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A minimal model for the onset of congestion in networks

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Introduction

Empirical facts:

- experimental measures of congestion in communication networks are very difficult

- TCP/IP protocols are conceived in such a way to avoid the onset of congestion

- computer scientists are interested in the optimization of the routing algorithm

- statistical physicists are interested in understanding how congestion can occur

Introduction

Numerical simulations (main ingredients):

- networked structure (e.g. Internet maps)
- define an information exchange process
  (packets are created at rate p and removed when a given destination is reached)
- define a routing protocol (static, dynamic, ...)
- a queue in each node (limited or unlimited capacity)

➡ define an ORDER PARAMETER to measure congestion

$$\rho(p) = \lim_{t \to \infty} \frac{N(t+\tau) - N(t)}{\tau p}$$

Free-flow vs. Congested phase

First attempts: Ohira & Sawatari,PRE 58 198 (1998)
Takayasu et al. (1996-2000)

Breakthrough: Echenique, Gomez-Gardenes, & Moreno, EPL 71, 325 (2005)

- traffic on the Internet map (AS level)

- mixed routing rule: shortest path + congestion avoidance

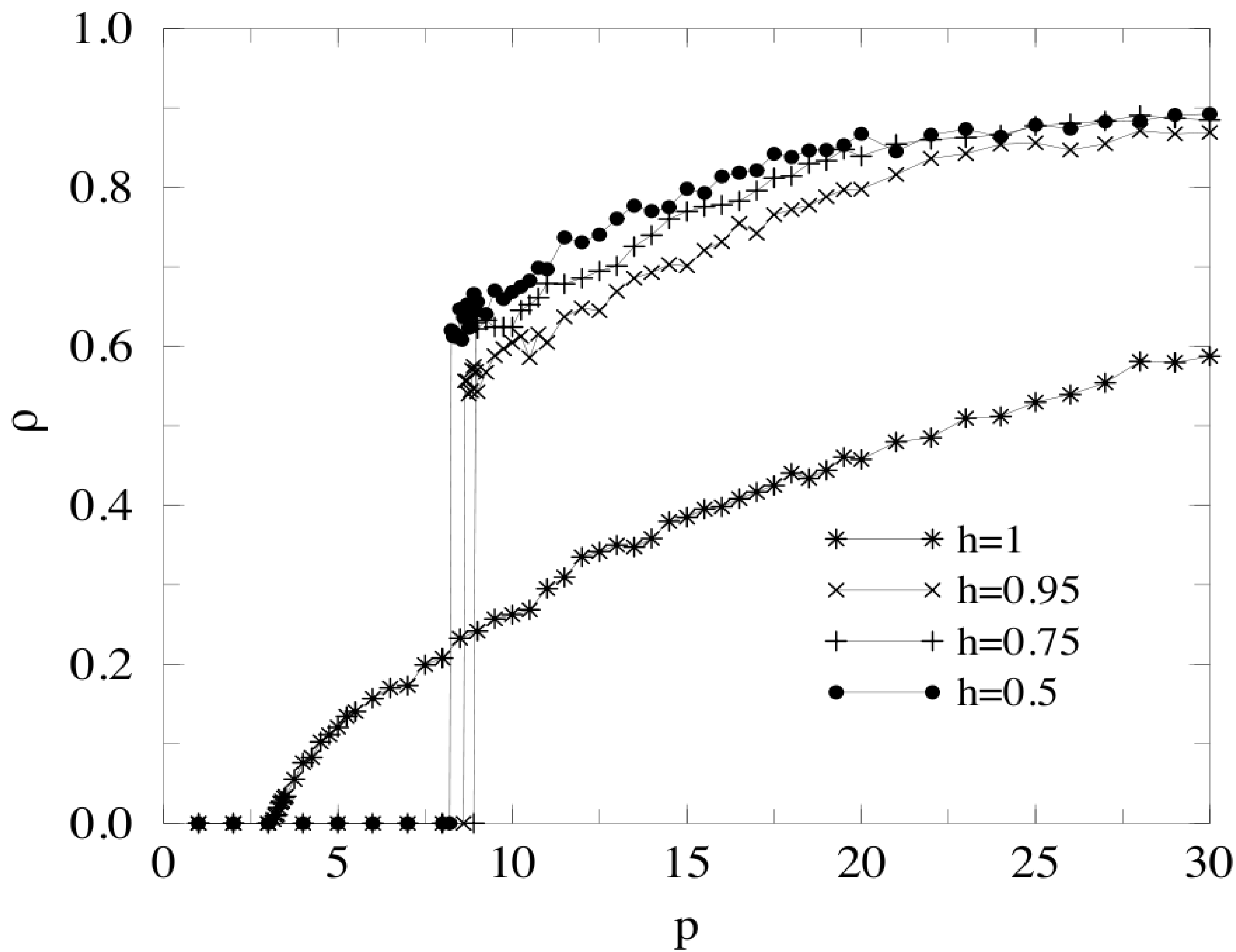
$$\delta_i(j \rightarrow k) = h\, d_{ik} + (1-h)\, n_i$$

