Locked Constraint Satisfaction Problems

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Our Question about CSPs: Where the really hard problems are?

(Cheeseman, Kanefsky, Taylor'91; Mitchell, Selman, Levesque'92)

- **Answer n. 1: Near to the satisfiability threshold.**
  (Cheeseman, Kanefsky, Taylor'91; Mitchell, Selman, Levesque'92)

- **More precisely (Answer n. 2): When the backbone appears discontinuously at the threshold.**
  (Monasson, Zecchina, Kirkpatrick, Selman, Troyansky'99)

- **Answer n. 3: In the clustered phase.**
  (Mézard, Parisi, Zecchina'02)
  The consequences of clustering are deep, but it itself does not imply hardness of finding a solution (rather hardness of sampling).

Conjecture (Answer n. 4):

When all the clusters are frozen the CSP is hard.

(Zdeborová, Krzakala'07)

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Description of frozen variables in K-SAT $\neq$ 

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Description of frozen variables in K-SAT $\neq$

You saw the presentations of
F. Ricci-Tersenghi and F. Krzakala

Task:

Challenge and understand the freezing.

My strategy:

Create a problem where frozen variables are “a piece of cake”.

Outcome:

The locked constraint satisfaction problems

(Zdeborová, Mézard’08)

- **Every cluster contains a single configuration.**
- **Very easy description of the space of solutions.**
- **But challenging from the algorithmic point of view.**

Pre-definition:

- The **occupation constraint satisfaction problems** *(Mora, PhD thesis'07)*
  - binary variables (1-occupied, 0-empty)
  - every constraint has K variables
  - given by a $K+1$ component binary vector A
  - the constraint is satisfied if and only if the number of occupied variables around it r is such that $A[r]=1$.

- **Examples**
  - 4-XOR-SAT without negations: A=(01010) or A=(10101)
  - NAE-SAT without negations (bi-coloring): A=(011...110)
  - positive 1-in-3 SAT (exact cover): A=(0100)

Definition of the locked CSPs

- **Def.:** A **constraint is locked** if and only if there are not two satisfying assignments at distance 1.

- **Def.:** A **constraint satisfaction problem is locked** if and only if all its constraints are locked and there are no variables of degree 1 (leaves).

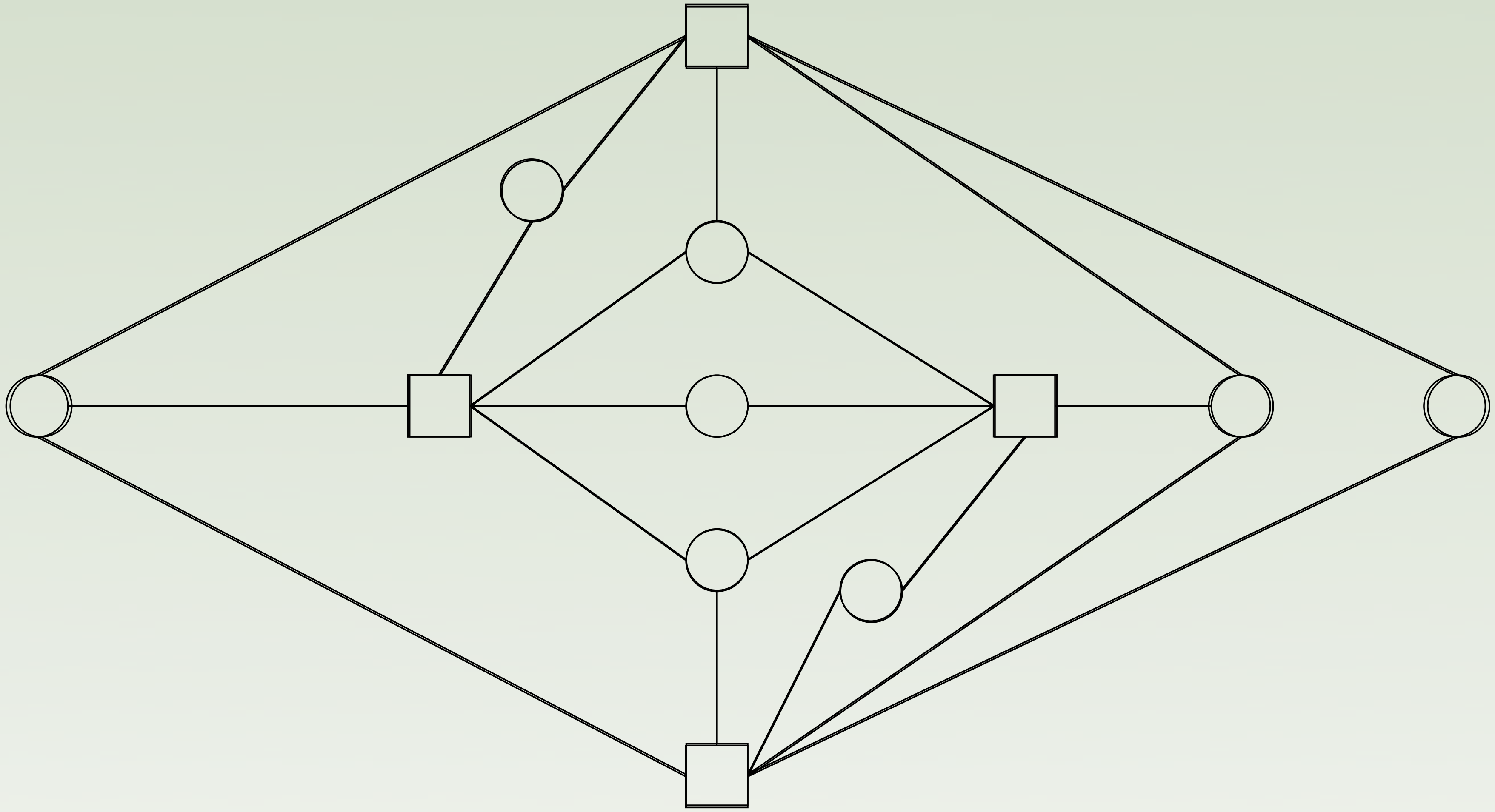
- The occupation problems are locked if and only if

  $A[i]*A[i+1]=0$ for all $i=0...K-1$.

