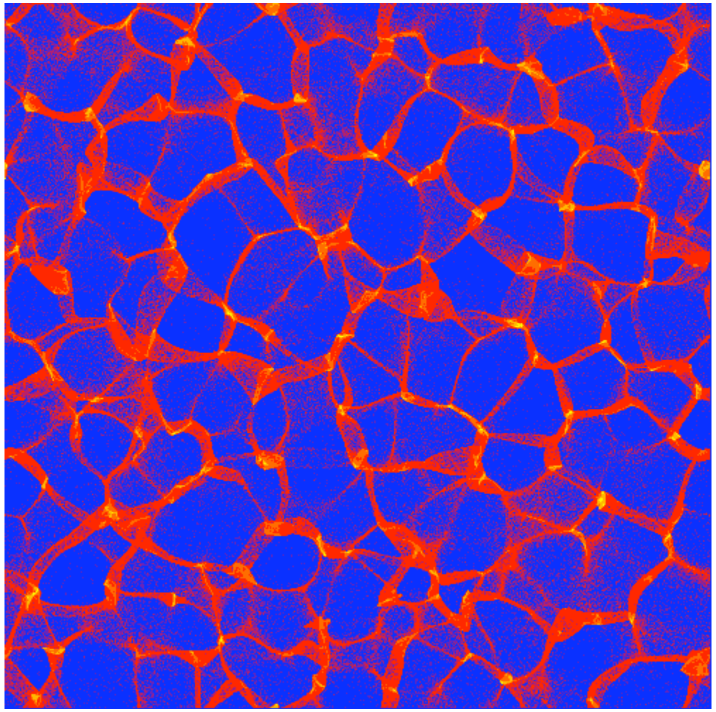
Falkovich, Fouxon & Stepanov, Nature 419 (2002)151
Wilkinson & Mehlig, Phys. Rev. E 68 (2003) 04010

Infinitesimal volume element $\delta\mathcal{V}_t$ contracts or expands on average. But nothing prevents it from collapsing to zero for an instant of time: singularity in particle density ρ.

Consider one spatial dimension: $\delta x_t = 0$ corresponds to singularity in $X = \delta v/\delta x$.