- each time step every node sends one packet to a neighbor

⇒ *Phase Transition from free-flow to congested phase*

continuous if h = 1, discontinuous if h < 1

Free-flow vs. Congested phase



Free-flow vs. Congested phase

1) Existence of congested phase:

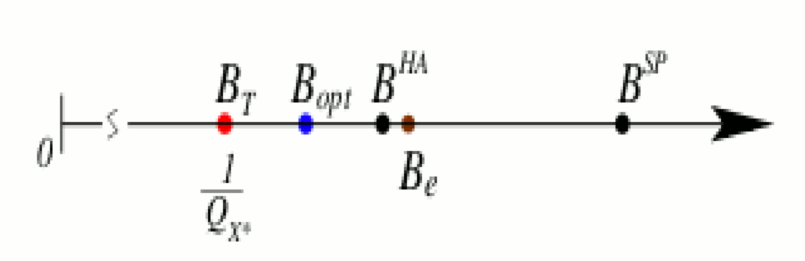
number of packets arriving at node *i* is on average $\frac{p B_i}{N-1}$

where B is the betweenness centrality $B_i = \sum_{s \neq t} \frac{\sigma_{st}(i)}{\sigma_{st}}$

one node is congested if $\frac{p B_i}{N-1} > 1$ → $p_c = \frac{N-1}{B_{max}}$

Sreenivasan et al (Phys. Rev. E 75, 036105 (2007)):
existence of "structural bottlenecks" independent of the routing process

→ the system will eventually be congested !!



Free-flow vs. Congested phase

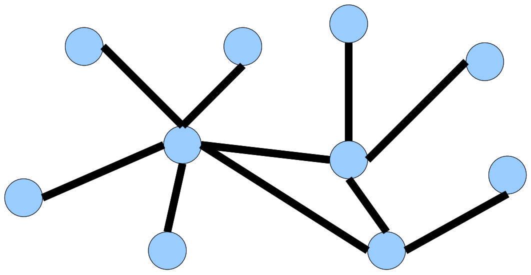
2) Continuous phase transition:

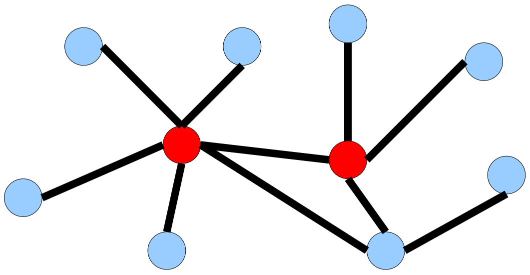
on hierarchical trees, Guimerà et al. PRL 86, 3196 (2001) showed that above p_c,

$$\rho = 1 - \frac{p_c}{p}$$

3) Discontinuous phase transition:

is due to a cooperative effect, packets move along “allowed paths” like in kinetically constrained models





when paths do not percolate anymore → jump into a jammed phase

Our model

A minimal model of routing:

- can be studied analytically,
- all main features defining congestion scenario

Rules:

• local routing (random walks + congestion avoidance bias)

• no source and destination, but creation rate p and deletion rate μ (deletion does not apply when packets are in the queue)

• node capacity n^*

Goal: 1) recursive equation on a given graph

2) average behavior for an ensemble of graphs

Equations for single graphs (I)

$$w(n_i \to n_i+1) = p + (1-\mu)(1-x_i\eta)\sum_{j\in i}\frac{1-\delta_{n_j,0}}{k_j} \qquad \text{for} \quad n_i \geq 0$$

$$w(n_i \to n_i-1) = 1-\frac{\eta}{k_i}\sum_{j\in i} x_j \qquad \text{for} \quad n_i > 0 \qquad x_i = \theta(n_i - n_i^*)$$

Remark:

if we assume that node probability distributions are factorizable, then, in the stationary state, $P_i(n)$ are (double) exponential