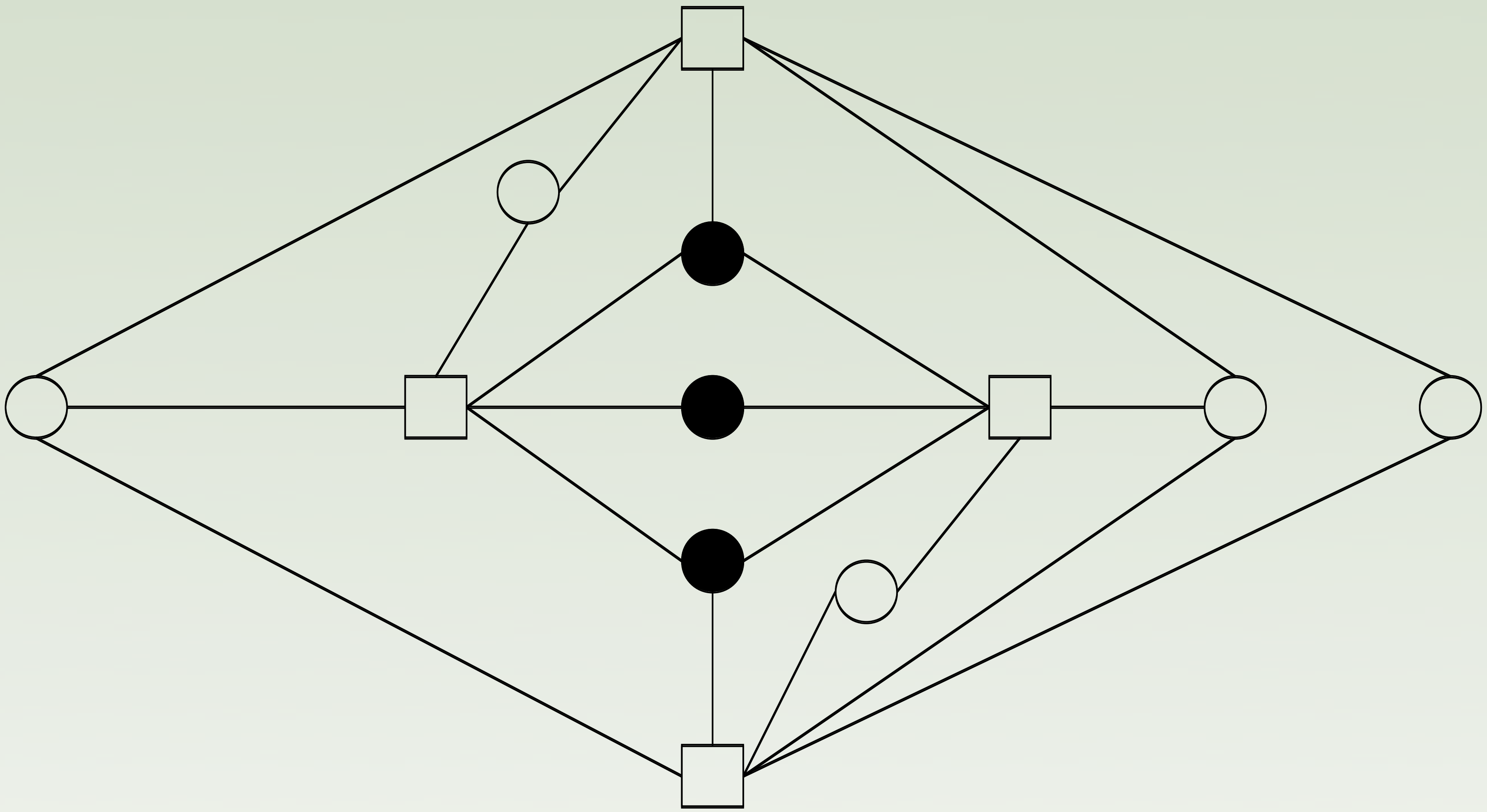
  and no leaves ....

NP-complete: For K>2 and except XOR-SAT.

