Finite-Size Scaling in Complex Networks

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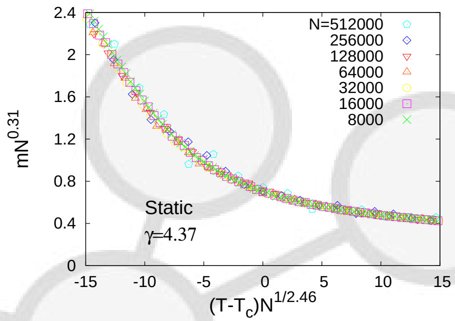
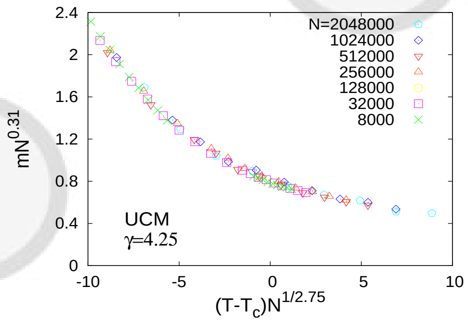
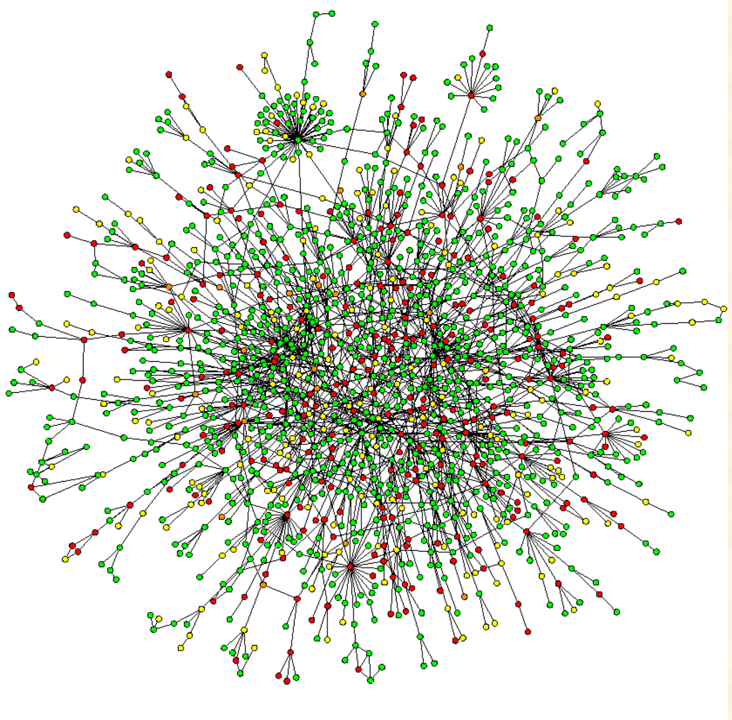
Outline

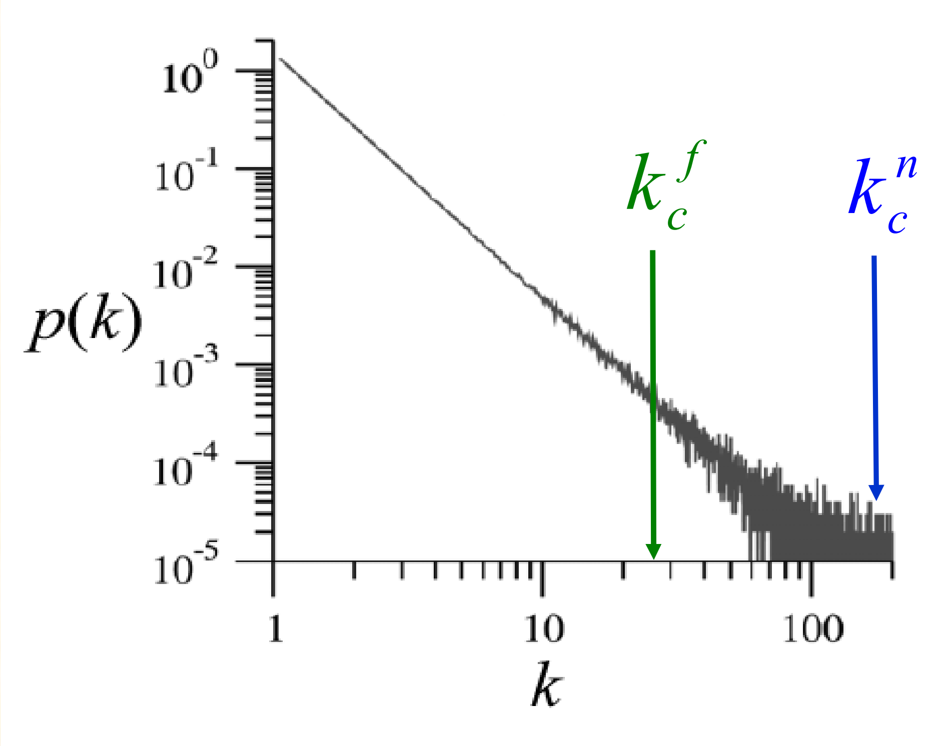
- Motivation & Review
- Heterogeneous MF for FSS & Numerical Tests
- Summary & Remarks





What was known & is unknown





- Courtesy to H. Jeong

What was known & is unknown

- Networks have no finite dimensionality ($d = \infty$), which can be considered as the case of $d > d_{\mathrm{uc}}$ (MF).

  The validity of the MF theory?

  Any critical phenomena in networks??

- Critical phenomena and scaling exponents may depend on the network heterogeneity, i.e., the value of the decay exponent γ in the degree distribution $P(k) \sim k^{-\gamma}$.

  FSS exponents in networks?

  Any cutoff dependence??

Controversial issues

for critical behavior of the Contact Process on scale-free networks

- **Critical Behavior of the CP on SFNs (MF versus non-MF)?**

Castellano and Pastor-Satorras (PRL `06): Non-MF behaviors on SFNs w/ large density fluctuations at highly connected nodes

However, it turns out that

all of their numerical results can be explained well by the proper MF treatment.

