The (many) phase transitions in random constraint satisfaction problems

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Nordita, PhysDIS workshop, 15-18 May 2008

Problem definition

Optimization problem

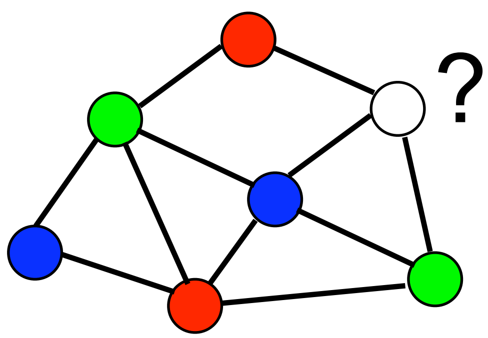
Find a configuration minimizing a cost function
$H(\vec{\sigma})$ = number of violated constraints

With $H_{\min} = 0$

Constraint Satisfaction Problem

Find a configuration of
N variables satisfying *M* constraints

q-colorability (q-COL) of a graph



N q-states Potts variables $\sigma_i \in \{1, 2, \ldots, q\}$

M pairwise interactions avoiding monochromatic edges

$$H(\vec{\sigma}) = \sum_{\langle ij \rangle} \delta_{\sigma_i, \sigma_j} \longleftarrow$$ counts the number of edges connecting vertices of the same color

K-Satisfiability (K-SAT)

N binary variables $\sigma_i \in \{-1, 1\}$

M constraints involving *K* variables each

each constraint (clause) prohibits 1 among the 2^K configurations of the *K* variables it contains, e.g.

$(\sigma_7 \vee \bar{\sigma}_4 \vee \sigma_{13})$ forbids $\sigma_7 = \mathrm{F},\ \sigma_4 = \mathrm{T},\ \sigma_{13} = \mathrm{F}$

$$H(\vec{\sigma}) = \sum_{a=1}^{M} \left| \frac{\sigma_{i_a(1)} - J_{a,1}}{2} \frac{\sigma_{i_a(2)} - J_{a,2}}{2} \dots \frac{\sigma_{i_a(K)} - J_{a,K}}{2} \right|$$

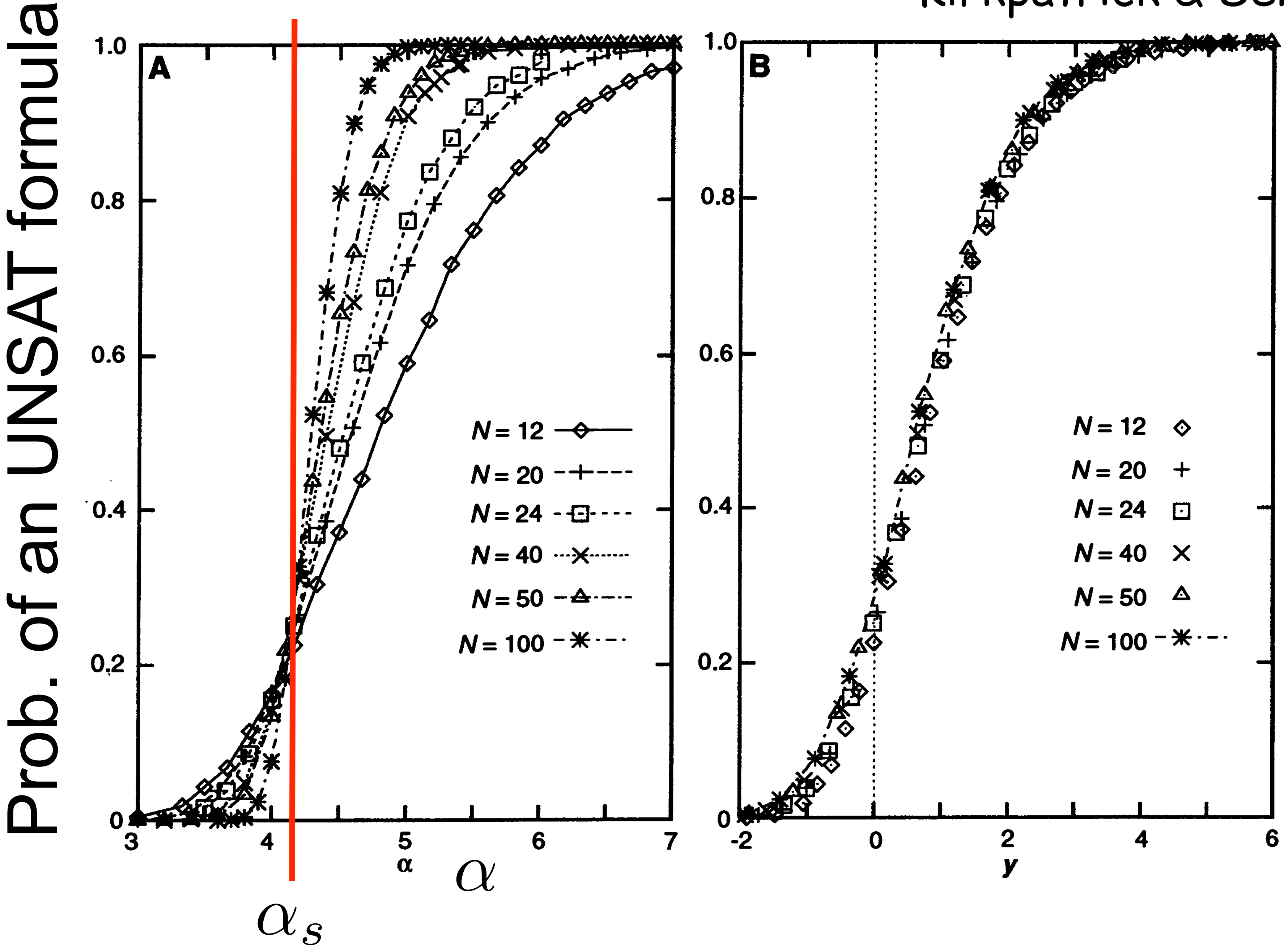
Random CSP

- *random q-col*
  - q-coloring a random graph with *M* links
- *random K-SAT*
  - *M* randomly generated clauses (constraints) of fixed length *K*

$$\alpha = M/N$$

SAT/UNSAT phase transition

Kirkpatrick & Selman, Science '94

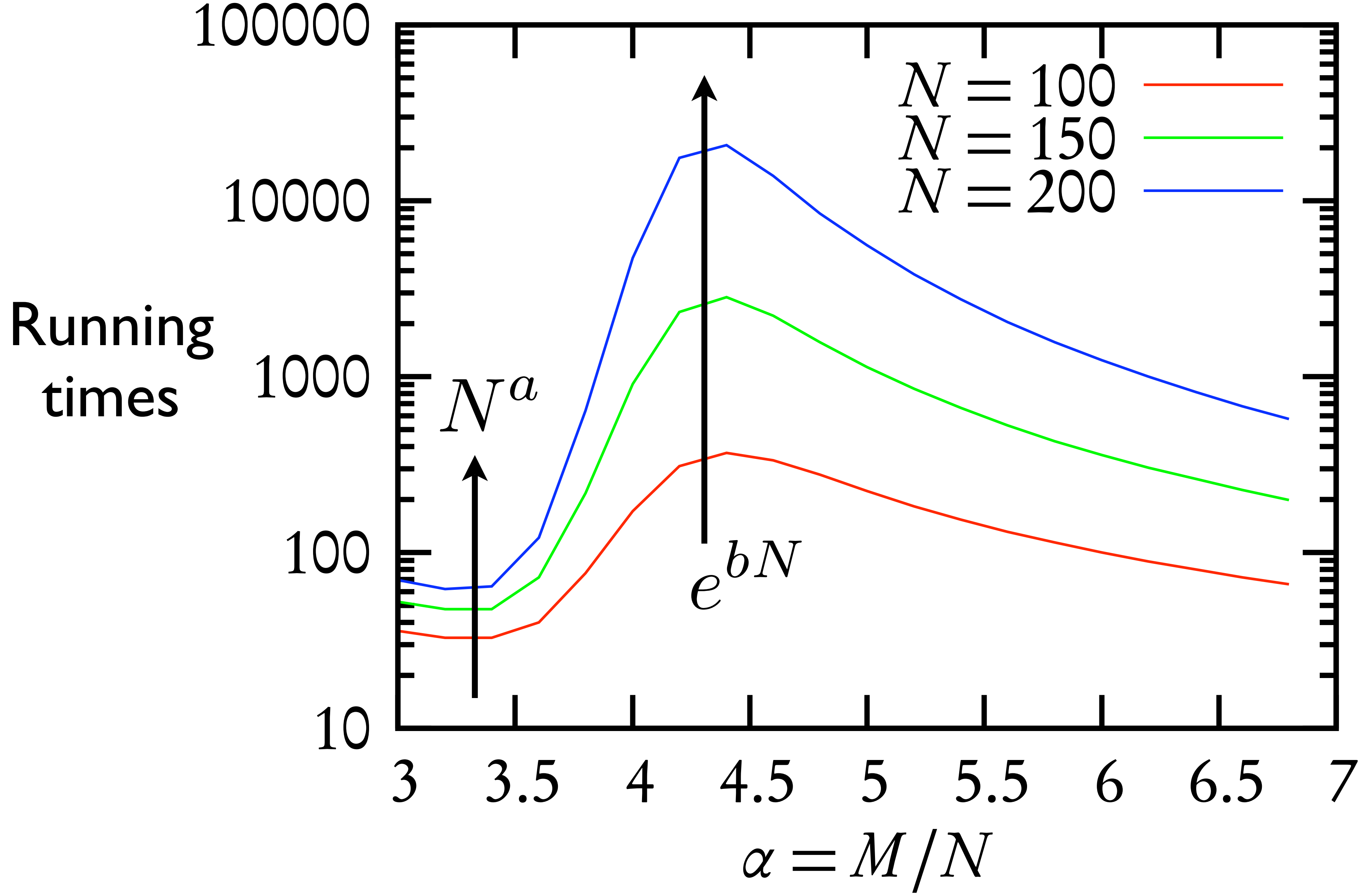


random
3-SAT

$$\alpha_s \sim 4.17$$

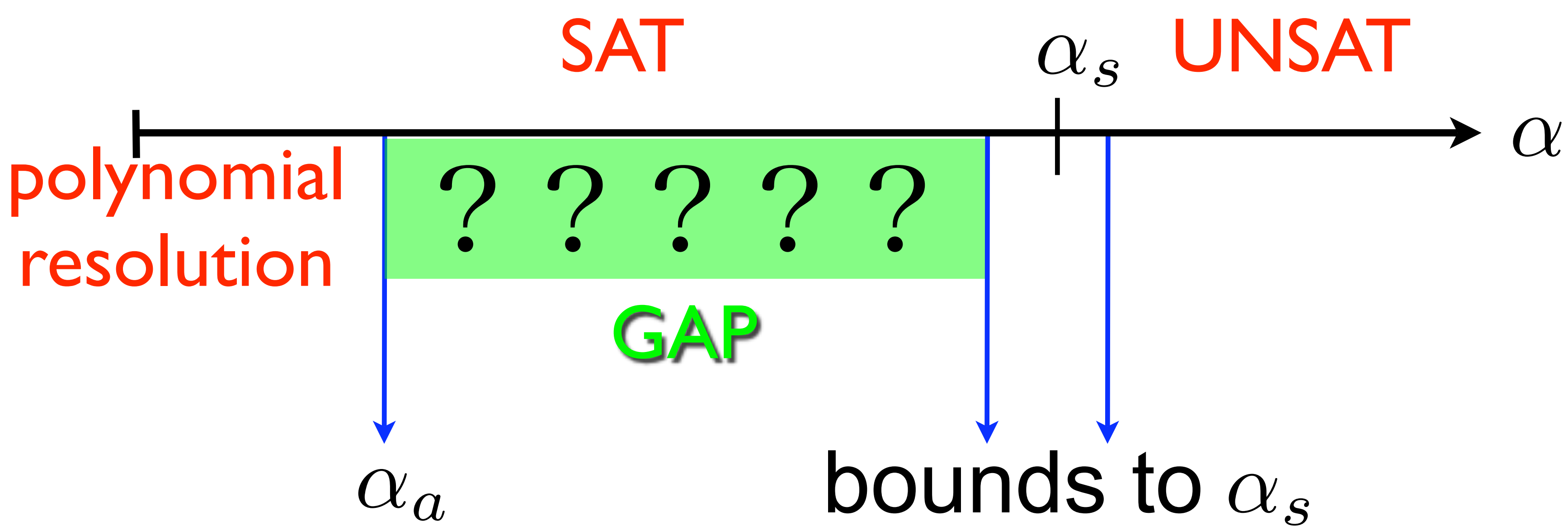
$$\nu \sim 1.5$$

Connection to computational complexity



Using a complete solving algorithm (DPLL)

A big gap!