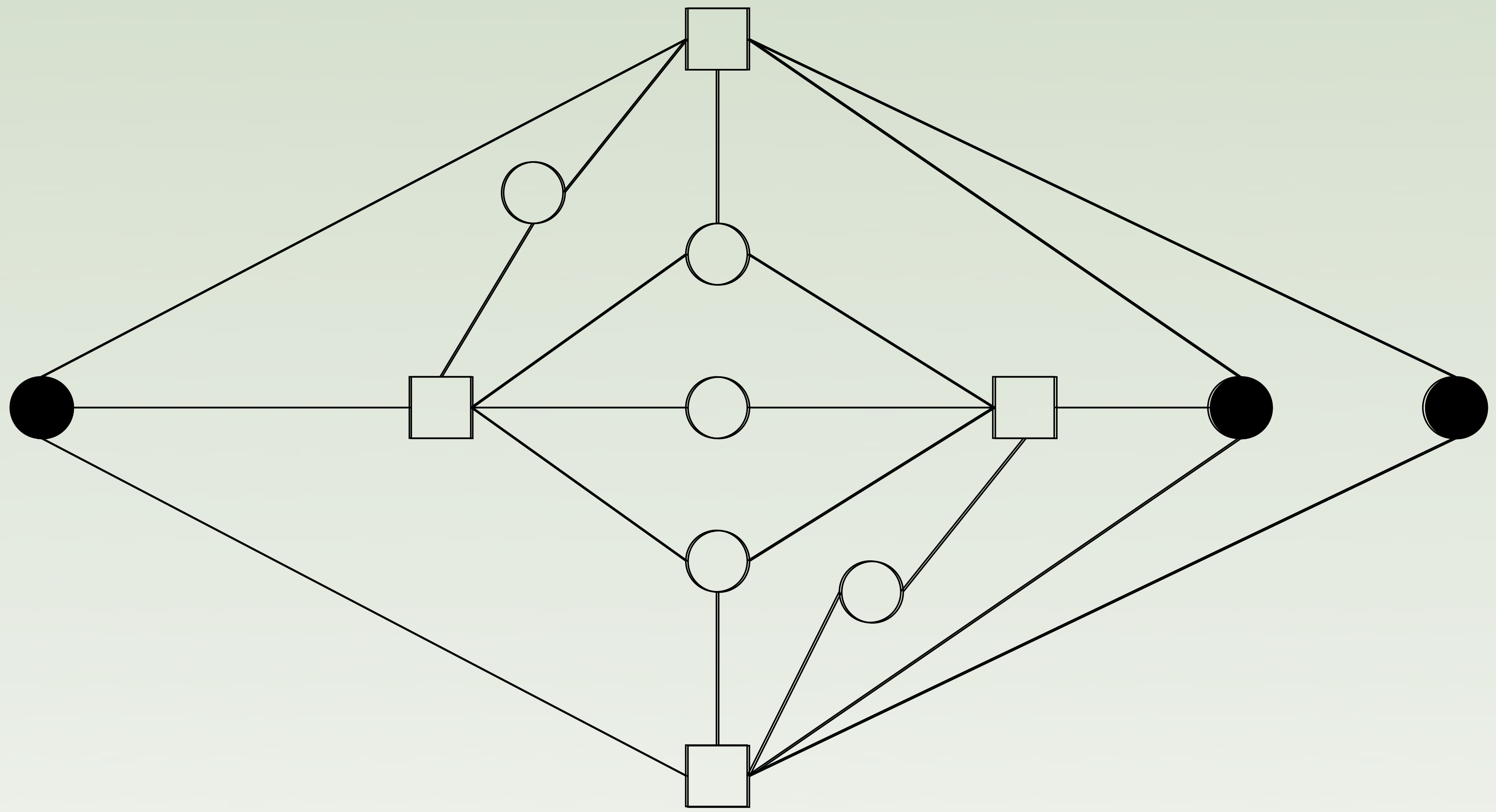
Example: A=(010100), 1-or-3-in-5 SAT



Example: A=(010100), 1-or-3-in-5 SAT



Example: A=(010100), 1-or-3-in-5 SAT



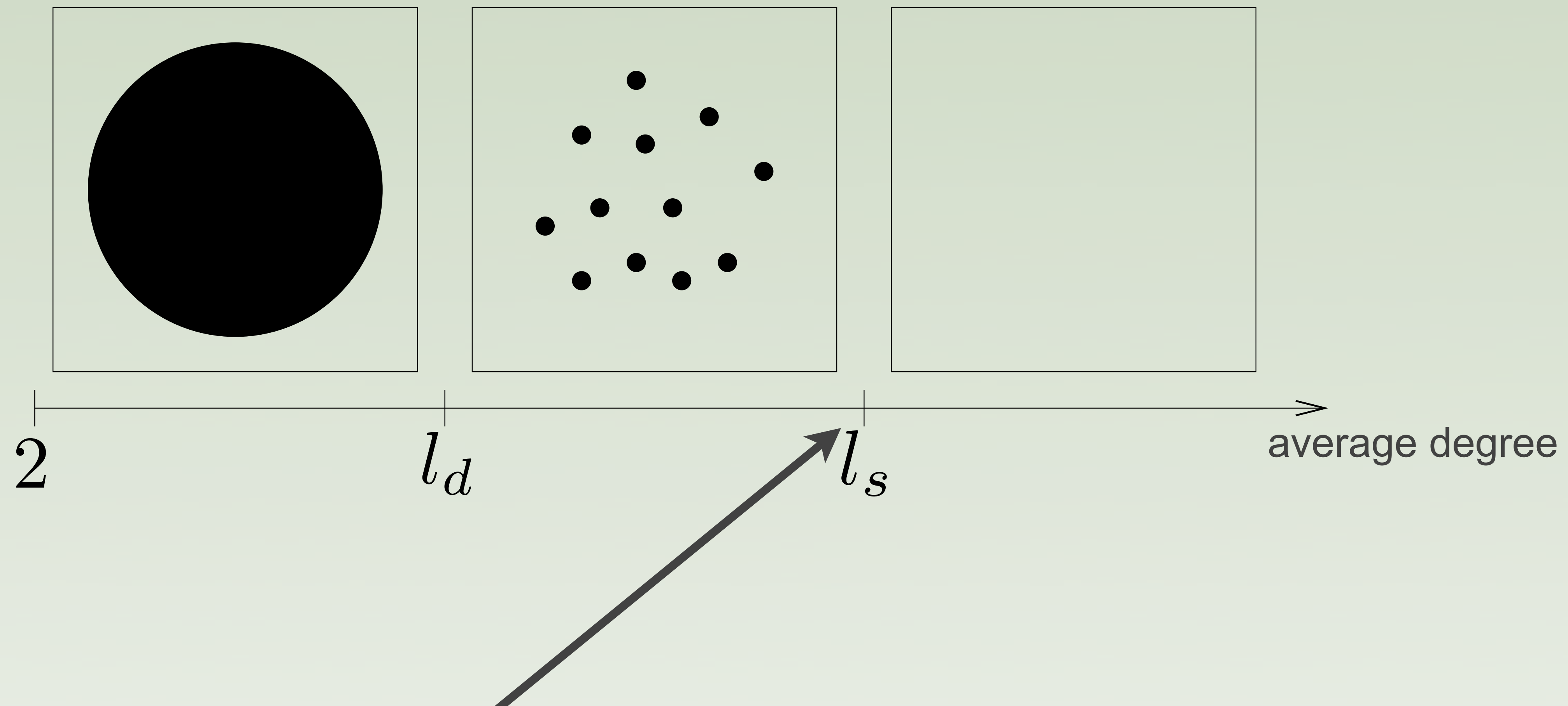
Basic properties:

- Given a solution at least a closed loop have to be flipped to find another solution. Distance at least log(N) between solutions.
  - Could serve as an alternative definition for the locked problems.

- Belief propagation (randomly initialized) asymptotically exact on random instances. Gives correct marginals, entropy, and satisfiability threshold.

- Every solution is a fixed point of belief propagation. Thus BP is a fixed point of survey propagation (jokers have no weight).

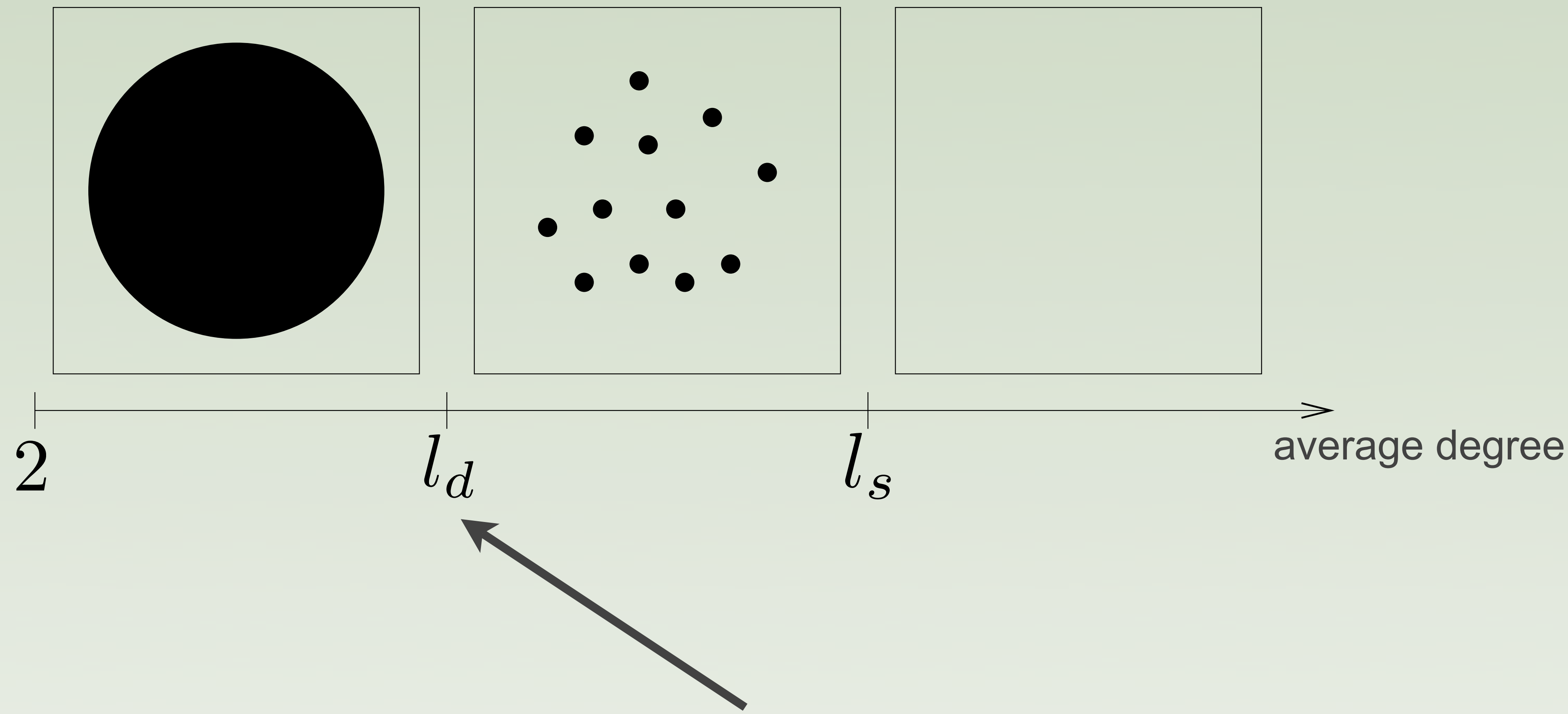
The phase diagram



Satisfiability threshold

- **Belief propagation** (replica symmetric) entropy **exact**.
- Coincides with the condensation transition.