In particular, the unbounded density fluctuations are not critical fluctuations, which are just due to the multiplicative nature of the noise in DP systems.
(Ha, Hong, and Park; Hong, Ha, and Park, PRL `07)

- **Upper cutoff dependence on FSS exponents?**

Natural cutoff ***vs.*** **Forced sharp cutoff**

Controversial issues

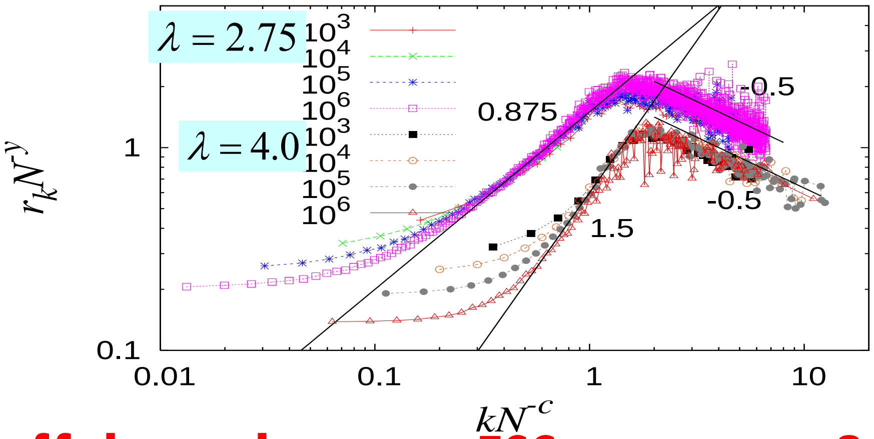
for critical behavior of the Contact Process on scale-free networks

- **Critical Behavior of the CP on SFNs (MF versus non-MF)?**

$$\mathbf{MF}: r_k \equiv \Delta\rho_k/\rho_k \sim N^y f(k/N^c),$$

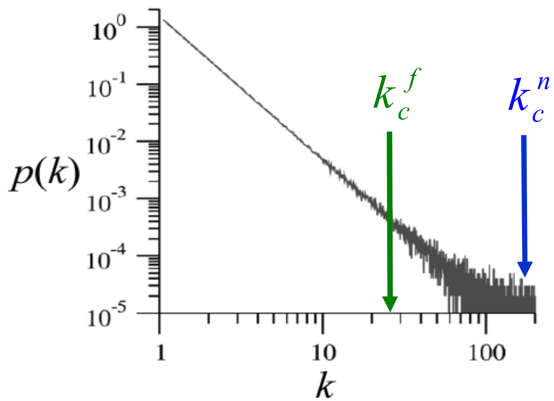
where $c = 1/\lambda$, $y = (\beta/\bar{\nu} - 1/\lambda)/2$, and

$$f(x) \sim \begin{cases} x^{(\lambda-1)/2} & \text{as } x \to 0, \\ x^{-1/2} & \text{as } x \to \infty. \end{cases}$$



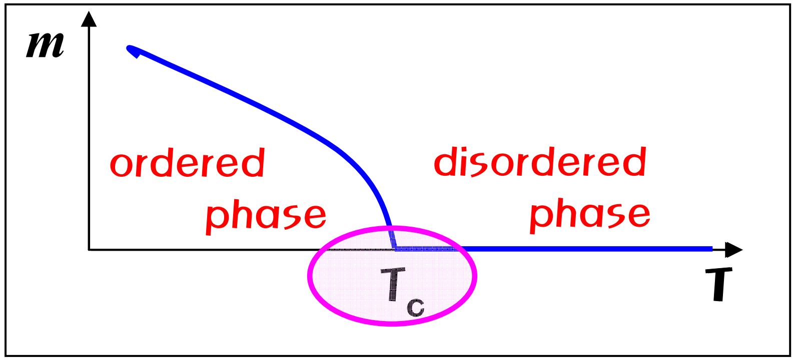
- **Upper cutoff dependence on FSS exponents?**

Natural cutoff *vs.* Forced sharp cutoff



Finite Size Scaling in Low Dimensions

Ferromagnetic Ising model $\mathcal{H} = -J\sum_{<ij>} S_i S_j - H\sum_i S_i$



$$m \sim (T_c - T)^{\beta}, \quad C_V \sim |T_c - T|^{-\alpha}$$
$$\chi \sim |T_c - T|^{-\gamma}, \quad \xi \sim |T_c - T|^{-\nu}$$

Scaling theory

$$f(\varepsilon, h) = b^{-d} f(b^{y_T}\varepsilon, b^{y_H} h)$$
$$\xi(\varepsilon, h) = b^{+1}\xi(b^{y_T}\varepsilon, b^{y_H} h)$$

$\varepsilon = (T_c - T)/T_c$

$h = H/T$

$$\alpha = (2y_T - d)/y_T$$
$$\beta = (d - y_H)/y_T$$
$$\gamma = (2y_H - d)/y_T$$

$$\nu = 1/y_T$$

HYPERSCALING

Correlation exponents versus **Thermodynamic exponents**

$$d/y_T = 2\beta + \gamma = 2 - \alpha$$

$$d\nu = 2\beta + \gamma = 2 - \alpha$$

Standard Finite Size Scaling Theory

$$f(\varepsilon, h, L^{-1}) = b^{-d} f(b^{y_T}\varepsilon, b^{y_H} h, b^1 L^{-1})$$

$$m = \left.\frac{\partial f}{\partial h}\right|_{h=0} = b^{-d+y_H} m(b^{y_T}\varepsilon, 0, bL^{-1}) = L^{-\beta y_T}\varphi(L^{y_T}\varepsilon)$$

$$y_T = 1/\nu$$

$$m = L^{-\beta y_T}\varphi(L^{y_T}\varepsilon) = L^{-\beta/\nu}\varphi(L^{1/\nu}\varepsilon)$$

It works **perfectly well** for "isotropic" equilibrium models in low dimensions and even nonequilibrium ("anisotropic") models with appropriate modifications!!!

FSS variable $u = L^{y_T}\varepsilon \sim (L/\xi)^{y_T}$

ξ competes with L

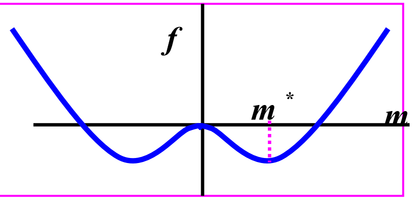
What happens if the hyperscaling breaks down? $\nu \neq 1/y_T$

A new length scale competing with system size ?