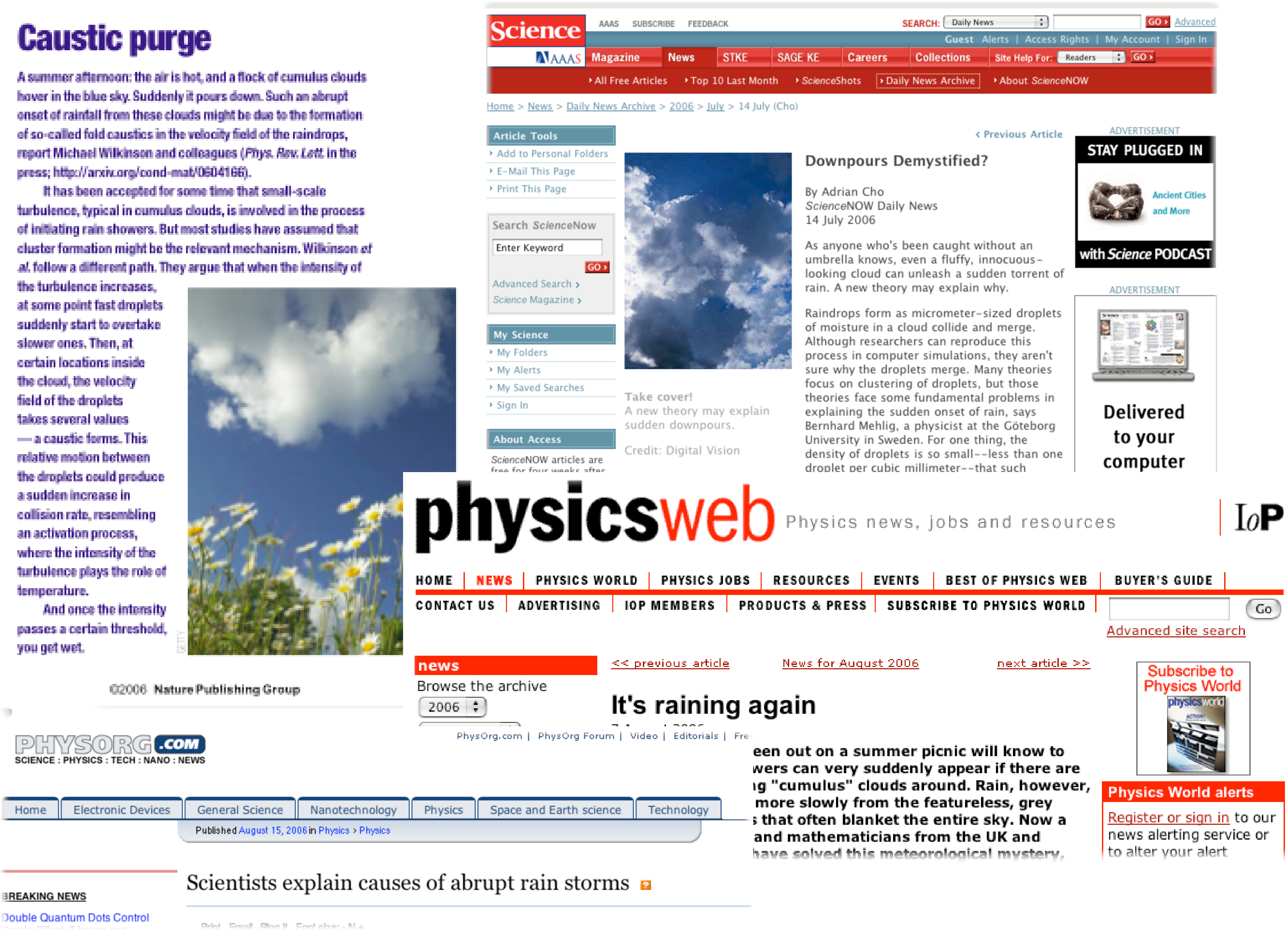
Caustics in two dimensions



Density of particles suspended in a random flow (compressible, $\boldsymbol{u} = -\boldsymbol{\nabla}\phi$)

Caustics of sun light in water

http://www.physics.utoronto.ca/~peet/



Caustic purge

A summer afternoon: the air is hot, and a flock of cumulus clouds hover in the blue sky. Suddenly it pours down. Such an abrupt onset of rainfall from these clouds might be due to the formation of so-called fold caustics in the velocity field of the raindrops, report Michael Wilkinson and colleagues (*Phys. Rev. Lett.* in the press; http://arxiv.org/cond-mat/0604166).

It has been accepted for some time that small-scale turbulence, typical in cumulus clouds, is involved in the process of initiating rain showers. But most studies have assumed that cluster formation might be the relevant mechanism. Wilkinson *et al.* follow a different path. They argue that when the intensity of the turbulence increases, at some point fast droplets suddenly start to overtake slower ones. Then, at certain locations inside the cloud, the velocity field of the droplets takes several values — a caustic forms. This relative motion between the droplets could produce a sudden increase in collision rate, resembling an activation process, where the intensity of the turbulence plays the role of temperature.

And once the intensity passes a certain threshold, you get wet.

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Take cover!
A new theory may explain sudden downpours.

Credit: Digital Vision

Downpours Demystified?

By Adrian Cho
*Science*NOW Daily News
14 July 2006

As anyone who's been caught without an umbrella knows, even a fluffy, innocuous-looking cloud can unleash a sudden torrent of rain. A new theory may explain why.

Raindrops form as micrometer-sized droplets of moisture in a cloud collide and merge. Although researchers can reproduce this process in computer simulations, they aren't sure why the droplets merge. Many theories focus on clustering of droplets, but those theories face some fundamental problems in explaining the sudden onset of rain, says Bernhard Mehlig, a physicist at the Göteborg University in Sweden. For one thing, the density of droplets is so small--less than one droplet per cubic millimeter--that such