➡ Node state is fully defined by only two variables in [0,1]

$$\begin{cases} q_i = Prob\{n_i = 0\} = P_i(0) \\ \chi_i = Prob\{x_i = 1\} \end{cases}$$

Equations for single graphs (II)

Equation for the queue length:

$$\dot{n}_i = p + (1-\mu)(1-\eta\chi_i)\sum_{j\in i}\frac{1-q_j}{k_j} - (1-q_i)\left[1-\frac{\eta}{k_i}\sum_{j\in i}\chi_j\right]$$

Limit for $n^* \to \infty$, *three classes of nodes:*

- *congested* $\chi_i = 1,\ q_i = 0$ → $\dot{n}_i > 0$
- *free* $\chi_i = 0,\ \dot{n}_i = 0$ → $q_i = Q_i(\vec{\chi}, \vec{q})$
- *fickle* $q_i = 0,\ \dot{n}_i = 0$ → $\chi_i = C_i(\vec{\chi}, \vec{q})$

Equations for single graphs (III)

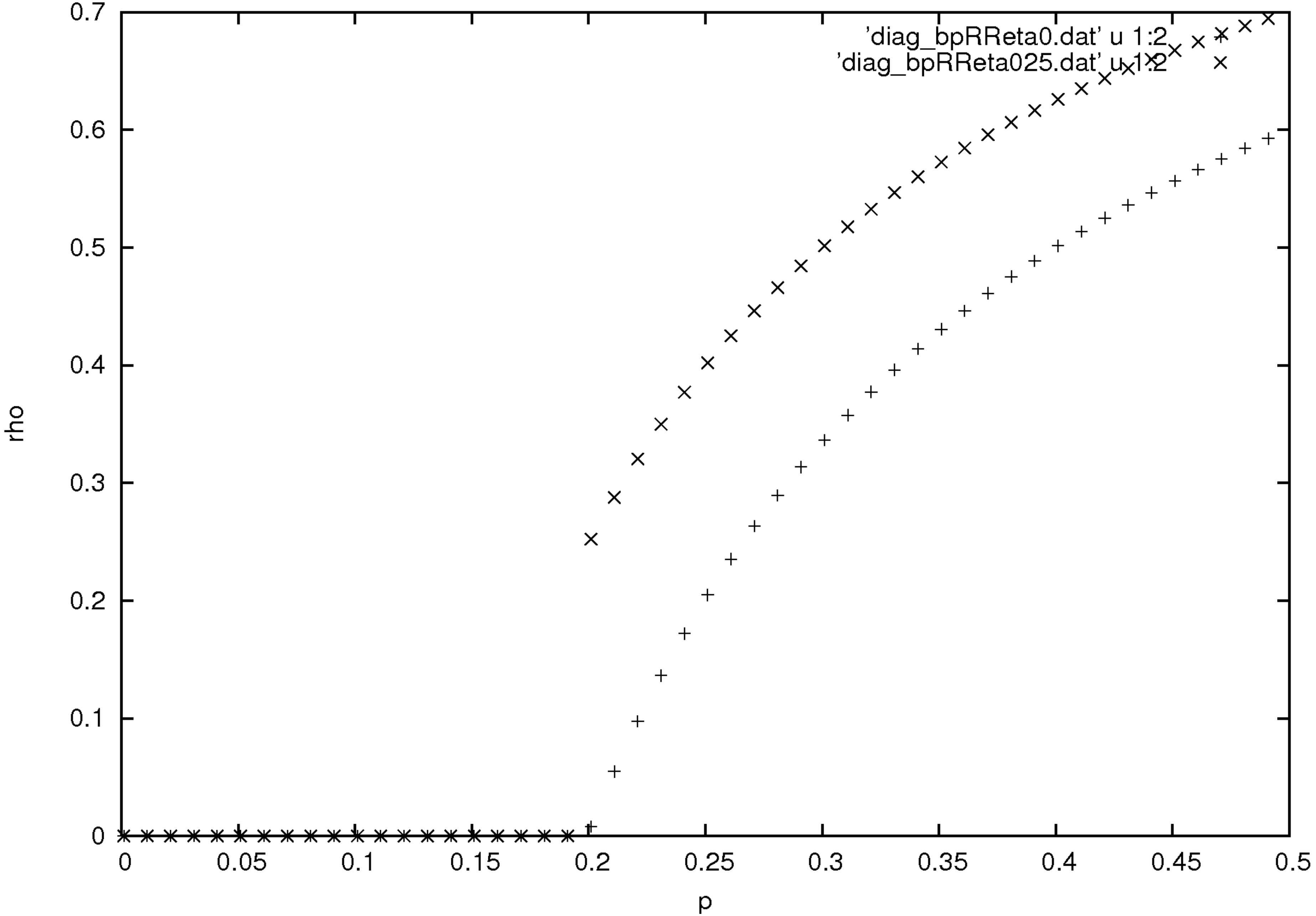
Recursion relations that can be solved on any specific graph

