Computing the Tutte polynomial in vertex-exponential time

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Outline

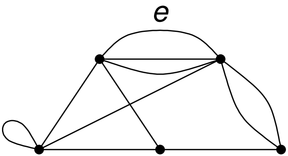
1. **Background and motivation**
   - Deletion–contraction invariants of graphs
   - The Tutte polynomial
   - Universality of the Tutte polynomial
2. **Computing the Tutte polynomial**
   - Deletion–contraction requires $\exp(\Omega(n \log n))$ worst case time
   - Our contribution: Algorithm with runtime $\exp(O(n))$
   - Assuming the Exponential Time Hypothesis, optimal up to constants[a]

a) R. Impagliazzo, R. Paturi, and F. Zane, Which problems have strongly exponential complexity? *J. Comput. Syst. Sci.* 63 (2001) 512–530

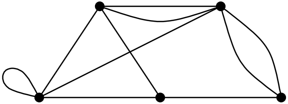
Deletion–contraction

- Let G be an undirected graph and let $e \in E$ be an edge
- Let $G \setminus e$ be the graph obtained from G by **deleting** e
- Let G/e be the graph obtained from G by **contracting** e; that is, by identifying the ends of e and then deleting e

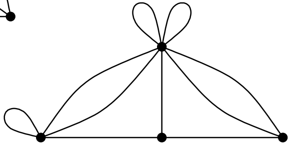
Example



G



$G \setminus e$



G/e

An example invariant: the chromatic polynomial

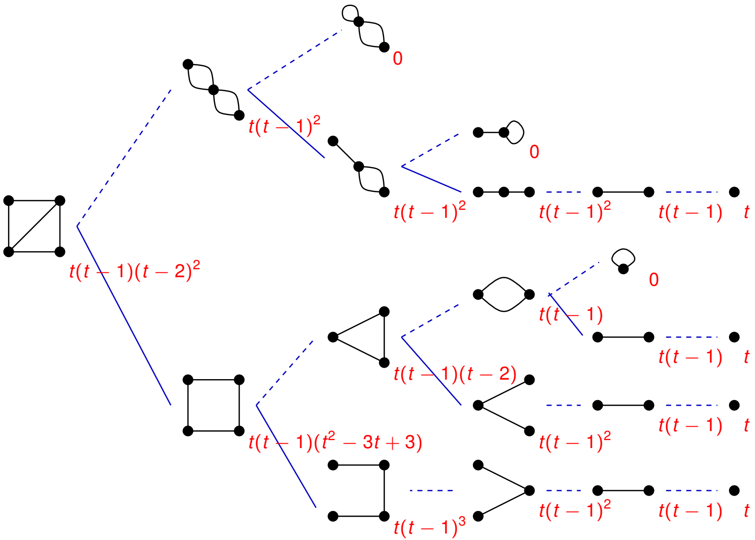
Let G be an undirected graph with n vertices

Denote by $P_G(t)$ the number of proper colorings of the vertices of G with $t = 1, 2, \ldots$ colors

Theorem
For all $e \in E$,

$$P_G(t) = \begin{cases} t^n & \textit{if } G \textit{ has no edges}; \\ 0 & \textit{if } e \textit{ is a loop}; \\ (t-1)P_{G/e}(t) & \textit{if } e \textit{ is a cut-edge}; \\ P_{G\setminus e}(t) - P_{G/e}(t) & \textit{otherwise} \end{cases}$$

Example: $P_G(t)$ with deletion–contraction



An example invariant: the reliability polynomial

Denote by $R_G(p)$ the probability that no connected component of G is disconnected if each edge is deleted independently with probability $0 \leq p \leq 1$

Theorem

For all $e \in E$,

$$R_G(p) = \begin{cases} 1 & \text{if } G \text{ has no edges;} \\ R_{G \setminus e}(p) & \text{if } e \text{ is a loop;} \\ (1-p)R_{G/e}(p) & \text{if } e \text{ is a cut-edge;} \\ pR_{G \setminus e}(p) + (1-p)R_{G/e}(p) & \text{otherwise} \end{cases}$$

An example invariant: the Ising–Potts partition function

For $q = 2, 3, \ldots$ and $b = \exp(-\beta) > 0$, let

$$Z_G(q,b) = \sum_{s: V \to \{1,2,\ldots,q\}} \exp\Big(\sum_{e \in E} \delta_e(s) \log b\Big)$$

where $\delta_e(s) = 1$ if s assigns the ends of e the same value; otherwise $\delta_e(s) = 0$