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It's raining again

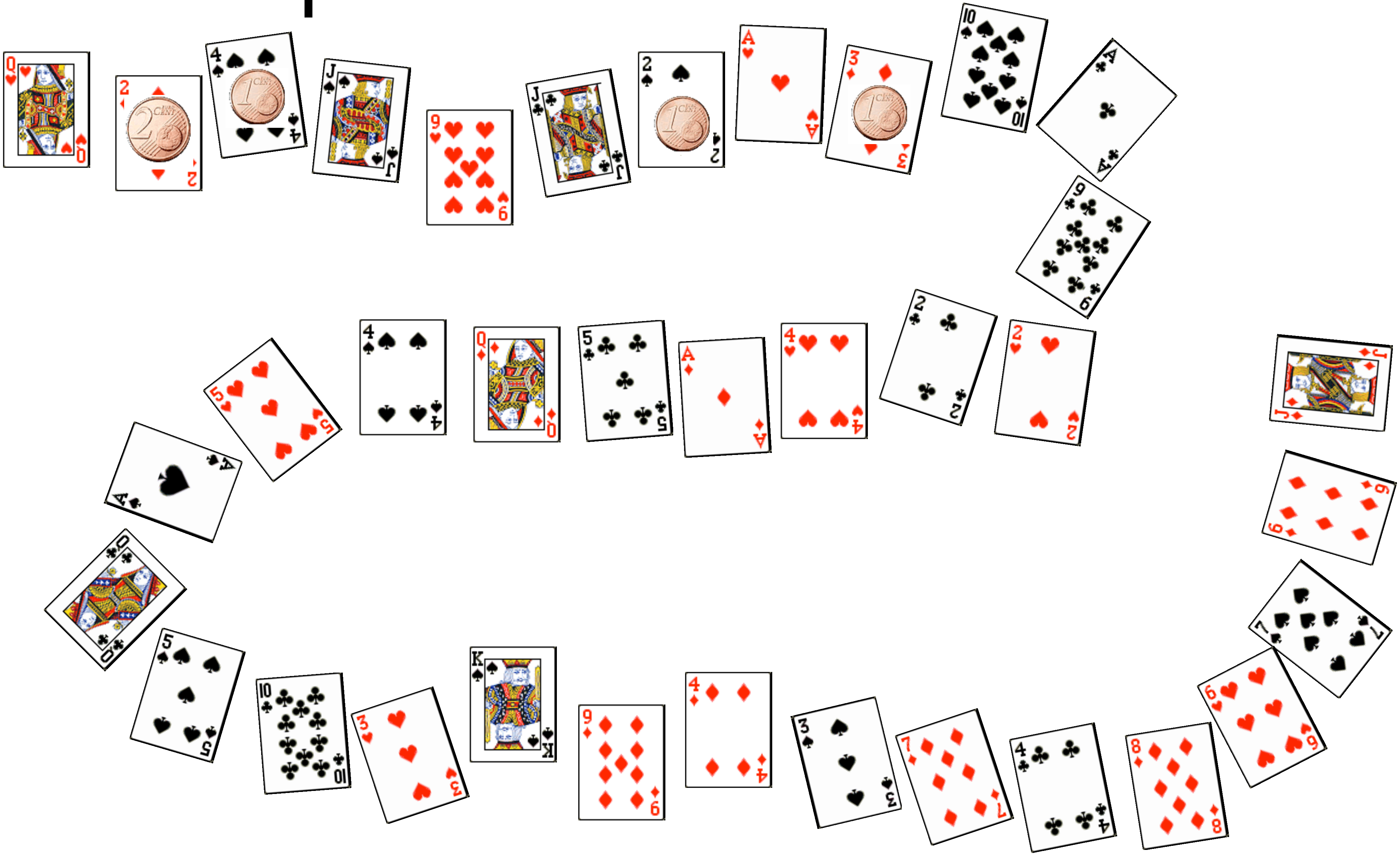
een out on a summer picnic will know to
wers can very suddenly appear if there are
ıg "cumulus" clouds around. Rain, however,
more slowly from the featureless, grey
; that often blanket the entire sky. Now a
and mathematicians from the UK and
have solved this meteorological mystery.

Published August 15, 2006 in Physics > Physics

Scientists explain causes of abrupt rain storms

An example

Gardner, Sci. Amer. 238 (1978) 19



c) Ancestral recombination graphs

Patterns of genetic variation in contiguous DNA segment close to a locus $\mathcal{B}$ recently subject to selective sweep.