K	α_a	α_s
10	172.65	707 ± 2
20	95263	726813 ± 4

stat. mech. approach

$$P_{\mathrm{GB}}(\vec{\sigma}) = \frac{e^{-\beta H(\vec{\sigma})}}{Z(\beta)} = \frac{1}{Z(\beta)} \prod_{a=1}^{M} \psi_a \left(\sigma_{i_a(1)}, \ldots, \sigma_{i_a(k)}\right)$$

compatibility functions (inference problems)

Limit $T \to 0, \beta \to \infty$

$$\mu(\vec{\sigma}) = \frac{1}{Z} \prod_{a=1}^{M} \mathbb{I}_a \left(\sigma_{i_a(1)}, \ldots, \sigma_{i_a(k)}\right)$$

indicator functions

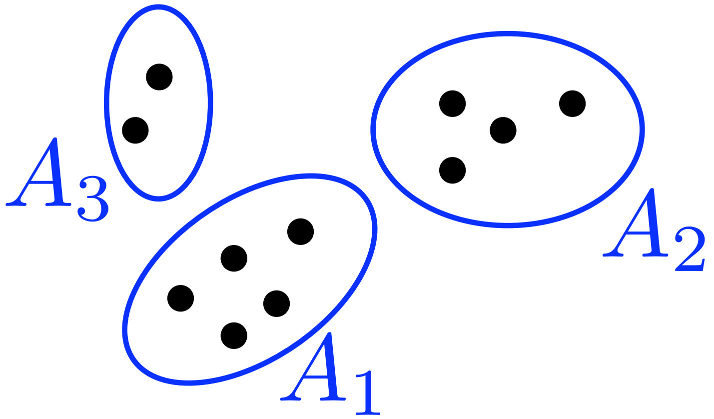
number of solutions

structure of solution-space

pure states decomposition of $\mu(\vec{\sigma})$.

$$w_\gamma = \sum_{\vec{\sigma} \in A_\gamma} \mu(\vec{\sigma})$$

$$w_1 > w_2 > w_3 > \ldots$$



- **RS**: most of the measure in a single cluster

$$\lim_{N \to \infty} w_1 = 1$$

- **d1RSB**: the measure divides in $e^{N\Sigma^*}$ clusters

- **1RSB**: the measure condensates in sub-exp number of clusters

$$\lim_{n \to \infty} \lim_{N \to \infty} \sum_{i=1}^{n} w_i = 1$$

Counting the states

Aim: compute $\Sigma_f(f,T)$ such that $\mathcal{N}(f,T) = e^{N\Sigma_f(f,T)}$

Define the replicated free-energy $\Phi(m,T)$

$$e^{-\beta m \Phi(m,T) N} \equiv \sum_\gamma Z_\gamma^m = \int e^{-\beta m f N + N\Sigma_f(f,T)} \mathrm{d}f$$

and by the Legendre transform

$$\Sigma_f(f,T) = \beta m f - \beta m \Phi(m,T)|_{f=\partial_m(m\Phi)}$$

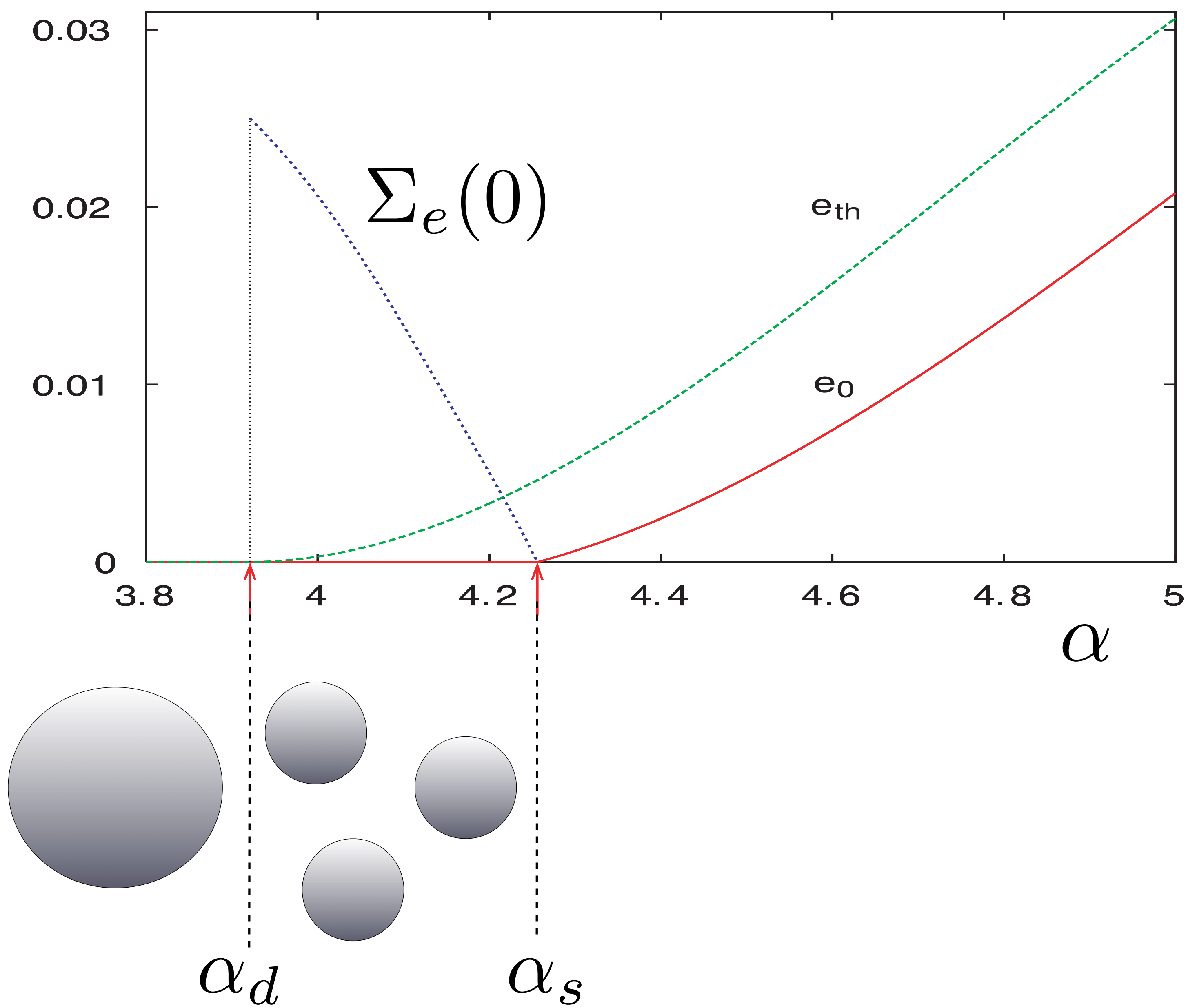
For $T \to 0$ with $\beta m = \mu$

$$\Sigma_e(e) = \mu e - \mu \Phi(\mu)|_{e=\partial_\mu(\mu\Phi)}$$

m is the Parisi parameter

Cavity solution for random K-SAT

Mézard, Parisi & Zecchina, Science '02



Entropic effects at very low temperatures

- taking first the limit $T \to 0$

  then $f = e - \cancel{Ts}$

  $$Z_\gamma = e^{-\beta N f_\gamma} \simeq e^{-\beta N e_\gamma}$$

  → ok if $e_\gamma > 0$

  → but if $e_\gamma = 0$ $Z_\gamma = 1$ always!

- consider only solutions ($e_\gamma = 0$)

  $$f = -Ts \qquad Z_\gamma = e^{-\beta N f_\gamma} \simeq e^{N s_\gamma}$$

  larger clusters count more!

New replicated potential

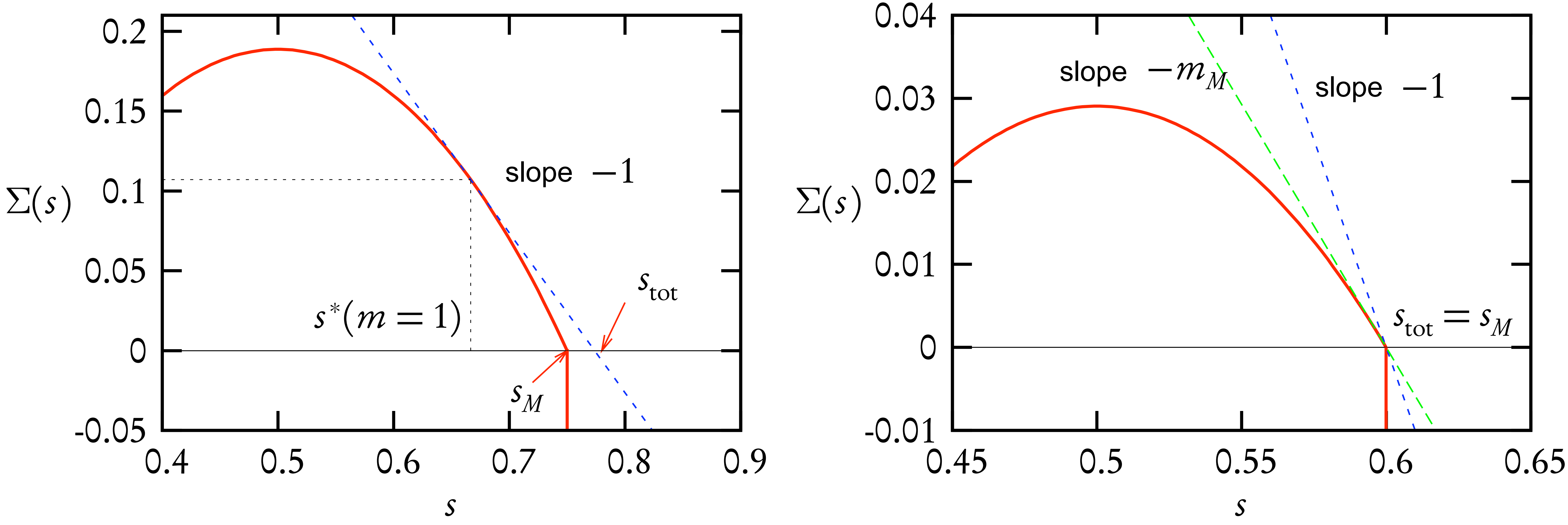
Mézard, Palassini & Rivoire, PRL '05
Krzakala, Montanari, Ricci-Tersenghi, Semerjian, Zdeborova, PNAS '07