Theorem

For all $e \in E$,

$$Z_G(q,b) = \begin{cases} q^n & \text{if } G \text{ has no edges;} \\ bZ_{G \setminus e}(q,b) & \text{if } e \text{ is a loop;} \\ (q-1+b)Z_{G/e}(q,b) & \text{if } e \text{ is a cut-edge;} \\ Z_{G \setminus e}(q,b) + (b-1)Z_{G/e}(q,b) & \text{otherwise} \end{cases}$$

The (classical) Tutte polynomial

- Let G be an undirected graph with n vertices, m edges, and c connected components

- The **Tutte polynomial**[a,b] of G is the two-variable polynomial

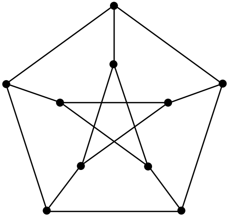
$$T_G(x,y) = \sum_{F \subseteq E} (x-1)^{c(F)-c}(y-1)^{c(F)+|F|-n},$$

where $c(F)$ is the number of connected components in the spanning subgraph of G with edge set F

a) W. T. Tutte, A ring in graph theory, *Proc. Cambridge Philos. Soc.* 43 (1947) 26–40
b) W. T. Tutte, A contribution to the theory of chromatic polynomials, *Canadian J. Math.* 6 (1954) 80–91
c) A. D. Sokal, The multivariate Tutte polynomial (alias Potts model) for graphs and matroids, *Surveys in Combinatorics, 2005*, Cambridge University Press, 2005, pp. 173–226

Example: the Petersen graph

	1	y	y^2	y^3	y^4	y^5	y^6
1	0	36	84	75	35	9	1
x	36	168	171	65	10		
x^2	120	240	105	15			
x^3	180	170	30				
x^4	170	70					
x^5	114	12					
x^6	56						
x^7	21						
x^8	6						
x^9	1						



The Tutte polynomial with deletion–contraction

Theorem

For all $e \in E$,

$$T_G(x,y) = \begin{cases} 1 & \text{if } G \text{ has no edges;} \\ yT_{G\setminus e}(x,y) & \text{if } e \text{ is a loop;} \\ xT_{G/e}(x,y) & \text{if } e \text{ is a cut-edge;} \\ T_{G\setminus e}(x,y) + T_{G/e}(x,y) & \text{otherwise} \end{cases}$$

Universality of the Tutte polynomial

Theorem (Recipe Theorem)[a,b]

Assume that f is a function from graphs to the multivariate polynomial ring $\mathbb{Z}[\alpha, \beta, \gamma, \lambda, \mu]$ such that, for all graphs G and $e \in E$,

$$f(G) = \begin{cases} \alpha^n & \text{if } G \text{ has no edges;} \\ \beta f(G \backslash e) & \text{if } e \text{ is a loop;} \\ \gamma f(G/e) & \text{if } e \text{ is a cut-edge;} \\ \lambda f(G \backslash e) + \mu f(G/e) & \text{otherwise.} \end{cases}$$

Then

$$f(G) = \alpha^c \lambda^{c+m-n} \mu^{n-c} T_G(\gamma\mu^{-1}, \beta\lambda^{-1})$$

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a) T. H. Brylawski, A decomposition for combinatorial geometries, *Trans. Amer. Math. Soc.* 171 (1972) 235–282

b) J. G. Oxley, D. J. A. Welsh, The Tutte polynomial and percolation, Graph Theory and Related Topics (J. A. Bondy, U. S. R. Murty, Eds.), Academic Press, 1979, pp. 329–339

Corollary (Chromatic polynomial)

$P_G(t) = (-1)^{n-c} t^c T_G(1-t, 0)$

Proof.

Use the recipe $\alpha = t$, $\beta = 0$, $\gamma = t - 1$, $\lambda = 1$, $\mu = -1$ $\square$

Corollary (Reliability polynomial)

$R_G(p) = p^{c+m-n}(1-p)^{n-c} T_G(1, 1/p)$

Proof.

Use the recipe $\alpha = 1$, $\beta = 1$, $\gamma = 1 - p$, $\lambda = p$, $\mu = 1 - p$ $\square$

Corollary (Ising–Potts partition function)

$Z_G(q, b) = q^c (b-1)^{n-c} T_G(q/(b-1) + 1, b)$

Proof.