Standard FSS theory is still valid ?

MF approach

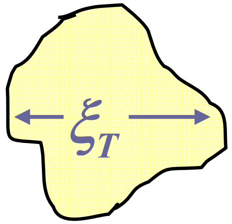
for the Ising model in regular lattices



ϕ^4 theory

$$f(m) = D(\cancel{\nabla m})^2 - \varepsilon m^2 + u m^4 + \cdots$$

Disordered droplet excitation out of uniformly ordered region



$$m^* \sim \varepsilon^{1/2} \quad (\beta = 1/2)$$

$$|f(m^*)| \sim \varepsilon^2$$

Droplet fluctuation

$$\xi_T^d \left(\Delta f\right) \sim k_B T \quad \Rightarrow \quad \xi_T \sim \left(\Delta f\right)^{-1/d} \sim \varepsilon^{-2/d}$$

$$\nu_T = 2/d = 1/y_T$$

$$d\nu_T = 2\beta + \gamma = 2 - \alpha$$

Gaussian fluctuation

$$m^2 / \xi_G^2 \sim \varepsilon m^2 \quad \Rightarrow \quad \xi_G \sim \varepsilon^{-1/2}$$

$$\nu_G = 1/2$$

In high dimensions $(d > d_{uc} = 4)$: $\quad \nu_G > \nu_T \quad (\xi_G >> \xi_T) \quad \Rightarrow \quad \nu_{MF} = \nu_G$

Hyperscaling is broken in the MF regime.

Finite Size Scaling in High Dimensions

$$f(\varepsilon, h, L^{-1}) = b^{-d} f(b^{y_T}\varepsilon, b^{y_H} h, b^1 L^{-1})$$

$$y_T \neq 1/\nu = 2$$

- In high dimensions, the hyperscaling breaks down and the MF theory is correct.
- *Dangerous irrelevant variable* hypothesis
- Two length scales: Gaussian Fluctuation vs. Thermodynamic Droplet Excitation

$$m = L^{-\beta y_T}\varphi(L^{y_T}\varepsilon)$$

with $y_T = d/2$, $y_H = 3d/4$

$$\alpha = (2y_T - d)/y_T = 0$$
$$\beta = (d - y_H)/y_T = 1/2$$
$$\gamma = (2y_H - d)/y_T = 1$$

$$d/y_T = 2\beta + \gamma = 2 - \alpha = 2$$

FSS variable $u = L^{y_T}\varepsilon \sim (L/\xi_T)^{y_T}$

$$\xi_T \sim \varepsilon^{-1/y_T} = \varepsilon^{-2/d}$$

Why do we care this droplet length?

For well-known equilibrium models and some nonequilibrium models, it is known that

this thermodynamic droplet length scale competes with system size in high dimensions and governs FSS as $\xi_T \to L$ **.**

It implies that **the FSS exponent for** $d > d_{uc}$ **is not** $\nu_{MF} = \nu_G = 1/2$ **but** $\nu_{FSS} = \nu_T = 2/d$, **equivalently to** $\bar{\nu} = \nu_{FSS} d = 2$ **for** $\xi_V = \xi_T^d \to N$.

$$m = N^{-\beta/\bar{\nu}} \psi(\varepsilon N^{1/\bar{\nu}}).$$

-Binder, Nauenberg, Privman, and Young, PRB (1985): 5D Ising model test

-Luebeck and Jassen, PRE (2005): 5D DP model test

-Botet, Jullien, and Pfeuty, PRL (1982): FSS in infinite systems

Generalization: FSS for the ϕ^q Theory

$$f(m) = -\varepsilon m^2 + u m^q + \cdots$$

$$m = N^{-\beta/\bar{\nu}} \psi(\varepsilon N^{1/\bar{\nu}})$$

$$\bar{\nu} \equiv d\nu_T = \nu_G d_{uc} = (1/2)*(2q)/(q-2) = q/(q-2) \quad : \ d-\text{independent}$$

$$\nu_T = \nu_G (d_{uc}/d) = d_{uc}/(2d)$$

- **Droplet excitation length scale**
- **Hyperscaling relation**

$$d\nu_T = 2\beta + \gamma$$

e.g., **Ising:** $q = 4, \nu_{MF} = 1/2,$ **and** $d_{UC} = 4$ **so that** $\bar{\nu} = 2.$

FSS of Ising model on SFNs

$$\bar{\nu} = d\nu_T = 2\beta + \gamma$$

$$P(k) \sim k^{-\lambda}$$

Using the phenomenological theory (Goltsev *et al*. PRE '03) for the Ising model in SF networks,

$$f(m) \simeq -\frac{\epsilon}{2}m^2 + \frac{u}{4}m^4 + c_\gamma |m|^{\lambda-1}.$$

For $\lambda > 5$, back to ϕ^4 theory

For $3 < \lambda < 5$,

$m \sim \varepsilon^{\beta}$ with $\beta = 1/(\lambda - 3)$

$\Delta f \sim \varepsilon^{1+2\beta}$

$\xi_T^d \sim (\Delta f)^{-1} \sim \varepsilon^{-\nu_T d} = \varepsilon^{-\bar{\nu}}$ with $\bar{\nu} = 1 + 2\beta$

$$\begin{cases} \beta = 1/2 \\ \bar{\nu} = 2 \end{cases} \quad \text{for } \lambda > 5$$

$$\begin{cases} \beta = 1/(\lambda-3) \\ \bar{\nu} = (\lambda-1)/(\lambda-3) \end{cases} \quad \text{for } 3 < \lambda < 5$$

$\gamma = 1$ for $\lambda > 3$

$$f(m) = D(\nabla m)^2 - \varepsilon m^2 + v|m|^{\lambda-1} \Rightarrow \bar{\nu} = \nu_G d_{uc}$$