The phase diagram



Dynamical (clustering) transition

- Iterative stability of the BP-like fixed point of the SP equations.
- Small noise reconstruction on trees.
- Small temperature erases the separation or not.
- Gap in the weight enumerator.

Sometimes thresholds are really simple

- **For A symmetric, A[i]=A[K-i], and the symmetry not broken, e.g. (01010), (00100), (0001000), (0010100), but not (010010).**

- **Entropy - rigorous via the 1st and 2nd moment:**

$$s(\bar{l}) = \log 2 + \frac{\bar{l}}{K} \log \left[2^{-K} \sum_{r=0}^{K} \delta(A_r - 1) \binom{K}{r} \right]$$

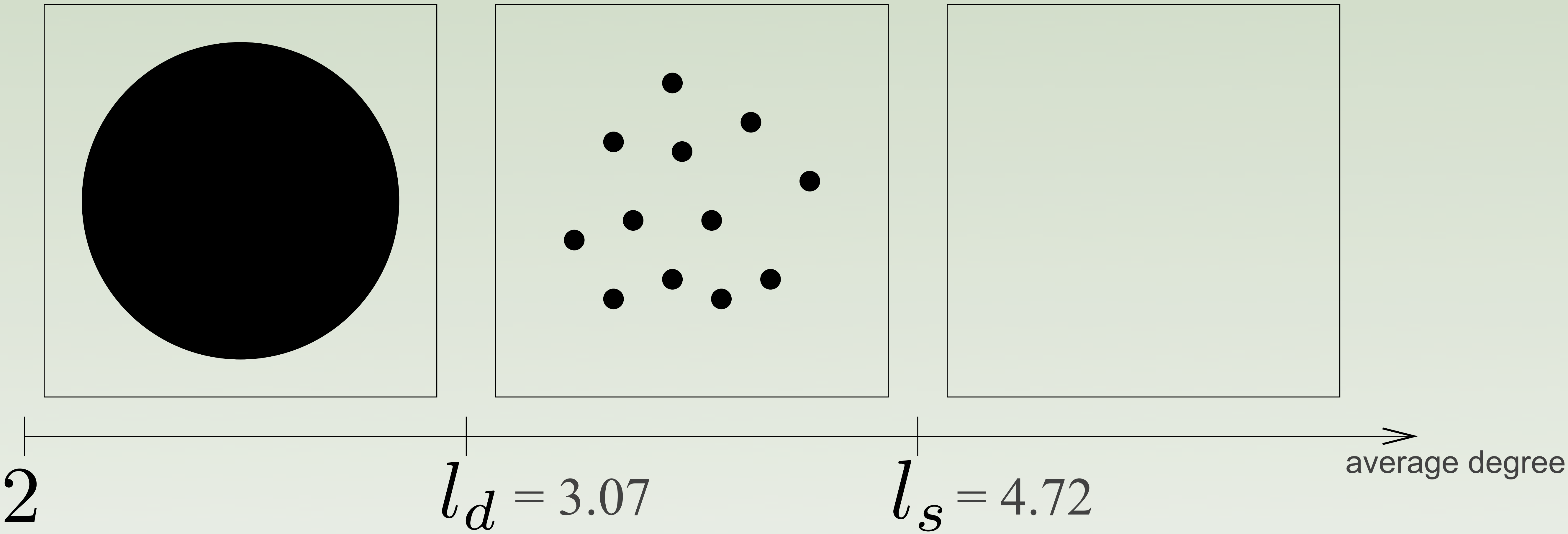
- **Clustering (dynamical) transition:**