$$e^{N\Psi(m)} = \sum_{\gamma} e^{mNs_\gamma + N\Sigma_s(s_\gamma)}$$

$$\Psi(m) = \max_s \left[\Sigma_s(s) + ms\right]$$

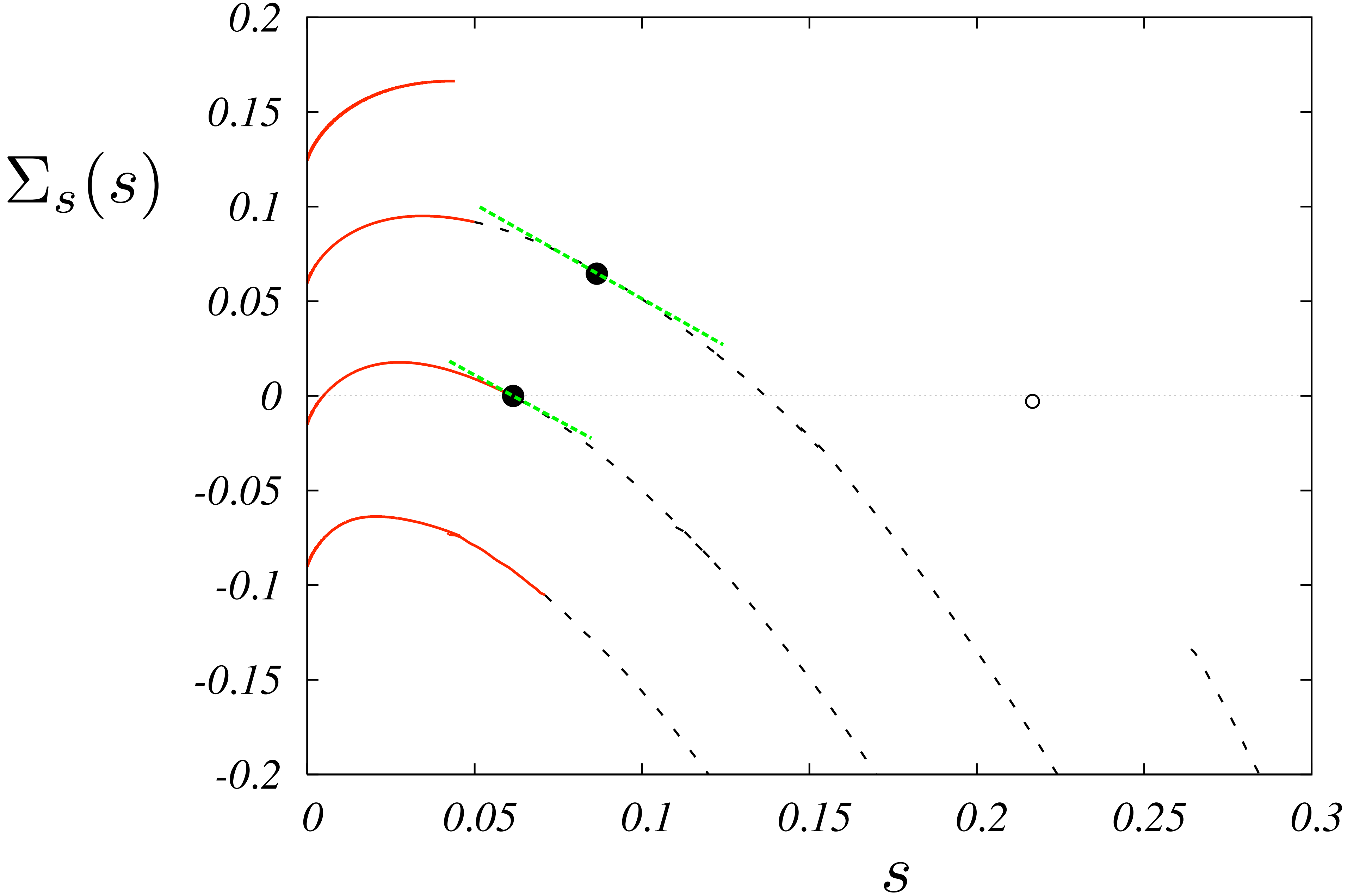
$m = 0 \longrightarrow$ most numerous clusters
(like with the energetic method)
$m = 1 \longrightarrow$ clusters dominating the measure
(if they exists, i.e. have $\Sigma > 0$)

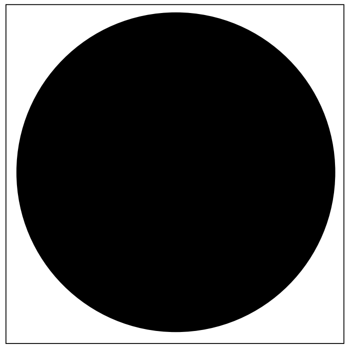
How to compute most probable states

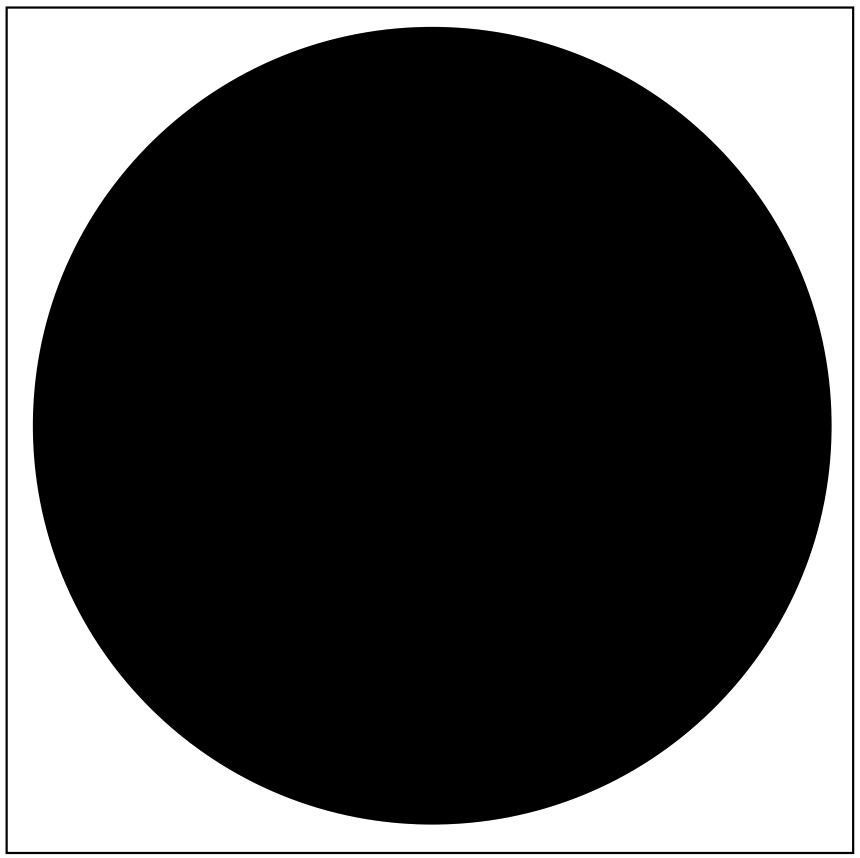


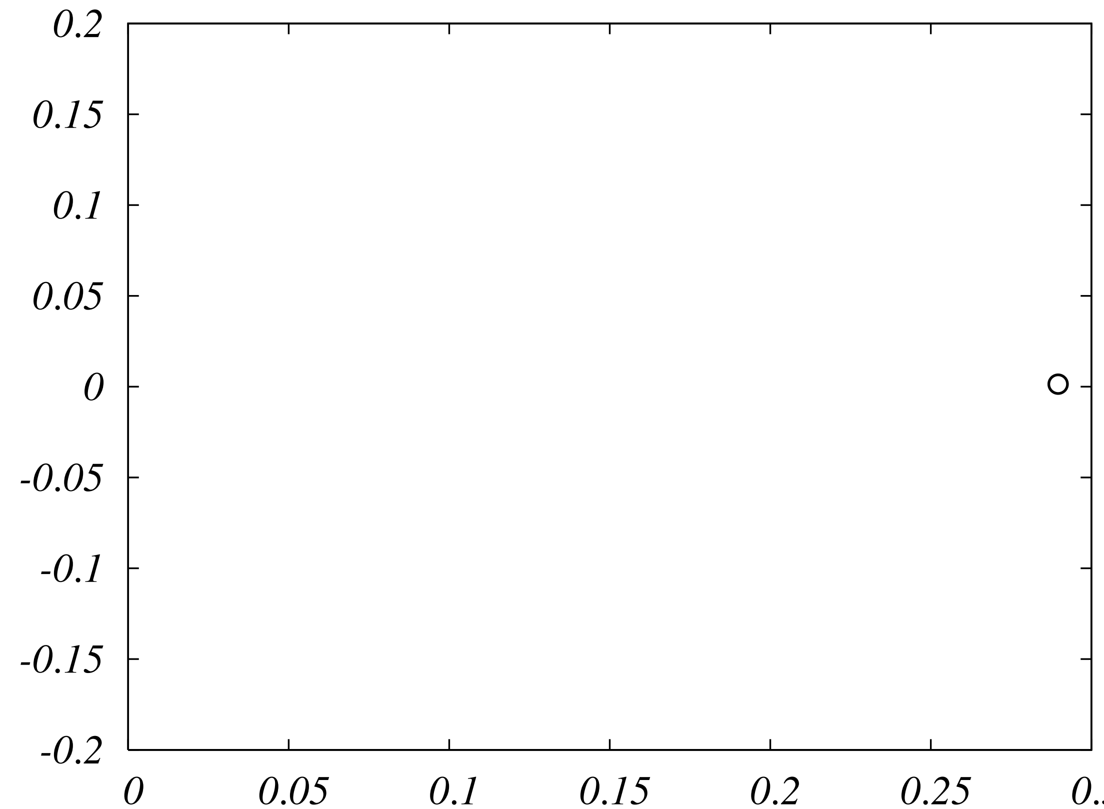
6-coloring random regular graphs

connectivity=17,18,19,20 (from top to bottom)