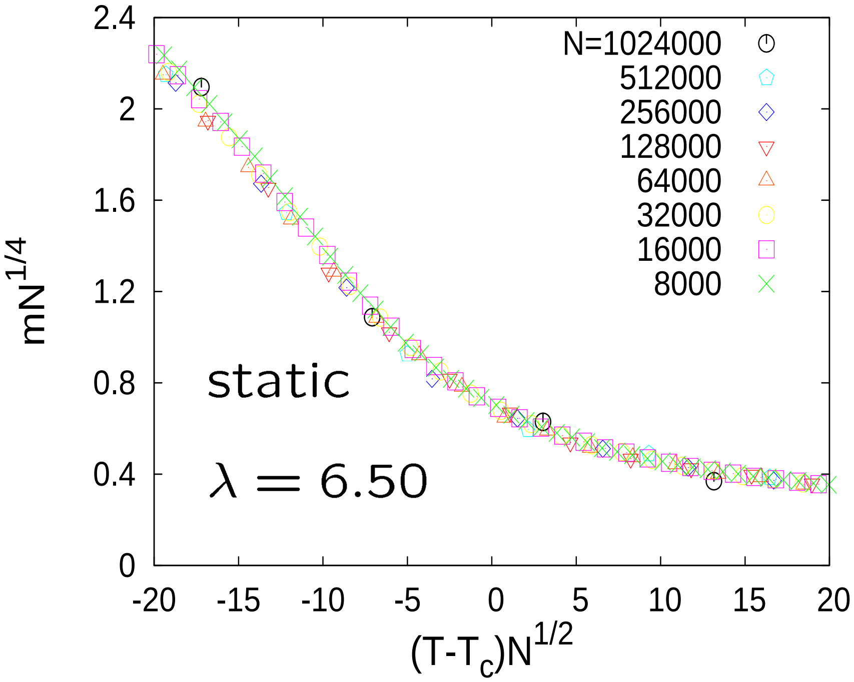
Our conjecture for FSS exponents is network (cutoff)-independent.

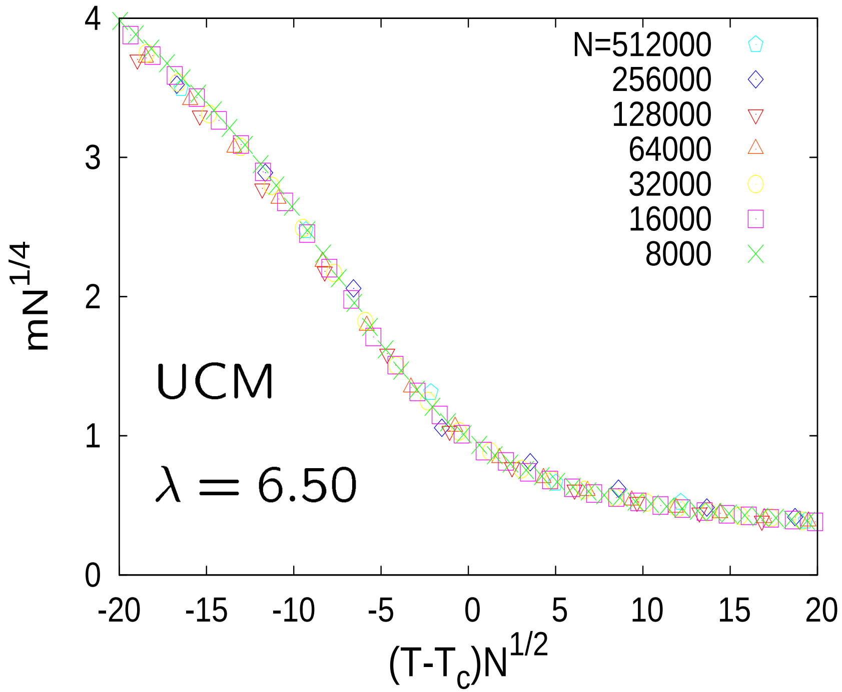
Ising

$$mN^{\beta/\bar{\nu}} = \psi(tN^{1/\bar{\nu}})$$

$\lambda = 6.50(>5)$

$\beta = 1/2, \quad \bar{\nu} = 2$





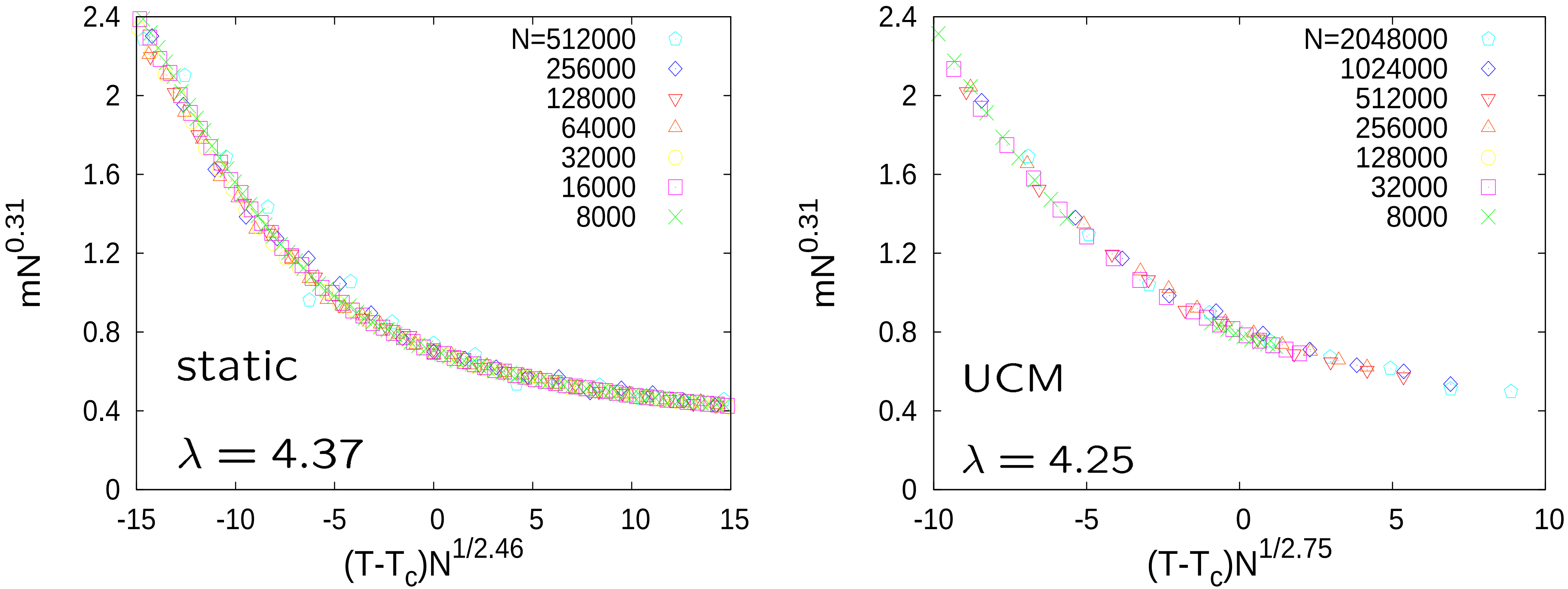
$(\beta/\bar{\nu},\ \bar{\nu})$	**Static**	**UCM**
Theory	**(¼, 2)**	**(¼, 2)**
Data	**(0.25(1), 2.0(1))**	**(0.25(1), 2.0(1))**