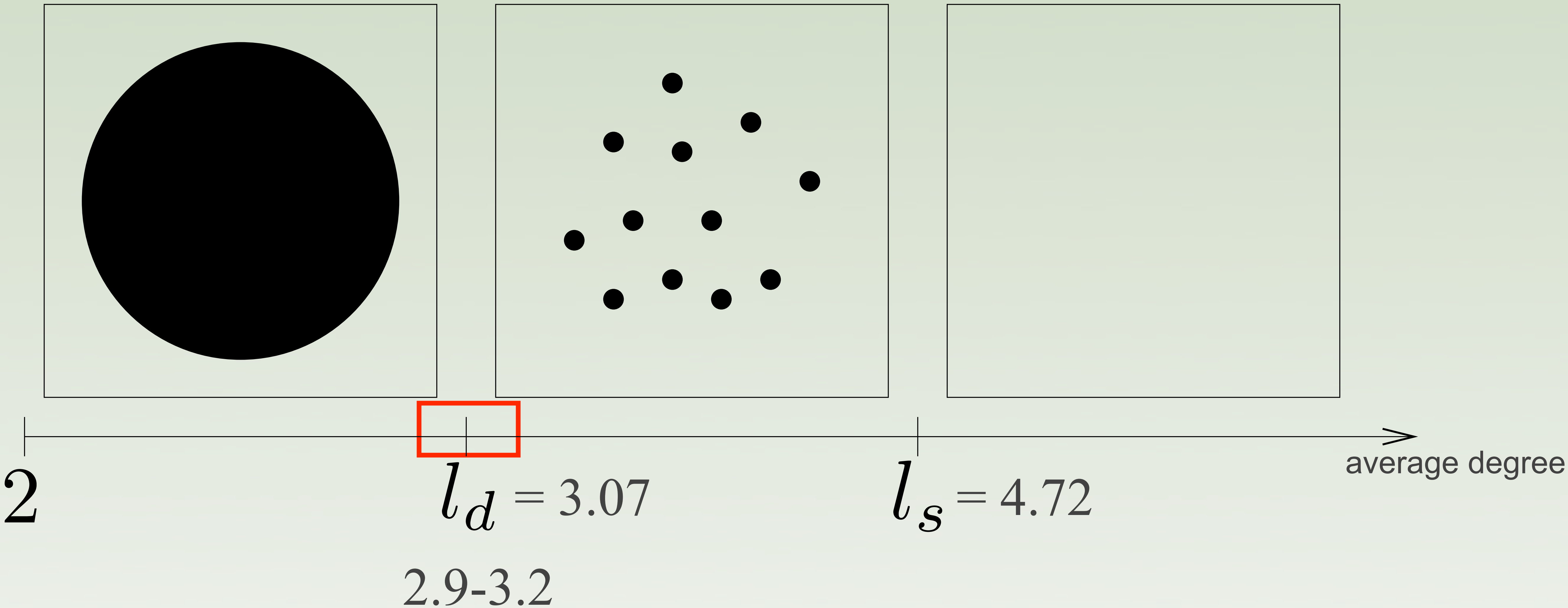
$$\frac{e^{c_d} - 1}{c_d} = K - 1 - \frac{\sum_{r=0}^{K-2} \delta(A_{r+1} - 1)\, \delta(A_{r-1})\, \delta(A_r)\, \binom{K-1}{r}}{\sum_{r=0}^{K-2} \delta(A_{r+1} - 1)\, \binom{K-1}{r}}$$

And algorithms?

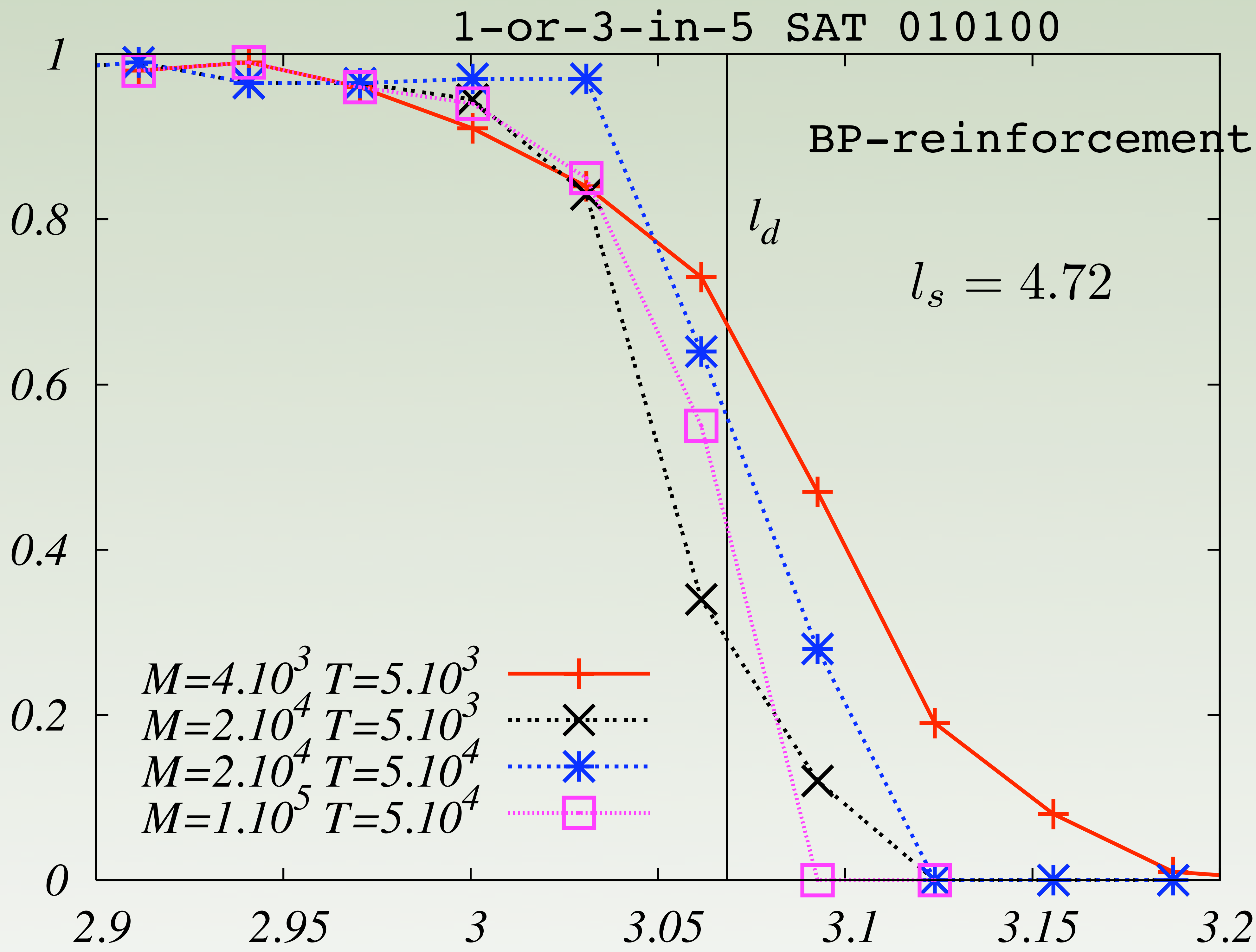
- The best algorithms I know for random 3-SAT
  - Survey Propagation decimation (4.25... *Mézard, Zecchina'02*)
  - Survey Propagation reinforcement (4.25... *Chavas, Furtlehner, Mézard, Zecchina'05*)
  - Stochastic locals search - focused SLS, ASAT (4.21... *Seitz, Alava, Orponen'05, Ardelius, Aurell'06*)