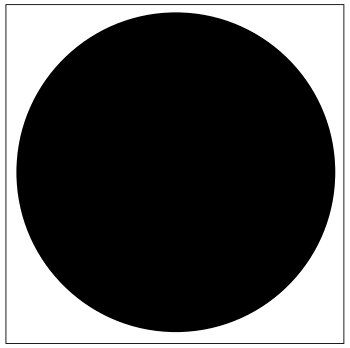


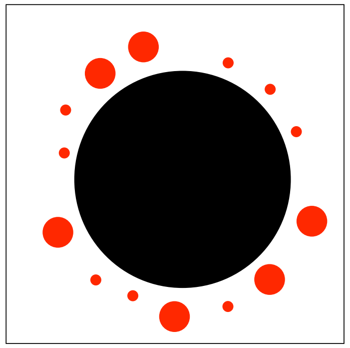


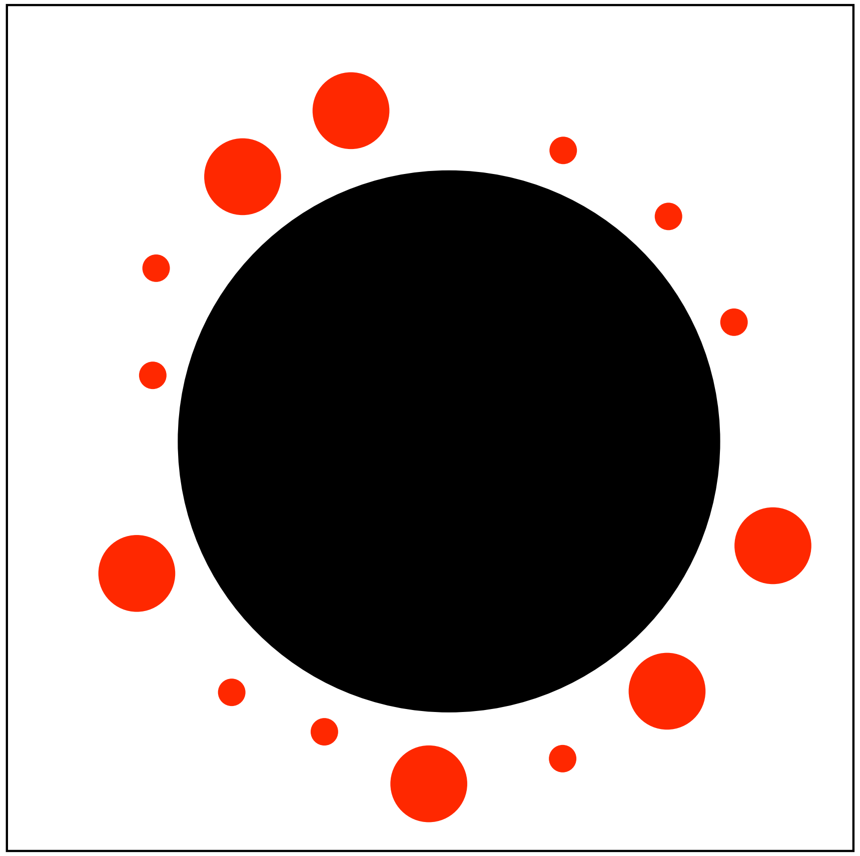


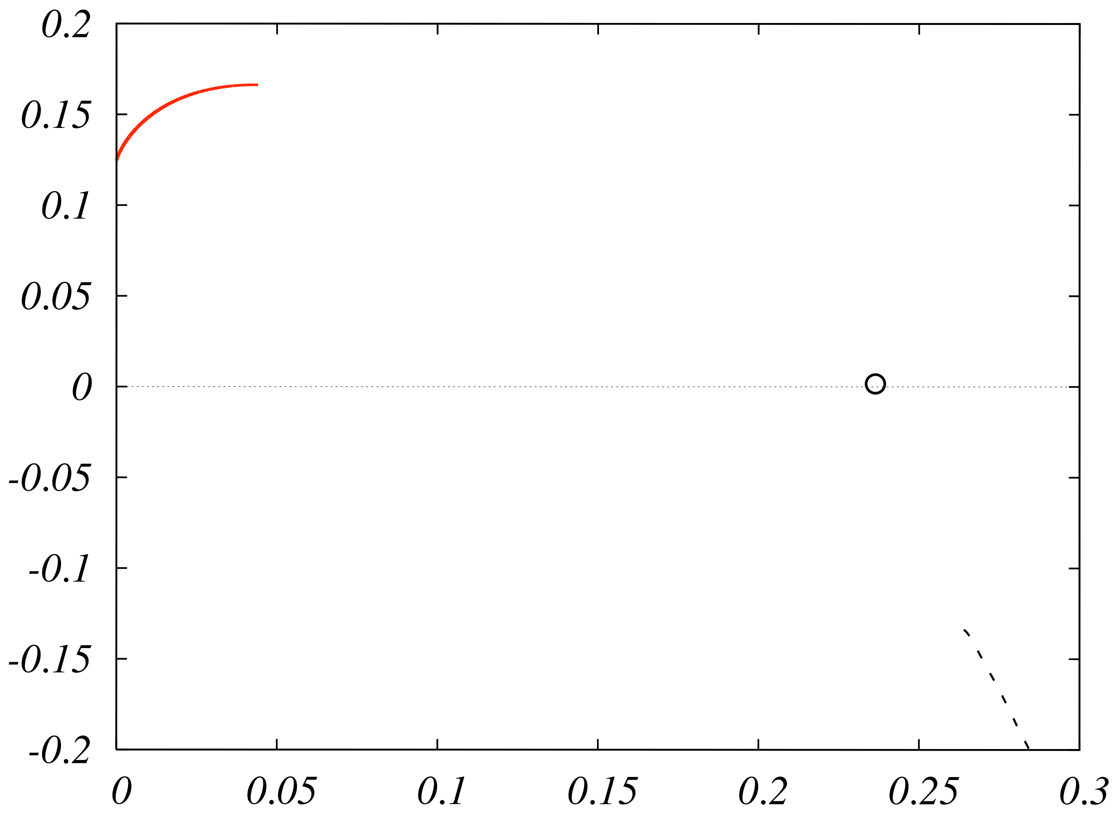
6 coloring of regular random graph

very low connectivity



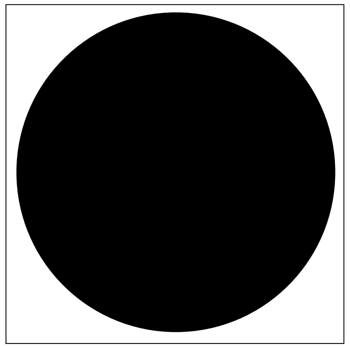


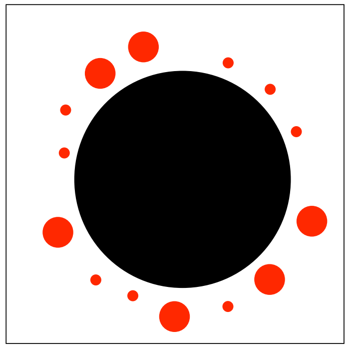


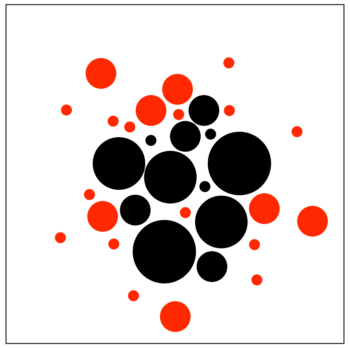


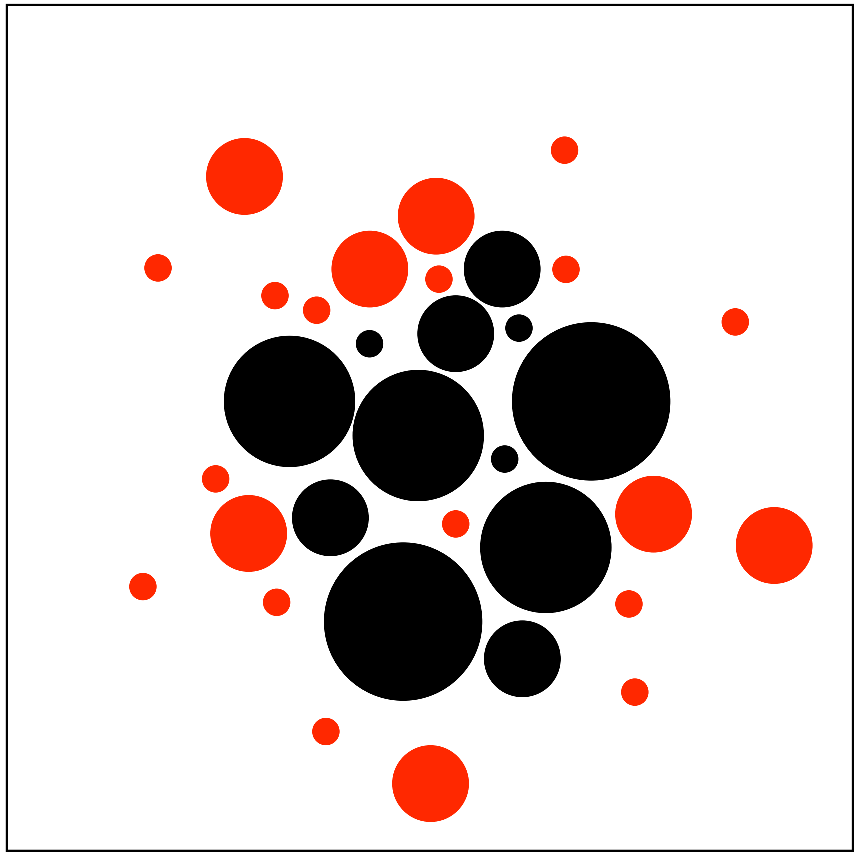
6 coloring of regular random graph

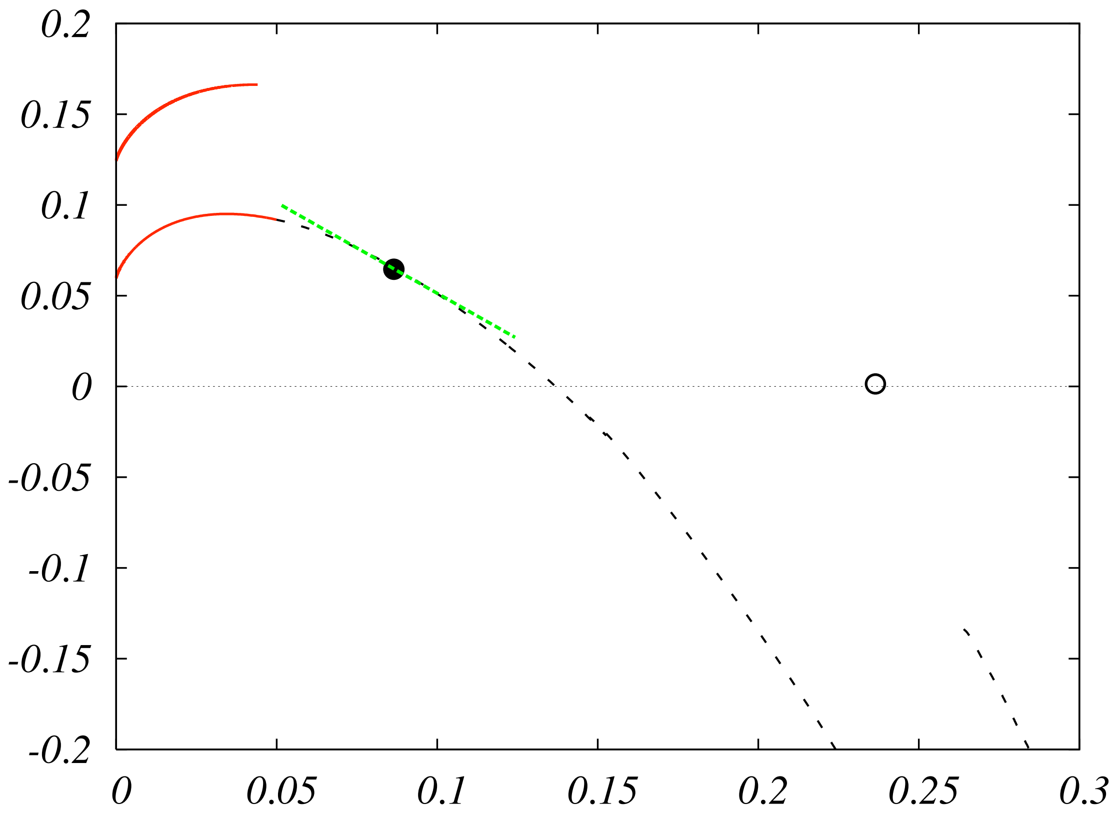