Ising

$$mN^{\beta/\bar{\nu}} = \psi(tN^{1/\bar{\nu}})$$

$3 < \lambda < 5$

$$\beta = 1/(\lambda - 3), \quad \bar{\nu} = (\lambda - 1)/(\lambda - 3)$$



$(\beta/\bar{\nu},\ \bar{\nu})$	**Static (4.37)**	**UCM (4.25)**
Theory	**(0.296, 2.46)**	**(0.308, 2.60)**
Data	**(0.31(2), 2.46(10))**	**(0.31(2), 2.75(20))**

FSS of CP/SIS on SFNs

Directed Percolation

$$d_t\rho = a\rho - b\rho^2 - \boxed{d\rho^\theta} + \sqrt{\rho}\eta(t)$$

Note that $\theta = \lambda - 1$ **for (CP) and** $\theta = \lambda - 2$ **for (SIS).**

In the CP, the stationary solution of the Fokker-Planck equation for $P(\rho, t)$, is similar to $P \sim \mathrm{e}^{-f^*}$ in the Ising case. Then the droplet excitation probability of ξ_V should obey

$$[P_{\rho^*} \sim \mathrm{e}^{-F(\rho^*)}]^{\xi_\mathrm{V}} \sim O(1).$$

Consequently, we get $\bar{\nu}_\mathrm{CP} = \frac{\lambda-1}{\lambda-2}$ for $2 < \lambda < 3$; 2 for $\lambda > 3$.

For the SIS case, the only one change $\lambda \rightarrow (\lambda - 1)$ **is needed.**

Phenomenological remedy for DP on SFNs

$$\frac{d\rho}{dt} = \varepsilon\rho - c\rho^2 - d\rho^{\theta-1} + \sqrt{\rho}\eta$$

$$\bar{\nu} = d\nu_T = \beta + \gamma - \max\{\nu_t - \beta', 0\}$$

For $\theta > 3$, back to ordinary theory

For $2 < \theta < 3$,

$$\rho^* \sim \varepsilon^{\beta} \quad \text{with} \quad \beta = 1/(\theta - 2)$$

$$\Delta f \sim \varepsilon\rho^* \sim \varepsilon^{1+\beta}$$

$$\xi_T^d \sim (\Delta f)^{-1} \sim \varepsilon^{-\bar{\nu}} \quad \text{with} \quad \bar{\nu} = 1 + \beta$$

$$\begin{cases} \beta = 1 \\ \bar{\nu} = 2 \end{cases} \quad \text{for } \theta > 3$$

$$\begin{cases} \beta = 1/(\theta-2) \\ \bar{\nu} = (\theta-1)/(\theta-2) \end{cases} \quad \text{for } 2 < \theta < 3$$

$$\gamma = 1, \quad \nu_t = 1 \quad \text{for } \theta > 2$$

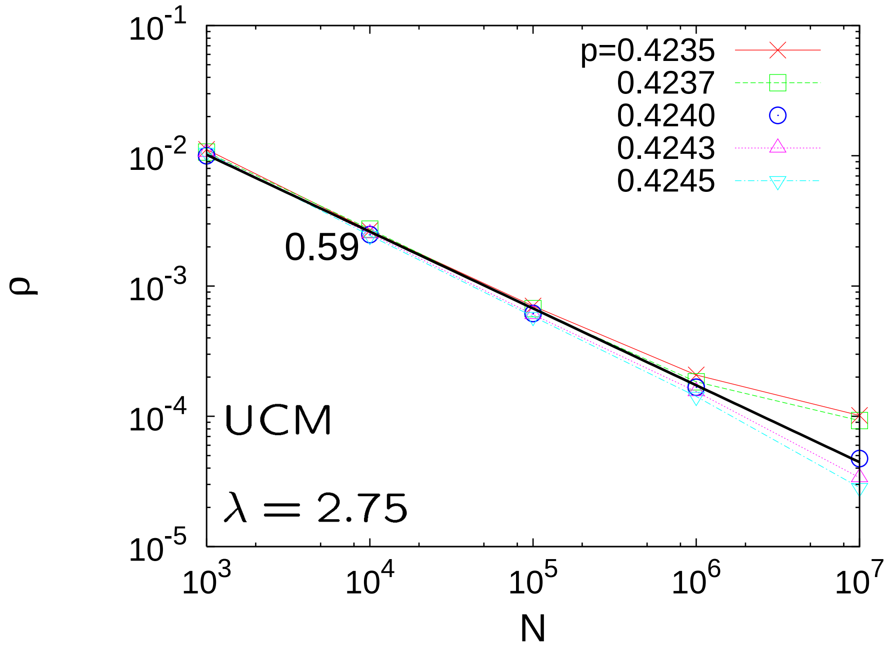
$$\frac{d\rho}{dt} = \nabla^2\rho + a\rho - c\rho^2 - d\rho^{\theta-1} + \sqrt{\rho}\eta \quad \Rightarrow \quad \bar{\nu} = \nu_G d_{uc}$$