- And for locked CSPs?
  - Survey Propagation = Belief Propagation
  - Decimation fails fatally - analytical understanding using the method of *(Montanari, Ricci-Tersenghi, Semerjian'07)*
  - BP reinforcement (next slide)
  - Stochastic locals search ASAT-like (next next slide)

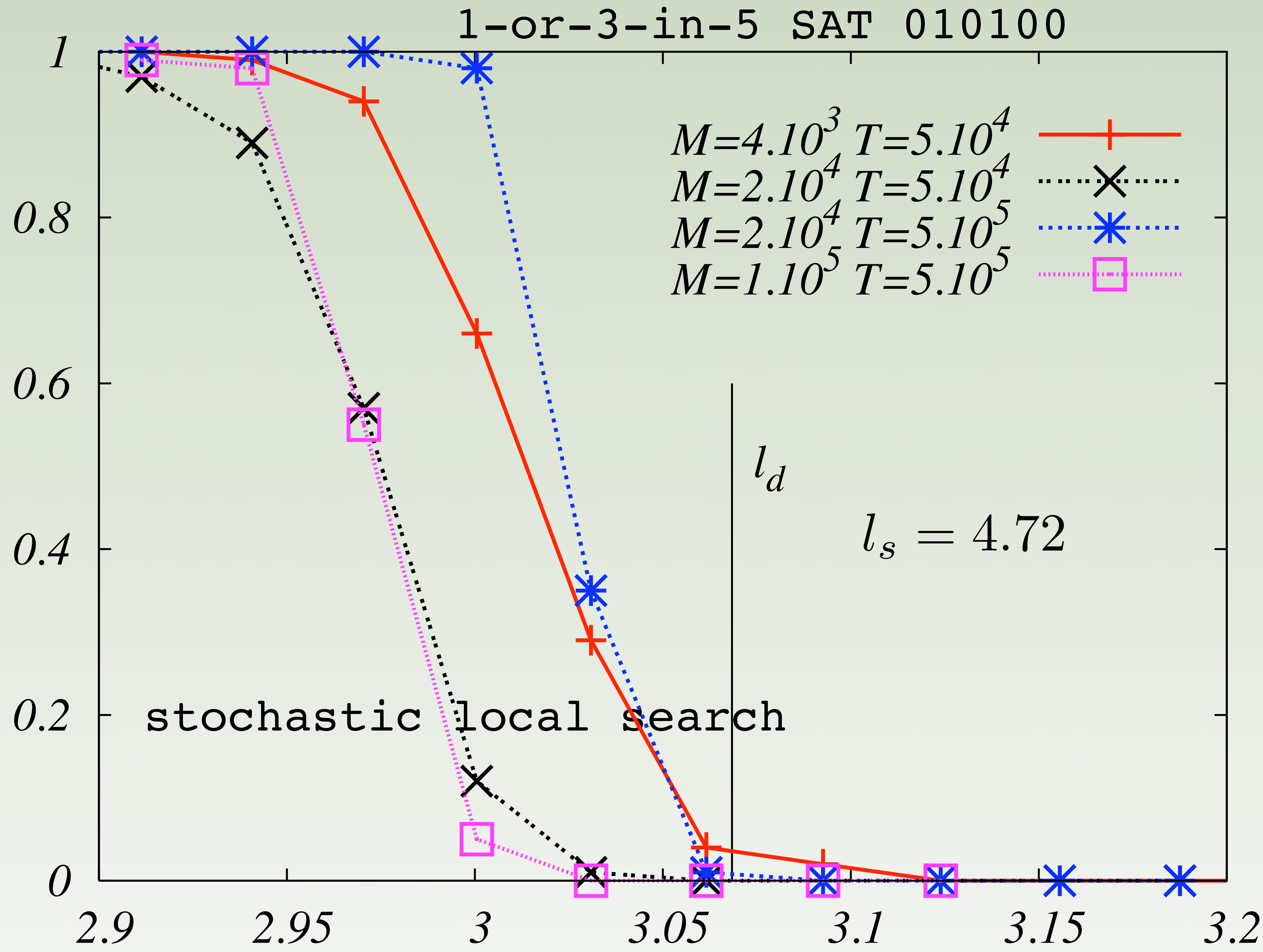

average degree
2
$l_d = 3.07$
$l_s = 4.72$



average degree
2
$l_d = 3.07$
$l_s = 4.72$
2.9-3.2



1-or-3-in-5 SAT 010100
BP-reinforcement
l_d
$l_s = 4.72$
$M=4.10^3$ $T=5.10^3$
$M=2.10^4$ $T=5.10^3$
$M=2.10^4$ $T=5.10^4$
$M=1.10^5$ $T=5.10^4$
1
0.8
0.6
0.4
0.2
0
2.9
2.95
3
3.05
3.1
3.15
3.2


1-or-3-in-5 SAT 010100
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0.8
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M=4.10^3 T=5.10^4
M=2.10^4 T=5.10^4
M=2.10^4 T=5.10^5
M=1.10^5 T=5.10^5
l_d
$l_s = 4.72$
stochastic local search

Conclusions

Does somebody have an algorithm which enters the clustered phase in these problems?