$$\begin{cases} \chi_i = max\left\{0, min\left[1, C_i(\vec{\chi}, \vec{q})\right]\right\} \\ q_i = max\left\{0, min\left[1, Q_i(\vec{\chi}, \vec{q})\right]\right\} \end{cases}$$

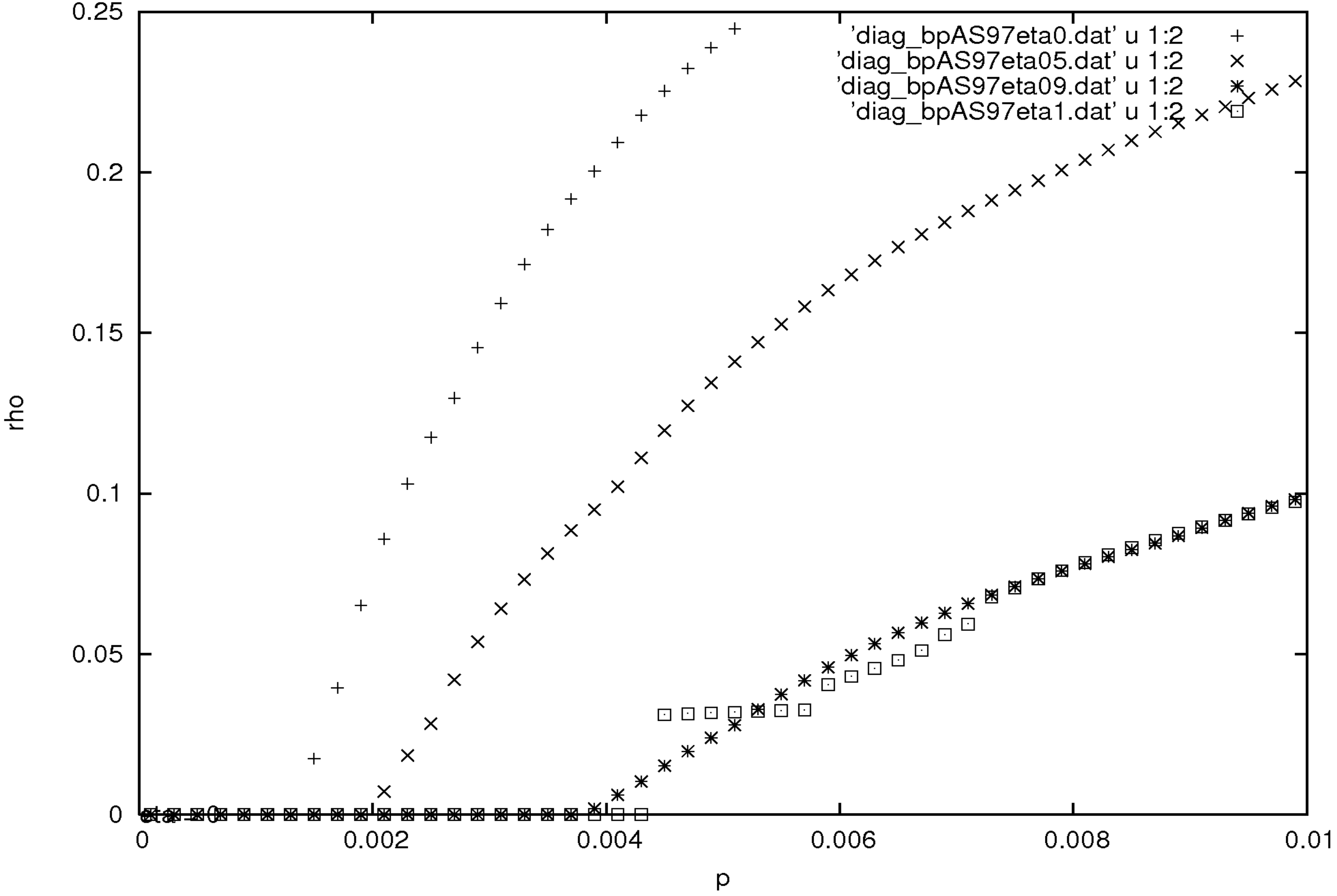
If a fixed point exists, we get the order parameter from

$$\rho(p) = \frac{1}{p} \sum_i \dot{n}_i$$

Results for single graphs



Results for single graphs



Equations for graph ensembles

The analysis can be extended to "ensembles of graphs",

- consider uncorrelated random graphs with degree distribution P(k)
- study the master equation for P(n|k)