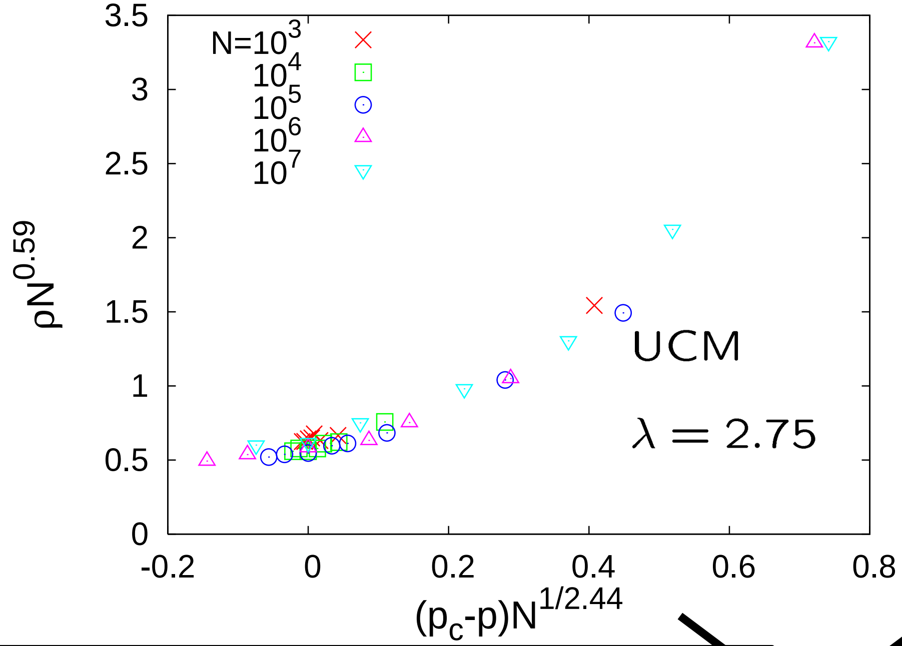
Our conjecture for FSS exponents is network (cutoff)-independent.

CP on UCM

$\lambda = 2.75 (< 3)$

$\beta = 1/(\lambda - 2), \quad \bar{\nu} = (\lambda - 1)/(\lambda - 2)$





$\beta = 1/(\lambda - 2), \quad \bar{\nu} = 2/(\lambda - 2)$

$(\beta/\bar{\nu},\ \bar{\nu})$	UCM	
	Ours	**C&P-S**
prediction	**(0.571, 2.33)**	**(½, 2.67)**
Data	**(0.59(2), 2.44(10))**	**(0.63(4), 2.4(2))**

Ours (PRL '08) vs. Castellano and Pastor-Satorros (PRL '06)

Network Cutoff Dependence Tests

Degree k should be bounded for finite N as

$$k_C \sim \begin{cases} N^{1/(\lambda-1)} & \text{for "static" natural cutoff} \\ N^{\frac{1}{2}} & \text{for "UCM" forced cutoff} \end{cases}$$

(Goh *et al.* PRL `01 for static; Cantanzaro *et al.* PRE `05 for UCM)

$$T_c \sim \frac{\langle k^2 \rangle}{\langle k \rangle} \quad \Rightarrow \quad T_c(N) - T_c(\infty) \sim k_c^{-(\lambda-3)} \sim N^{-\delta(\lambda-3)} \sim N^{-1/\bar{\nu}}$$

$$\begin{cases} \bar{\nu} = (\lambda-1)/(\lambda-3) & \text{for } \delta = 1/(\lambda-1) & \text{[natural cutoff]} \\ \bar{\nu} = 2/(\lambda-3) & \text{for } \delta = 1/2 & \text{[UCM]} \end{cases}$$

Numerical results show
no cutoff dependence
for Ising and DP models
for any λ .

$$\begin{cases} \bar{\nu} = (\lambda-1)/(\lambda-3) & \text{for Ising } (3<\lambda<5) \\ \bar{\nu} = (\lambda-2)/(\lambda-3) & \text{for SIS } (3<\lambda<4) \end{cases}$$

Take-home messages

- "Mean-Field theory" is still valid on scale-free networks:
  The heterogeneity(λ)-dependent MF theory is working perfectly well.
- We conjecture finite size scaling (FSS) exponents for various models on scale-free networks and other exponential networks, which are all confirmed reasonably well in numerical simulations. NOISE !!
- Our FSS conjecture is based on droplet excitation argument and/or hyperscaling.
- No cutoff dependence on finite size scaling exponents
  if not too strong!

Ongoing issues

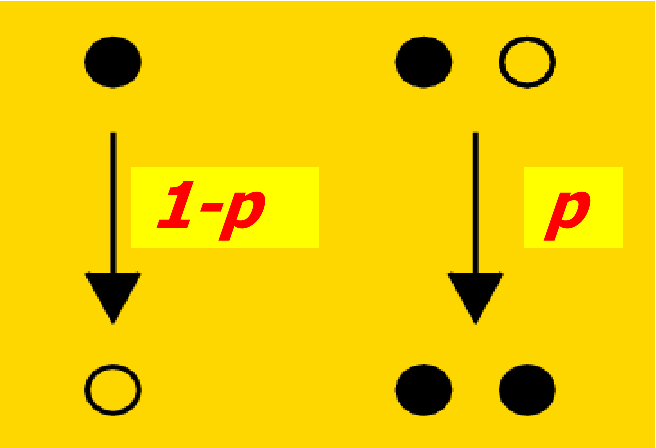
*** Degree (k)-dependent FSS & its cutoff dependence**

*** FSS in Networks (Quenched vs. Annealed)**

Thank You!