Zdeborová, Mézard, arXiv:0803.2955v1
and more to come ....

BP reinforcement: Main idea

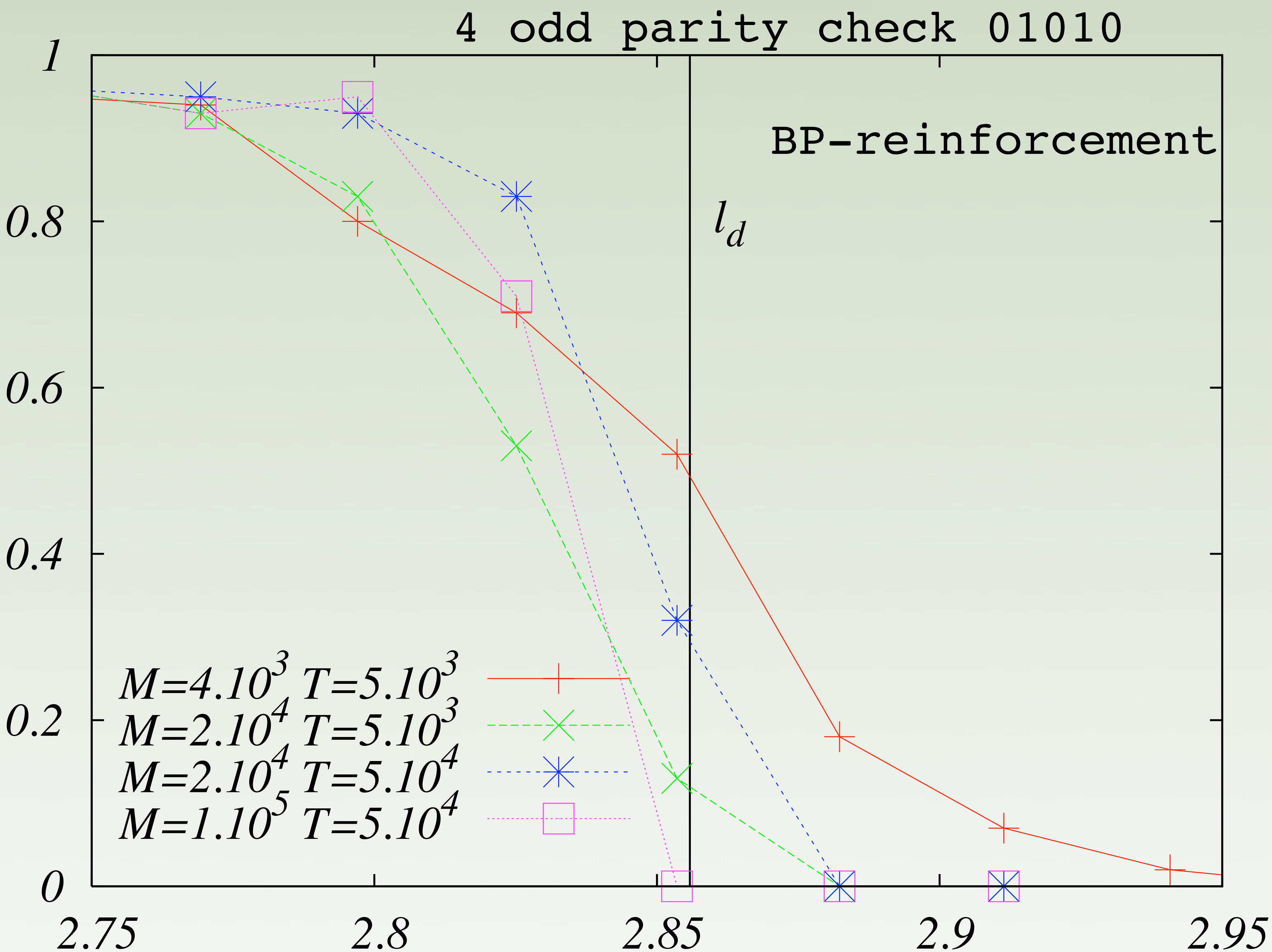
$$\psi_{s_i}^{a\to i} = \frac{1}{Z^{a\to i}} \sum_{A_{s_i+\sum s_j}=1} \prod_{j\in a-i} \chi_{s_j}^{j\to a}$$

$$\chi_{s_i}^{i\to a} = \frac{1}{Z^{i\to a}} \Phi_{s_i}^{i} \prod_{b\in i-a} \psi_{s_i}^{b\to i}$$

$$\Phi_1^i = (\pi)^{l_i-1}, \quad \Phi_0^i = (1-\pi)^{l_i-1}, \quad \text{if} \quad \chi_0^i > \chi_1^i\,,$$
$$\Phi_1^i = (1-\pi)^{l_i-1}, \quad \Phi_0^i = (\pi)^{l_i-1}, \quad \text{if} \quad \chi_0^i \le \chi_1^i\,,$$

$$0 < \pi < 0.5$$

XOR-SAT



4 odd parity check 01010
BP-reinforcement
l_d
M=4.10^3 T=5.10^3
M=2.10^4 T=5.10^3
M=2.10^4 T=5.10^4
M=1.10^5 T=5.10^4
1
0.8
0.6
0.4
0.2
0
2.75
2.8
2.85
2.9
2.95