Now, $\chi_i \rightarrow \chi_k \qquad q_i \rightarrow q_k$

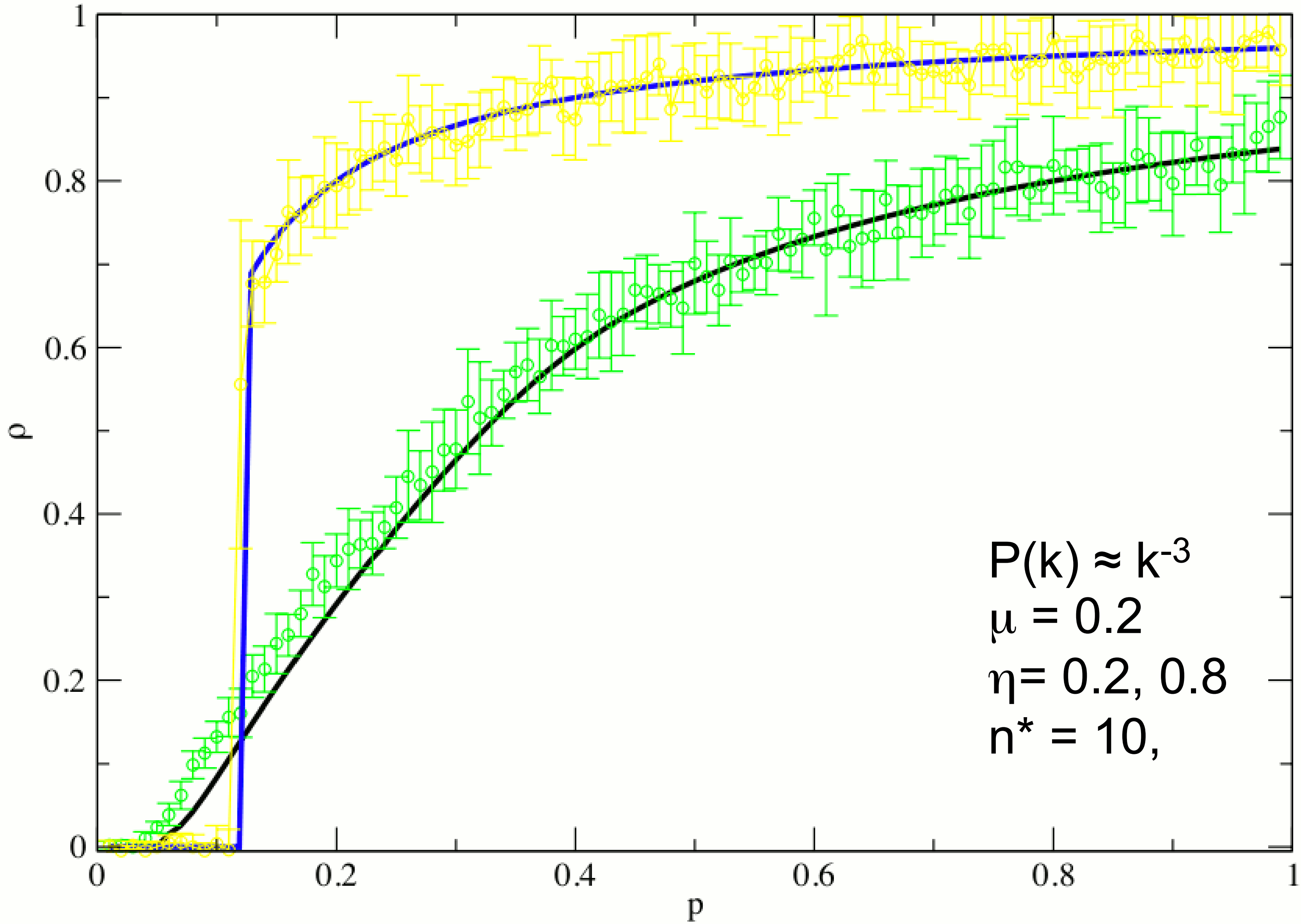
Increasing p, ∃ *k** such that

- for *k* < *k**, uncongested, q_k, χ_k determined from master eq.
- for *k* > *k**, congested, $q_k = 0, \chi_k = 1$
- for *k**, $q_k \simeq 0, \chi_k = 1, \langle n_k \rangle = 0$