Contact Process (CP)

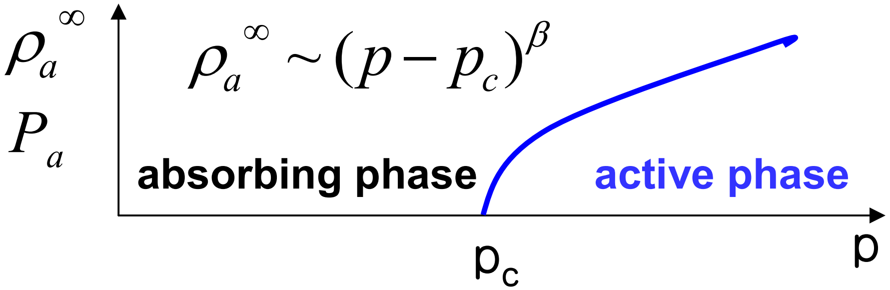
- Courtesy to HP



A typical **single absorbing state** model

absorbing state:
vacuum [single absorbing state]

Dynamic Rule:
a particle is annihilated with prob. (1-p) **(A → 0)**
or creates another one at a n.n. site with prob. p **(A → 2A)**



$$\xi \sim |p - p_c|^{-\nu_\perp} \qquad \tau \sim |p - p_c|^{-\nu_\parallel}$$

Directed Percolation (DP) universality class

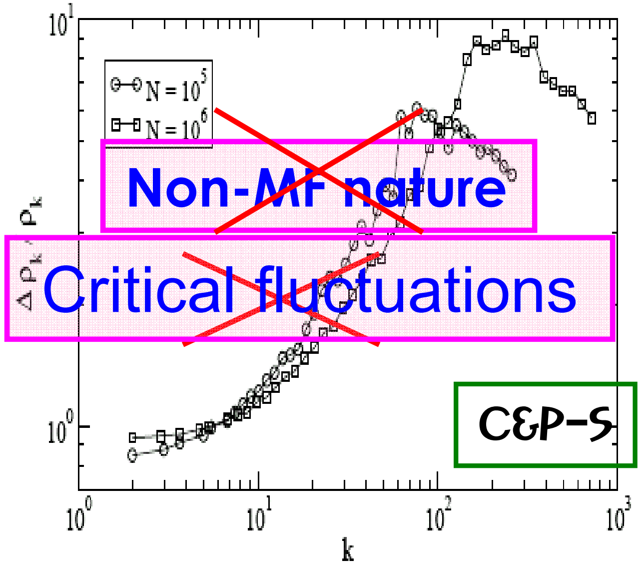
DP conjecture:

Continuous transitions from an active phase into a single absorbing state should belong to DP class.

Unbounded density fluctuations in DP systems

Relative density fluctuations at sites of degree k

$$r_k = \Delta\rho_k / \rho_k$$



site fluctuations | **multiplicative nature of noise**

N_k : # of sites of degree k

M_k : # of occupied sites of degree k

$$\Delta M_k \sim \sqrt{M_k} \quad \text{with} \quad \rho_k = M_k / N_k$$

$$\Rightarrow r_k = \Delta\rho_k / \rho_k = 1/\sqrt{N_k \rho_k} = 1/\sqrt{N P_k \rho_k}$$

$$\Rightarrow r_k \sim (N\rho)^{-1/2} k^{(\lambda-1)/2} \quad \text{with} \quad \rho_k \sim \rho k$$

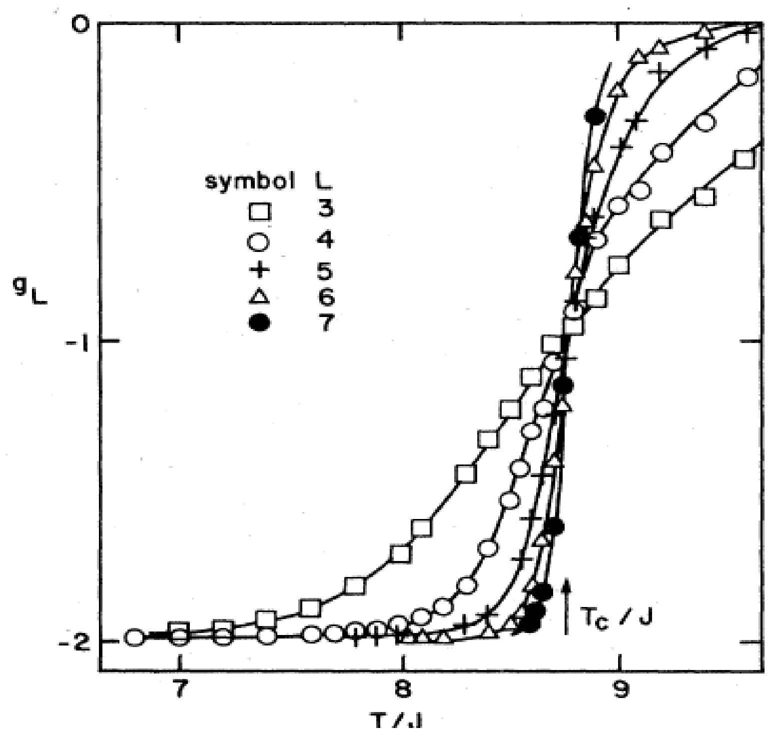
$$r_k = N^{(\alpha-1/\lambda)/2} f\left(kN^{-1/\lambda}\right) \quad \text{with} \quad f(x) \sim x^{(\lambda-1)/2} \quad \text{for small } x$$

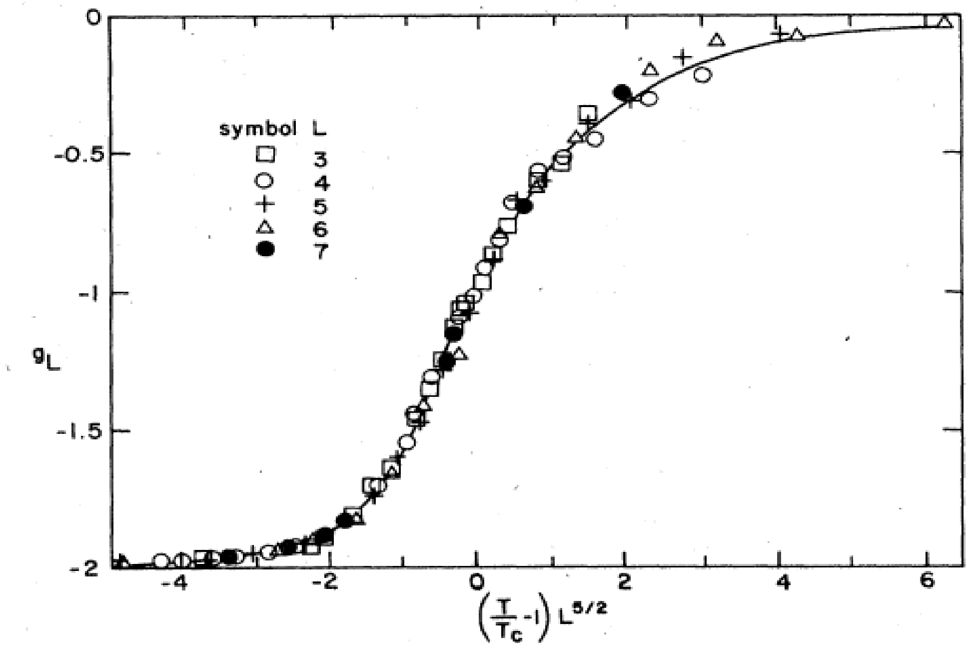
$$\sim x^{-1/2} \quad \text{for large } x$$