connectivity c=17





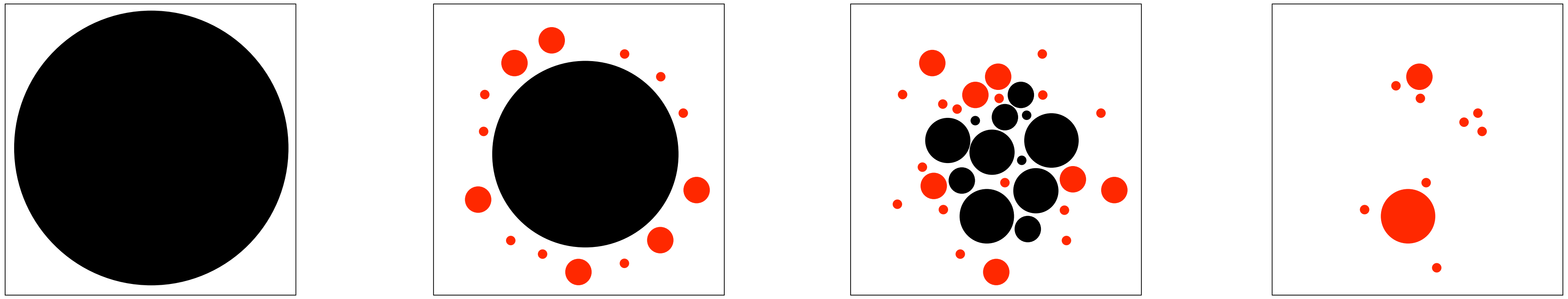


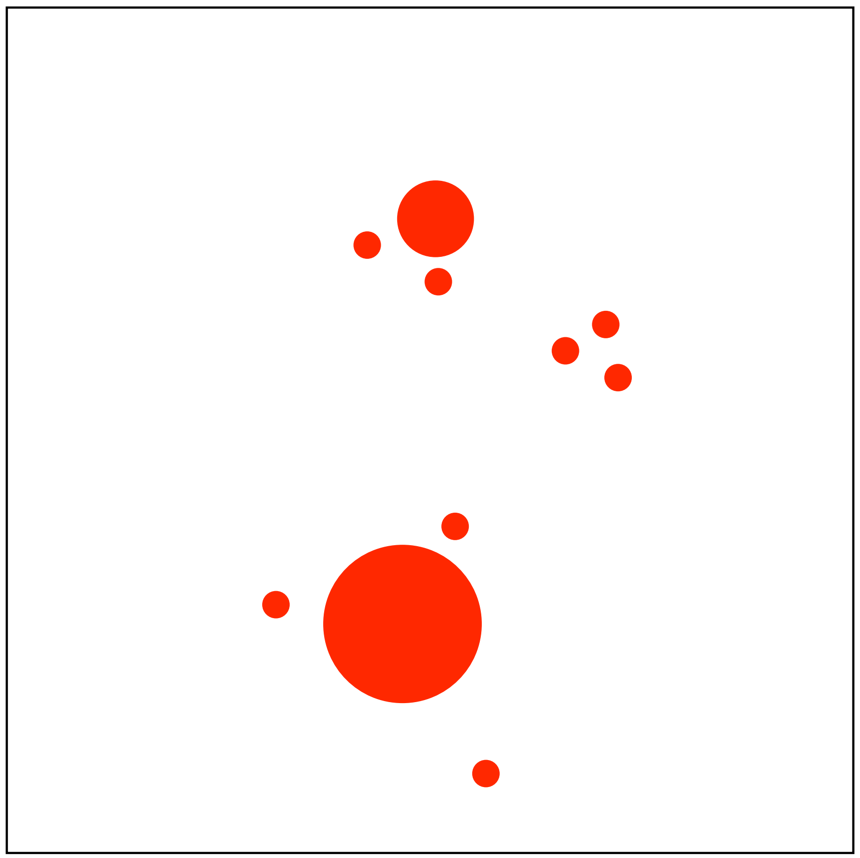


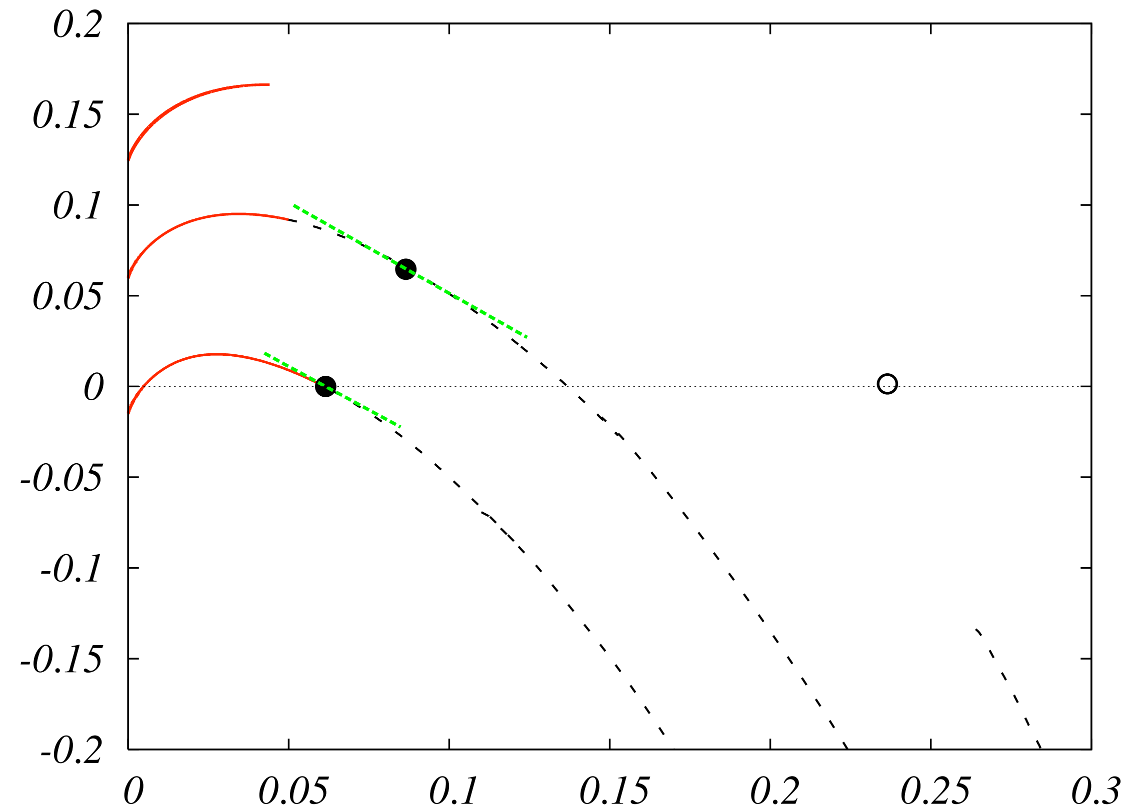


6 coloring of regular random graph

connectivity c=18

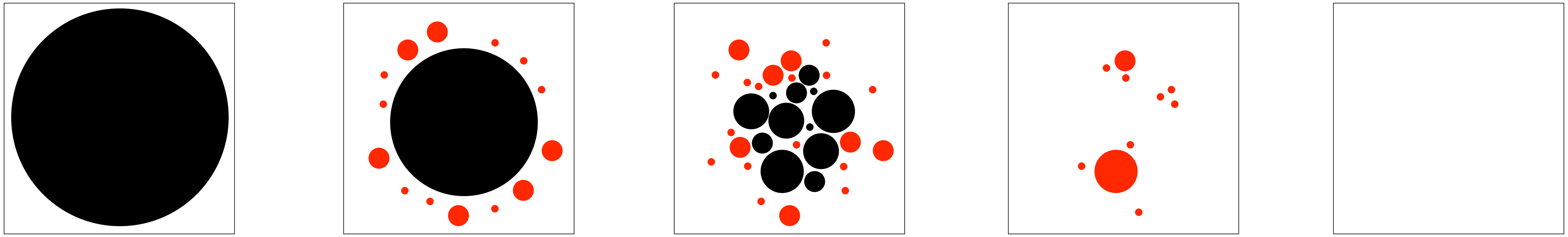


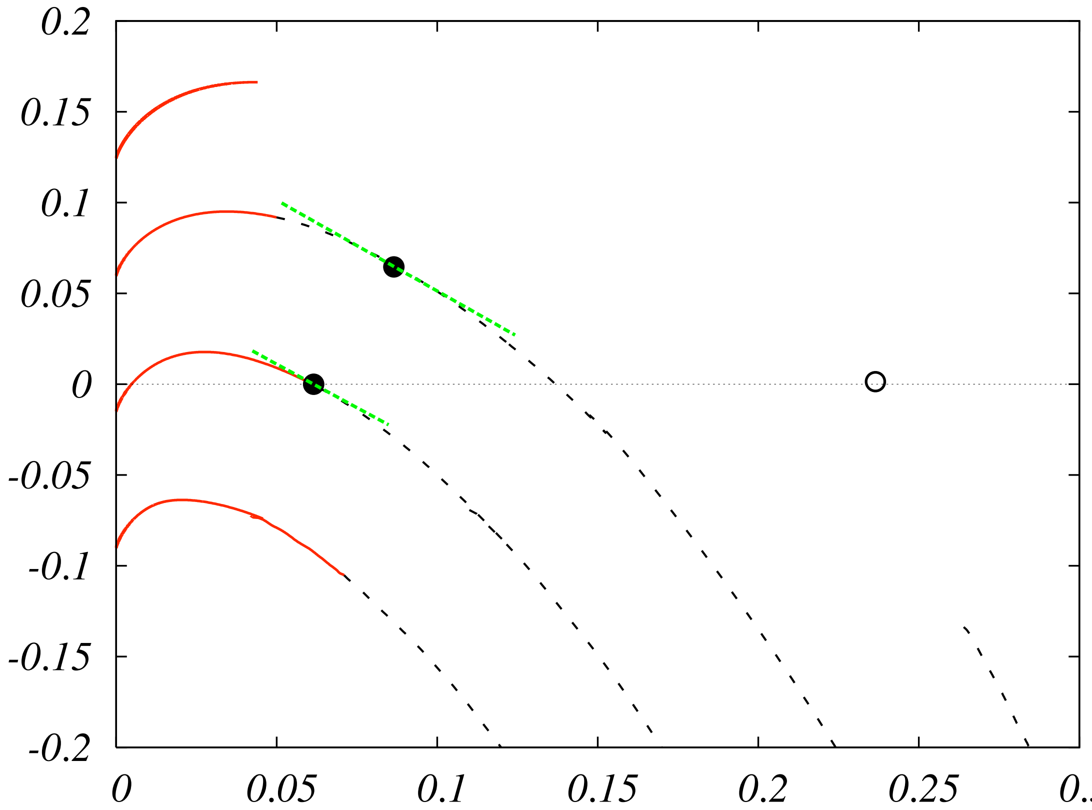




6 coloring of regular random graph

connectivity c=19





6 coloring of regular random graph

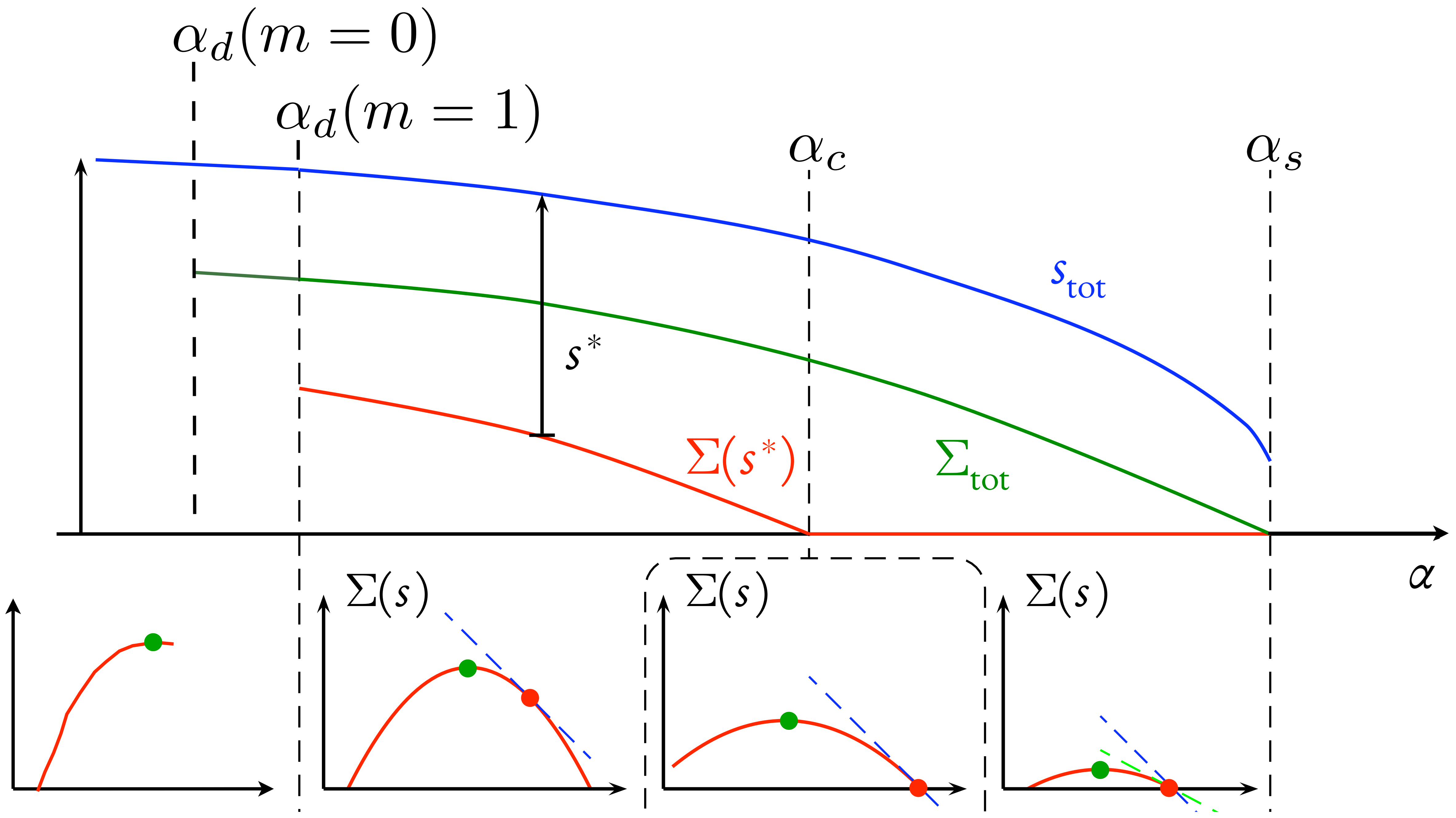
connectivity c=20

Random K-SAT revised