Use the recipe $\alpha = q$, $\beta = b$, $\gamma = q - 1 + b$, $\lambda = 1$, $\mu = b - 1$ $\square$

Computing the Tutte polynomial

- Given G, computing $T_G(x, y)$ is a #P-hard problem[a,b]
- It is also very difficult to approximate $T_G(x, y)$ on most points (x, y)[c]
- Deletion–contraction computes $T_G(x, y)$ for a connected G in time $\tau(G)n^{O(1)}$
- The number of spanning trees $\tau(G) \leq n^{n-2}$, with equality for complete graphs

a) F. Jaeger, D. L. Vertigan, D. J. A. Welsh, On the computational complexity of the Jones and Tutte polynomials, *Math. Proc. Camb. Phil. Soc.* 108 (1990) 35–53

b) D. J. A. Welsh, *Complexity: Knots, Colourings and Counting*, Cambridge University Press, 1993

c) L. A. Goldberg, M. Jerrum, Inapproximability of the Tutte polynomial, Proceedings of the 39th Annual ACM Symposium on Theory of Computing (San Diego, CA, June 11–13, 2007), Association for Computing Machinery, 2007, pp. 459–468

Our contribution[a]

- An algorithm with $2^n n^{O(1)}$ running time
- Three ideas:
  1. Suffices to enumerate spanning subgraphs of G by
     - number of connected components and
     - number of edges
  2. Enumerate via recursion over induced subgraphs
  3. "Fast subset convolution" to expedite the recursion

a) A. Björklund, T. Husfeldt, P.K., M. Koivisto; arxiv:0711.2585

First idea: enumerate spanning subgraphs

- Denote by $s_{k,l}(G)$ the number of spanning subgraphs of G with
  - k connected components, and
  - l edges
- Collecting terms,

$$
\begin{aligned}
T_G(x, y) &= \sum_{F \subseteq E} (x-1)^{c(F)-c} (y-1)^{c(F)+|F|-n} \\
&= \sum_{k=1}^{n} \sum_{l=0}^{m} s_{k,l}(G)(x-1)^{k-c}(y-1)^{k+l-n}
\end{aligned}
$$

- Thus, it is trivial to compute $T_G(x, y)$ given $s_{k,l}(G)$ for all k, l

Second idea: recursion on induced subgraphs

- Denote by $G[W]$ the subgraph induced by $W \subseteq V$ in G
- Suppose we know $s_{k,l}(G[U])$ for all $\emptyset \subsetneq U \subsetneq W$ and all k, l
- A two-step recursion solves $s_{k,l}(G[W])$ for all k, l:

1. For $k = 2, 3, \ldots, n(G[W])$ and $l = 0, 1, \ldots, m(G[W])$, set

$$s_{k,l}(G[W]) = \frac{1}{k} \sum_{t=0}^{m(G[W])} \sum_{\emptyset \subsetneq U \subsetneq W} s_{1,t}(G[U]) s_{k-1,l-t}(G[W \setminus U])$$

2. For $k = 1$ and $l = 0, 1, \ldots, m(G[W])$, set

$$s_{1,l}(G[W]) = \binom{m(G[W])}{l} - \sum_{k=2}^{n(G[W])} s_{k,l}(G[W])$$

Third idea: fast subset convolution

- A direct evalution of the recursion takes $\Omega(3^n)$ time
- The bottleneck is

$$s_{k,l}(G[W]) = \frac{1}{k} \sum_{t=0}^{m(G[W])} \sum_{\emptyset \subsetneq U \subsetneq W} s_{1,t}(G[U]) s_{k-1,l-t}(G[W \setminus U])$$

- This can be expedited to $2^n n^{O(1)}$ time using fast subset convolution[a]

$$(f * g)(W) = \sum_{U \subseteq W} f(U) g(W \setminus U)$$

a) A. Björklund, T. Husfeldt, P. K., M. Koivisto, Fourier meets Möbius: fast subset convolution, Proceedings of the 39th Annual ACM Symposium on Theory of Computing (San Diego, CA, June 11–13, 2007), Association for Computing Machinery, New York, 2007, pp. 67–74

Our result in more detail

Denote by $\sigma(G) \leq 2^n$ the number of connected vertex sets in G

Theorem (Björklund, Husfeldt, K., Koivisto)[a]

The Tutte polynomial of an n-vertex graph G can be computed

(a) *in time and space* $\sigma(G)n^{O(1)}$*;*

(b) *in time* $3^n n^{O(1)}$ *and polynomial space; and*

(c) *in time* $3^{n-s}2^s n^{O(1)}$ *and space* $2^s n^{O(1)}$ *for any integer s,* $0 \leq s \leq n$.

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a) A. Björklund, T. Husfeldt, P.K., M. Koivisto; arxiv:0711.2585