Scaling with $\rho \sim N^{-\alpha}$ with $\alpha = \beta/\nu_\perp$ (criticality), 0 (active), 1 (absorbing)

Simulation data for 5D Ising model

Binder, Nauenberg, Privman, Young , PRB (1985)





$$\nu' = d_{\mathrm{uc}} /(2d) = 2/5$$

$$tL^{1/\nu'} = tL^{5/2}$$

Ising model on scale-free networks

Langevin eq. for Ising system on regular lattice

$$\frac{dm(\vec{x},t)}{dt} = D\nabla^2 m + tm - cm^3 + \eta(\vec{x},t)$$

Degree distribution

$$P(k) \sim k^{-\lambda} \quad (\lambda : \text{degree exponent})$$

Dorogovtsev-type remedy for SFN MF theory

$$\frac{dm(\vec{x},t)}{dt} = D\nabla^2 m + tm - cm^3 - dm^{\lambda-2} + \eta(\vec{x},t)$$

Conjecture !!!

$$\begin{cases} \beta = 1/2 & \text{for } \lambda > 5 \\ \beta = 1/(\lambda - 3) & \text{for } 3 < \lambda < 5 \end{cases}$$

$$\begin{cases} \bar{\nu} = \nu_{\mathrm{MF}}^{G} d_{uc} = (1/2) * 4 = 2 & \text{for } \lambda > 5 \\ \bar{\nu} = (\lambda - 1)/(\lambda - 3) & \text{for } 3 < \lambda < 5 \end{cases}$$

power counting

$x \sim [\kappa]^{-1},\ t \sim [\kappa]^{-2},\ m \sim [\kappa]^{\beta/\nu} = [\kappa]^{1}$

at $d = d_{uc},\quad m^3 \sim \eta \Rightarrow [\kappa]^3 \sim [\kappa]^{(2+d_{uc})/2}$

power counting

$x \sim [\kappa]^{-1},\ t \sim [\kappa]^{-2},\ m \sim [\kappa]^{\beta/\nu} = [\kappa]^{2/(\lambda-3)}$

at $d = d_{uc},\quad m^{\lambda-2} \sim \eta \Rightarrow [\kappa]^{2(\lambda-2)/(\lambda-3)} \sim [\kappa]^{(2+d_{uc})/2}$

Fokker-Plank equation (Ito sense) for $P(\rho,t)$

$$\frac{d\rho}{dt} = \varepsilon\rho - c\rho^2 + \sqrt{\rho}\eta$$

$$\frac{dP}{dt} = \frac{\partial}{\partial\rho}\left[-\left(\varepsilon\rho - c\rho^2\right)P + \frac{\partial}{\partial\rho}(\rho P)\right]$$

Steady-state probability distribution

$$P^{\infty} \sim e^{\int d\rho\left(\varepsilon\rho - c\rho^2\right)/\rho} \sim e^{\varepsilon\rho - c\rho^2/2} \sim e^{-f(\rho)}$$

$$\varepsilon > 0 \quad \rho^{*} \sim \varepsilon^{1}$$

$$(\beta = 1)$$

Droplet excitation (in the active phase)

$$\left[\frac{P^{\infty}(\rho = 0)}{P^{\infty}(\rho = \rho^{*})}\right]^{\xi_T^d} \sim e^{-\Delta f\, \xi_T^d} \sim O(1)$$

$$\Delta f \sim \varepsilon\rho^{*} \sim \varepsilon^{2}$$

$$\xi_T \sim \varepsilon^{-\bar{\nu}}$$

$$\beta = 1, \quad \gamma = 1, \quad \bar{\nu} = 2$$

Directed Percolation (DP) Systems

Langevin eq. for DP systems on regular lattice

$$\frac{d\rho(\vec{x},t)}{dt} = D\nabla^2\rho + t\rho - c\rho^2 + \sqrt{\rho}\eta(\vec{x},t)$$

Dorogovtsev-type remedy for SFN MF theory for SIS (can be derived)

$$\frac{d\rho(\vec{x},t)}{dt} = D\nabla^2\rho + t\rho - c\rho^2 - d\rho^{\lambda-2} + \sqrt{\rho}\eta(\vec{x},t)$$

⇨

$$\begin{cases} \beta = 1 & \text{for } \lambda > 4 \\ \beta = 1/(\lambda - 3) & \text{for } 3 < \lambda < 4 \end{cases}$$

$$\begin{cases} \bar{\nu} = \nu^{G}_{\mathrm{MF}} d_{\mathrm{uc}} = (1/2) * 4 = 2 & \text{for } \lambda > 4 \\ \bar{\nu} = (\lambda - 2)/(\lambda - 3) & \text{for } 3 < \lambda < 4 \end{cases}$$

power counting

$x \sim [\kappa]^{-1},\ t \sim [\kappa]^{-2},\ \rho \sim [\kappa]^{\beta/\nu} = [\kappa]^2$

at $d = d_{uc}$, $\rho^2 \sim \sqrt{\rho}\eta \Rightarrow [\kappa]^4 \sim [\kappa]^{1+(d_{uc}+2)/2}$

power counting

$x \sim [\kappa]^{-1},\ t \sim [\kappa]^{-2},\ \rho \sim [\kappa]^{\beta/\nu} = [\kappa]^{2/(\lambda-3)}$

at $d = d_{uc}$, $\rho^{\lambda-2} \sim \sqrt{\rho}\eta \Rightarrow [\kappa]^{2(\lambda-2)/(\lambda-3)} \sim [\kappa]^{1/(\lambda-3)+(d_{uc}+2)/2}$