Krzakala, Montanari, Ricci-Tersenghi, Semerjian, Zdeborova, PNAS '07
Montanari, Ricci-Tersenghi, Semerjian, JSTAT '08

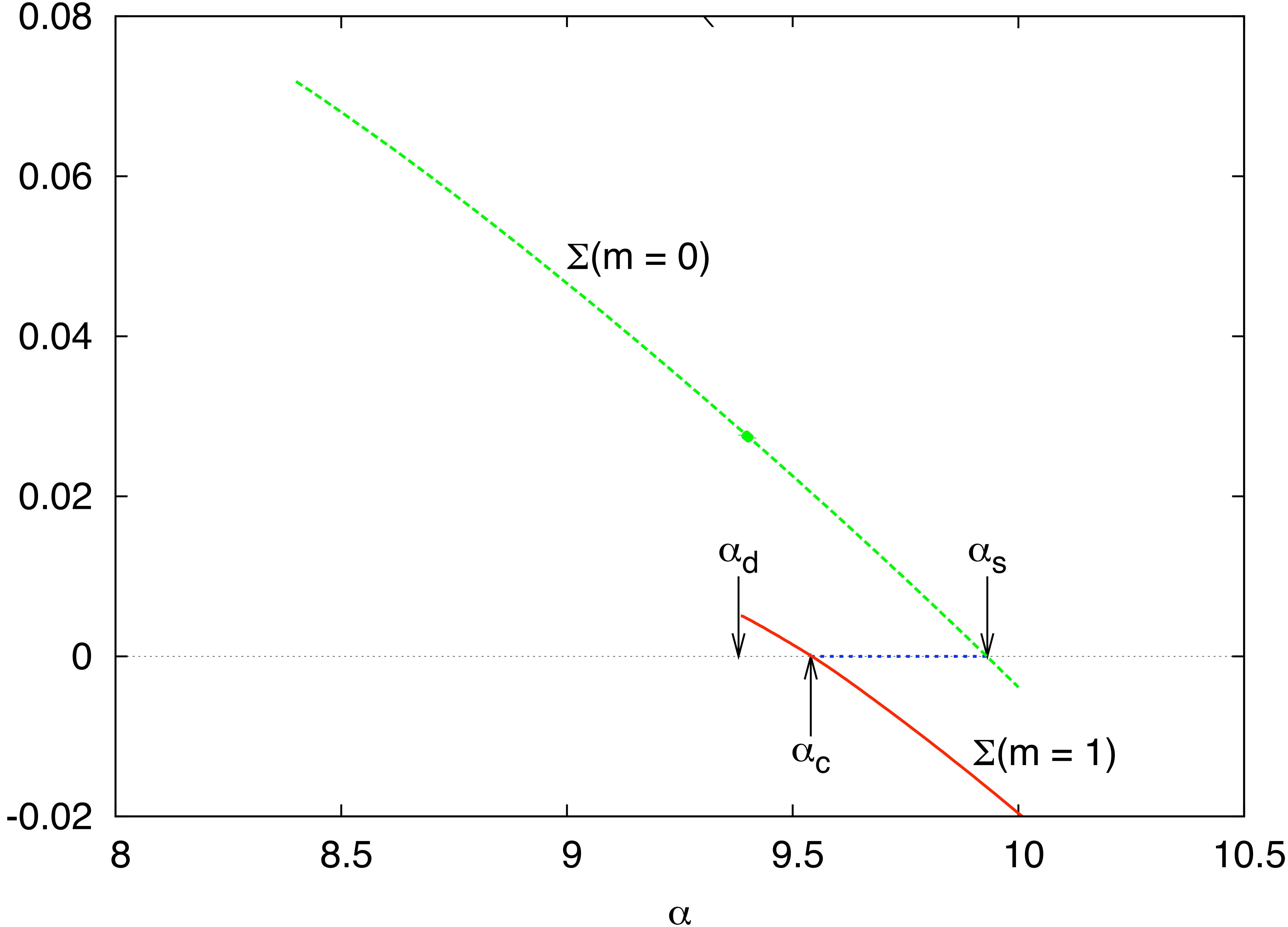
- We have computed $\Sigma_s(s,\alpha)$ for *K*=3 and *K*=4
- It is numerically very demanding: on each link there is a population of messages, to be updated and re-weighted at each iteration step, until convergence.
- For *m*=0 and *m*=1 equations simplify a lot
  - simpler messages (couple or triples) per link