Individual

ACTTTCGGAA
ACTTTCGCAA
ACTGTCGGAA
ACTGTCGCAA

position along chromosome

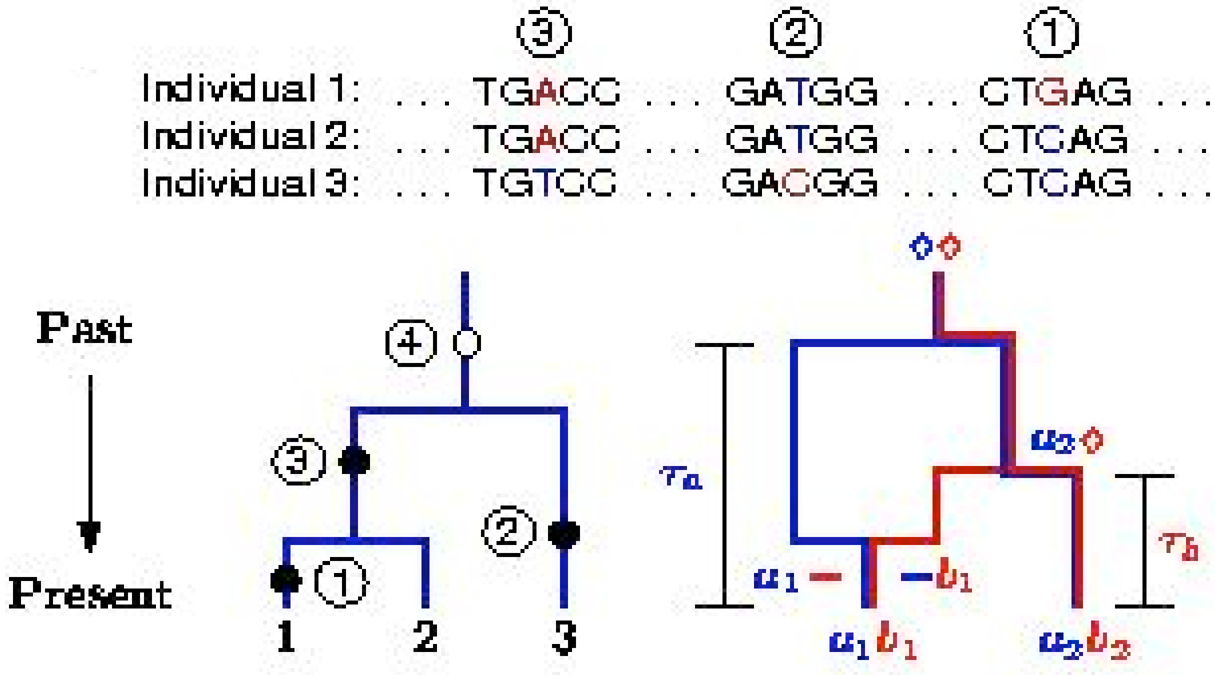
To the right: shows patterns of mutation (`genetic mosaic´) for sample from neutral coalescent model.

How is this pattern affected by recent sweep at locus $\mathcal{B}$ close to segment?

Durrett & Schweinsberg (2004);

The coalescent process

Gene histories determine degree of association between patterns of genetic variation at different loci



Nordborg (2001) in: *Handbook of Statistical Genetics*, ch. 7

d) Variable-range hopping

Temperature dependence of DC conductivity in disordered solid

Mott, J. Non-crystalline Solids 1 (1968) 1

$$\sigma(T) \sim \exp\left[-\left(\frac{T_0}{T}\right)^{1/(d+1)}\right]$$

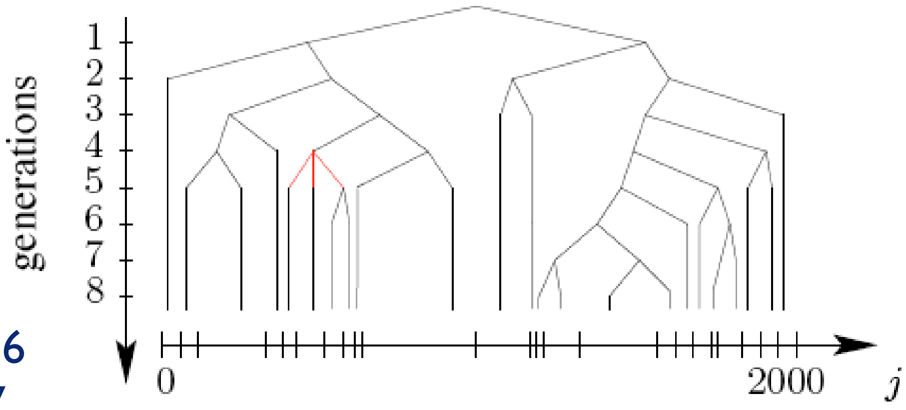
Competition between near and far electron hops (variable range).

Simplest model analogous to random resistor network (percolation).

Ambegoakar, Langer & Halperin, Phys. Rev. B 4 (1971) 2612

Long-standing problem: no exact solution.

Characterise sequence of hops as a hierarchical stick-breaking process.



Mehlig & Wilkinson, Prog. Theor. Phys. Suppl. 166 (2007) 136
Wilkinson, Mehlig & Bezuglyy, Phil. Mag. Lett. 88 (2008) 327

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A. Eriksson & B. Mehlig, Genetics 169 (2005) 1175
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