Results



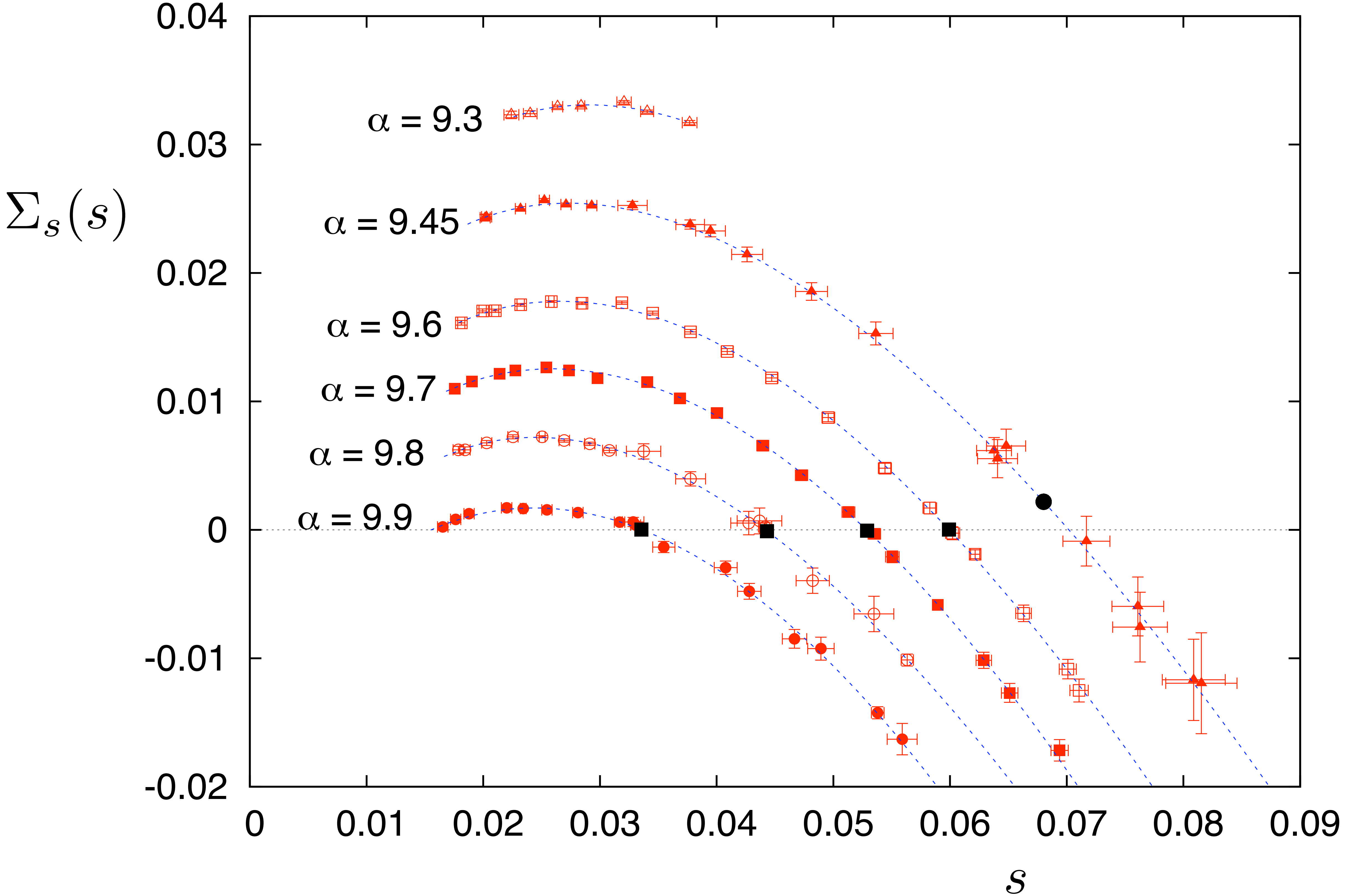
$\alpha_d(m=0)$
$\alpha_d(m=1)$
α_c
α_s
s_{tot}
s^*
$\Sigma(s^*)$
Σ_{tot}
α
$\Sigma(s)$
$\Sigma(s)$
$\Sigma(s)$

random 4-SAT



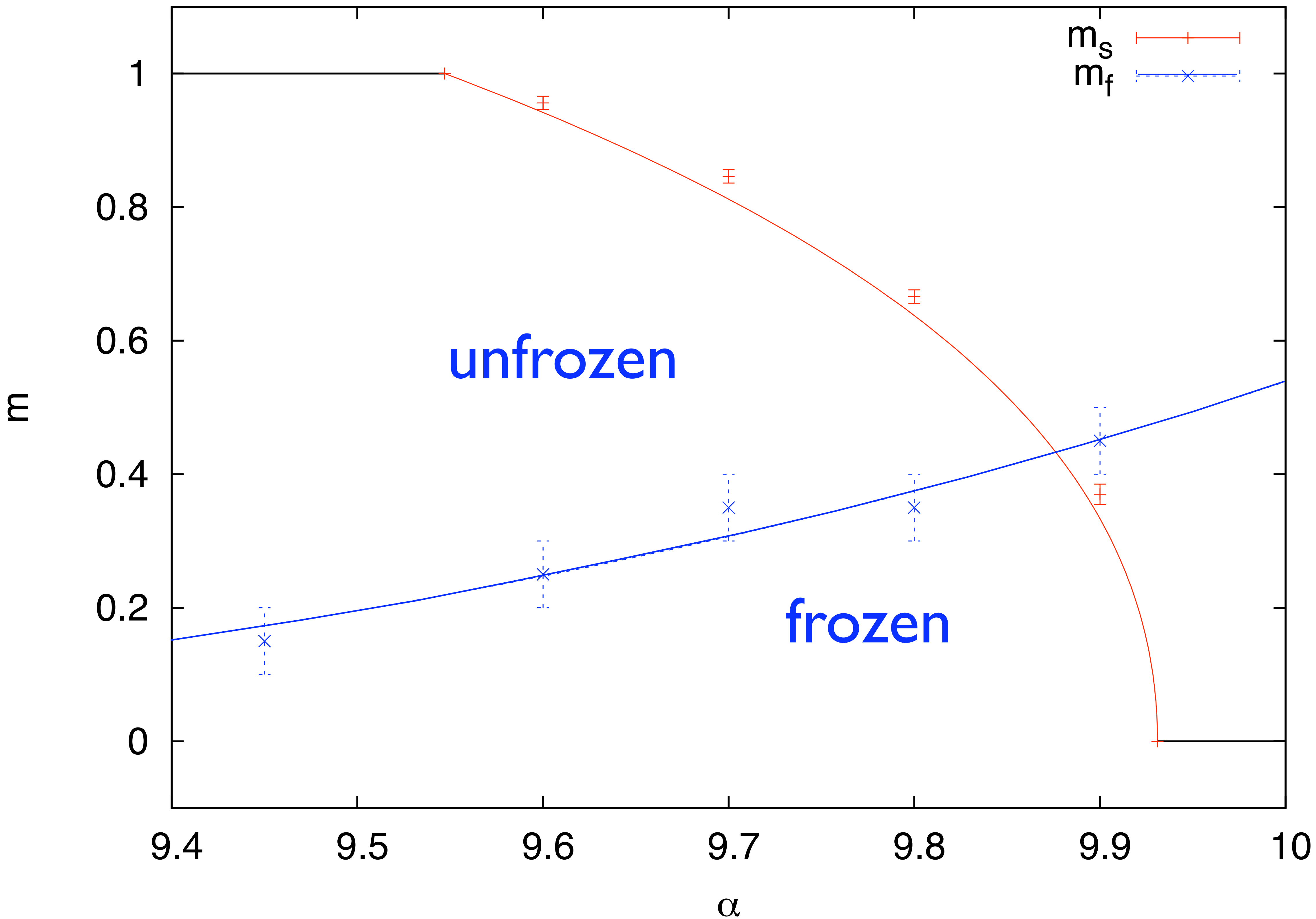
0.08
0.06
0.04
0.02
0
-0.02
8
8.5
9
9.5
10
10.5
α
Σ(m = 0)
Σ(m = 1)
α_d
α_s
α_c

random 4-SAT

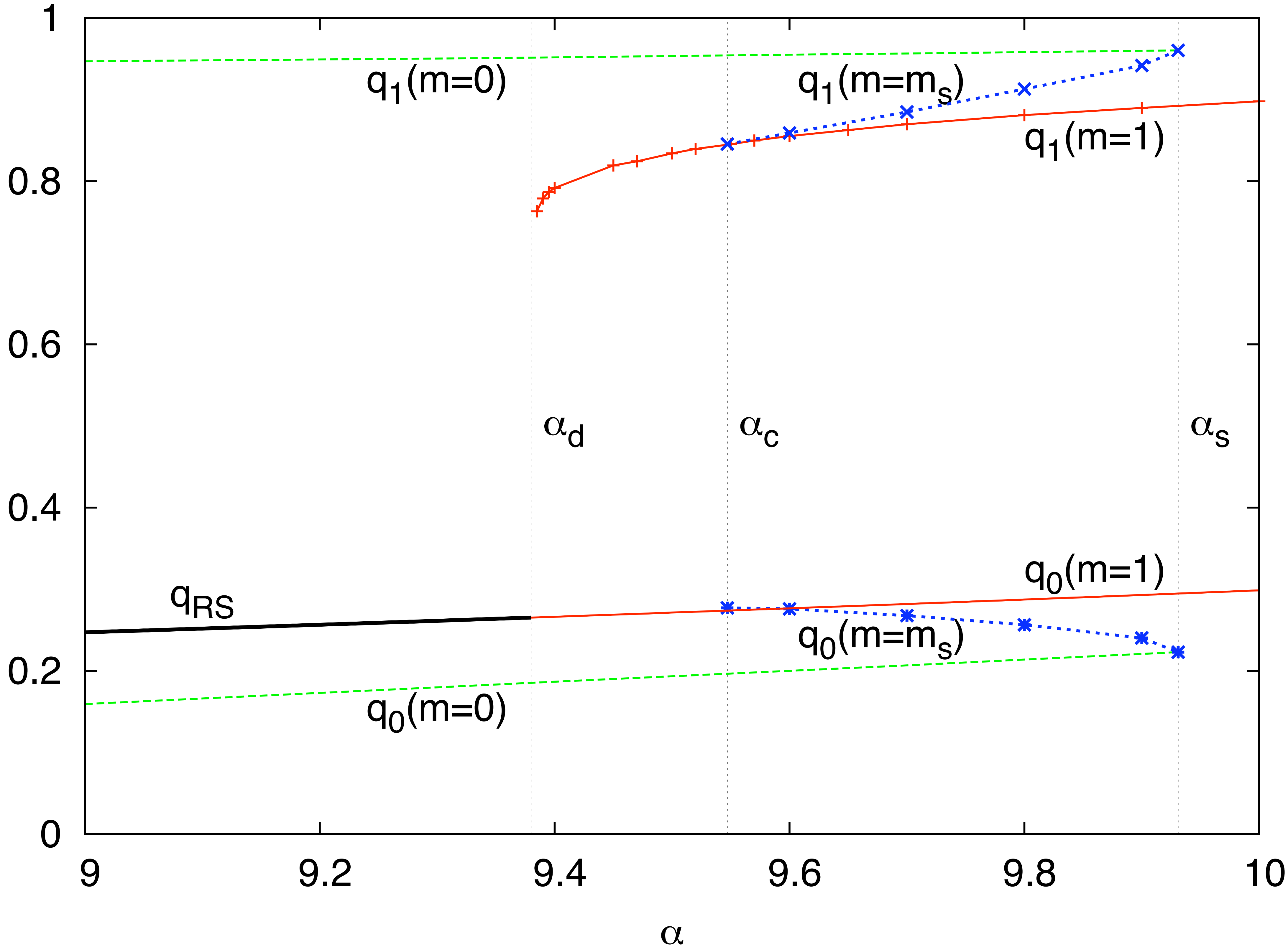


$\Sigma_s(s)$
s
α = 9.3
α = 9.45
α = 9.6
α = 9.7
α = 9.8
α = 9.9
0.04
0.03
0.02
0.01
0
-0.01
-0.02
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09

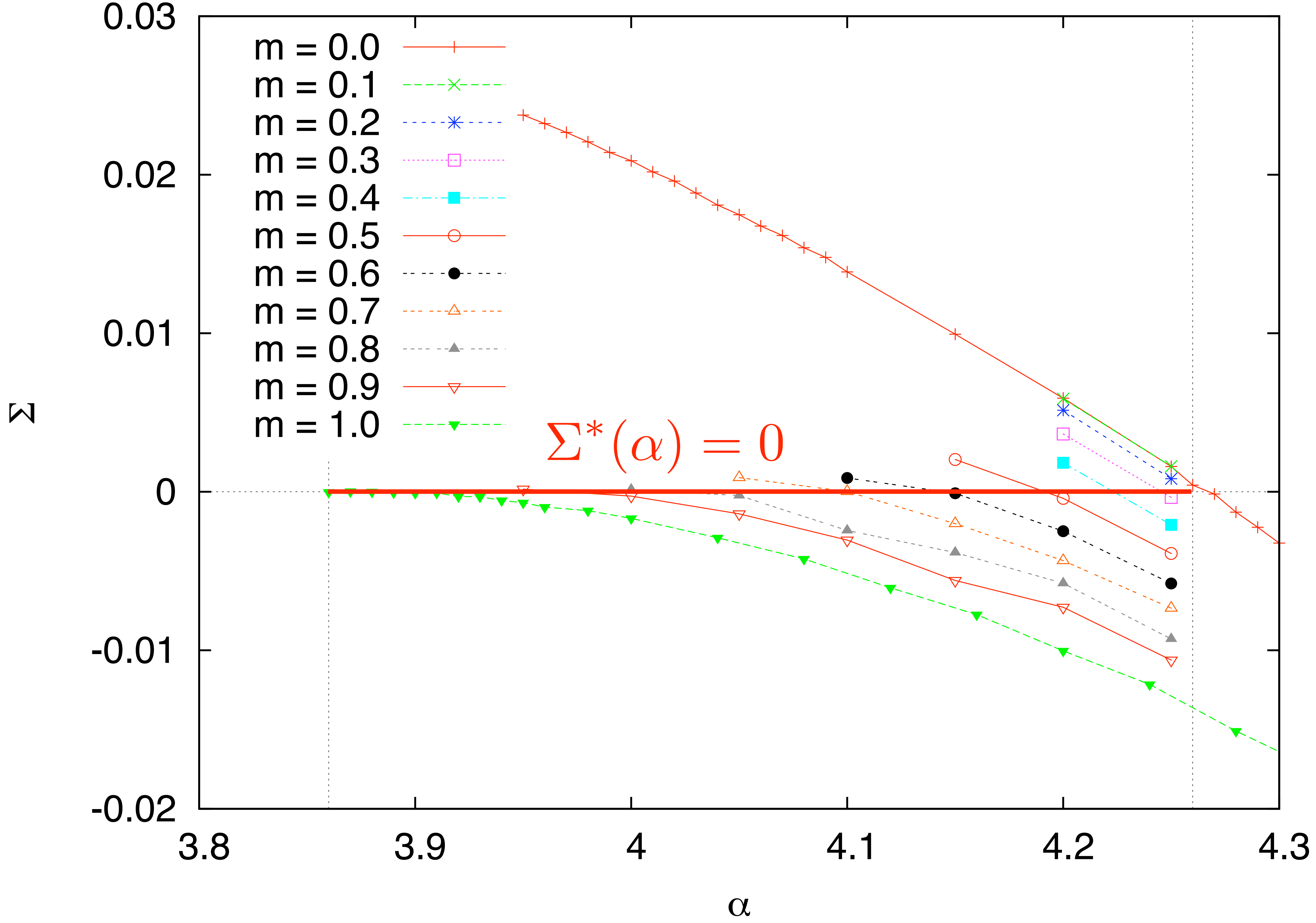
random 4-SAT



random 4-SAT

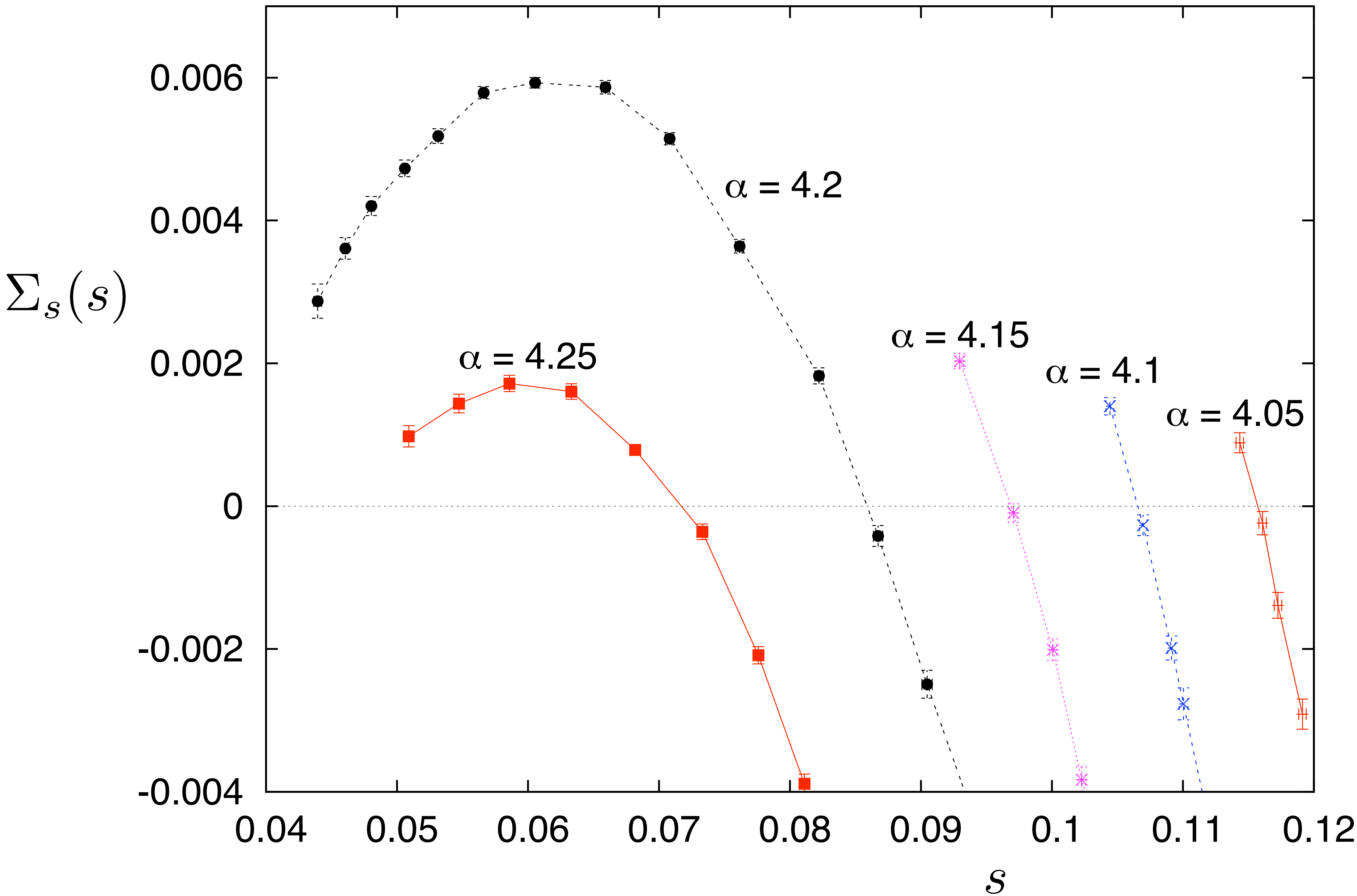

1
0.8
0.6
0.4
0.2
0
9
9.2
9.4
9.6
9.8
10
α
q1(m=0)
q1(m=ms)
q1(m=1)
αd
αc
αs
qRS
q0(m=1)
q0(m=ms)
q0(m=0)

random 3-SAT



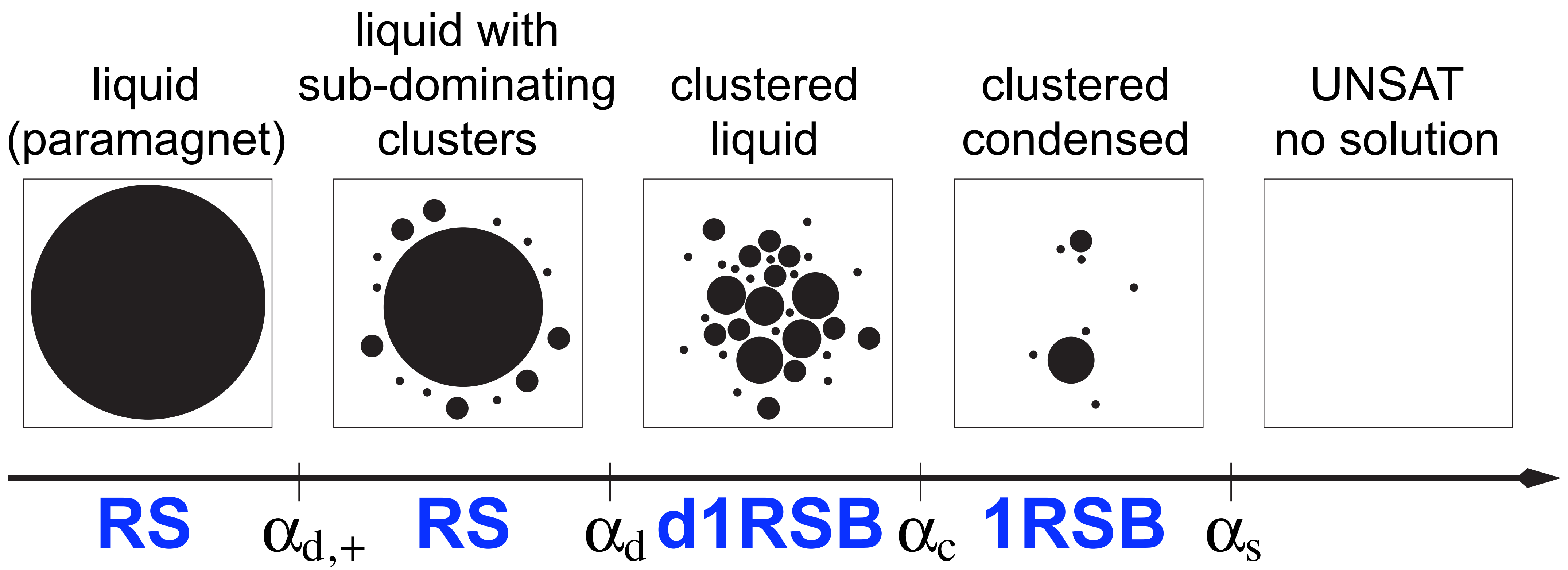
0.03
0.02
0.01
0
-0.01
-0.02
3.8
3.9
4
4.1
4.2
4.3
m = 0.0
m = 0.1
m = 0.2
m = 0.3
m = 0.4
m = 0.5
m = 0.6
m = 0.7
m = 0.8
m = 0.9
m = 1.0
$\Sigma^*(\alpha) = 0$
Σ
α

random 3-SAT



0.006
0.004
0.002
0
-0.002
-0.004
$\Sigma_s(s)$
α = 4.2
α = 4.25
α = 4.15
α = 4.1
α = 4.05
0.04
0.05
0.06
0.07
0.08
0.09
0.1
0.11
0.12
s

Summary



k	α_d	α_c	α_s	α_f
3	3.86	3.86	4.267	> 4.25
4	9.38	9.547	9.931	9.88
5	19.16	20.80	21.117	*
6	36.53	43.08	43.37	39.87

the largest for large *K*

$$\alpha_d \sim \frac{\log(k)}{k} 2^k$$

$$\alpha_c \sim \alpha_s \sim 2^k$$

Main open problems

- Stability of 1RSB solutions (technical point)
- Closing the gap between algorithmic threshold and SAT/UNSAT threshold
  - improvements in analysis of algorithms (decimation, reinforcement, etc.)
- Non-random structures, like those present in real world problems
  - beyond Bethe approximation (effects of loops, Kikuchi approximation, etc.)