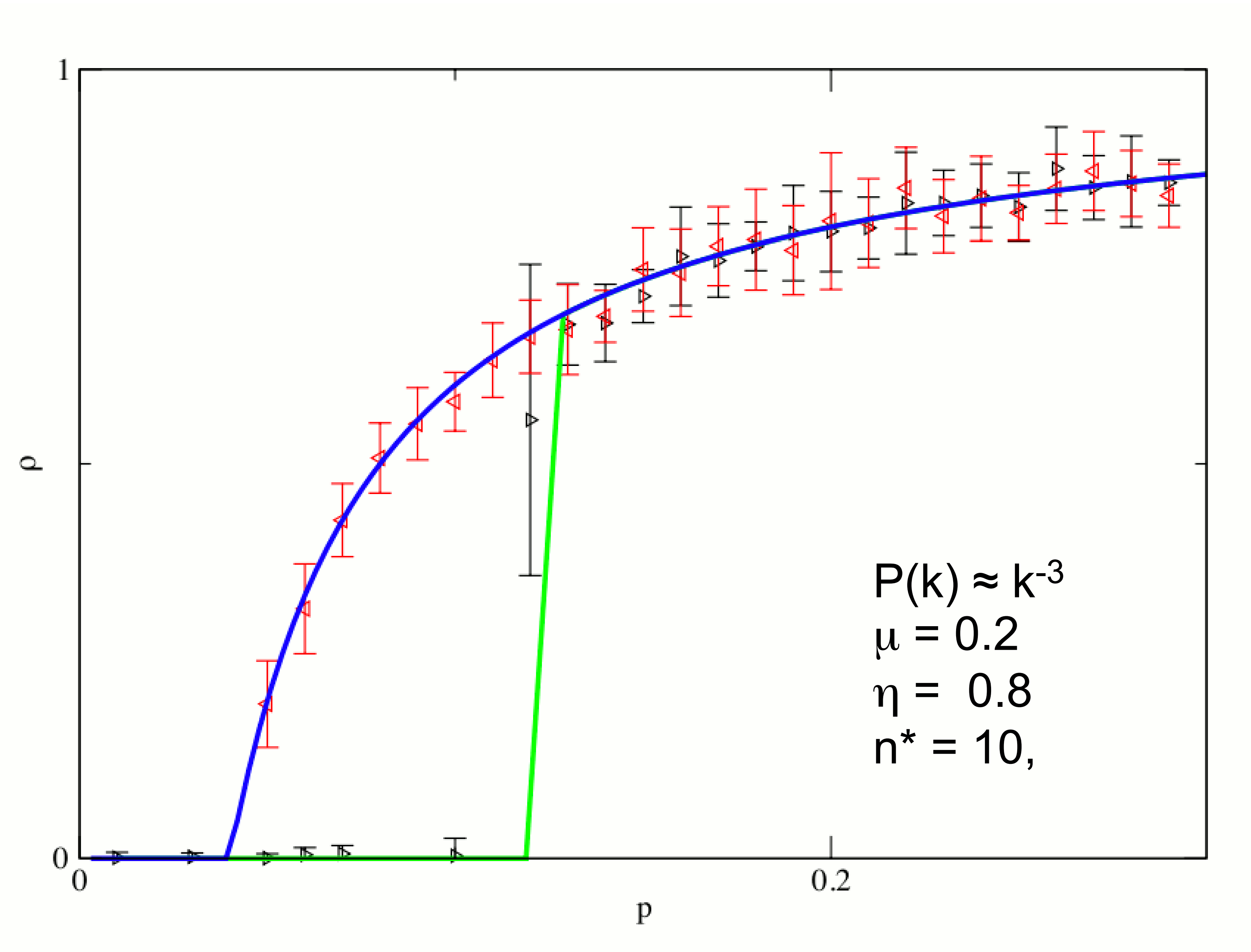
We obtain a set of three coupled equations for $\langle q \rangle, \langle \chi \rangle,$ and *k**

from which we get *k*(p),* then ρ

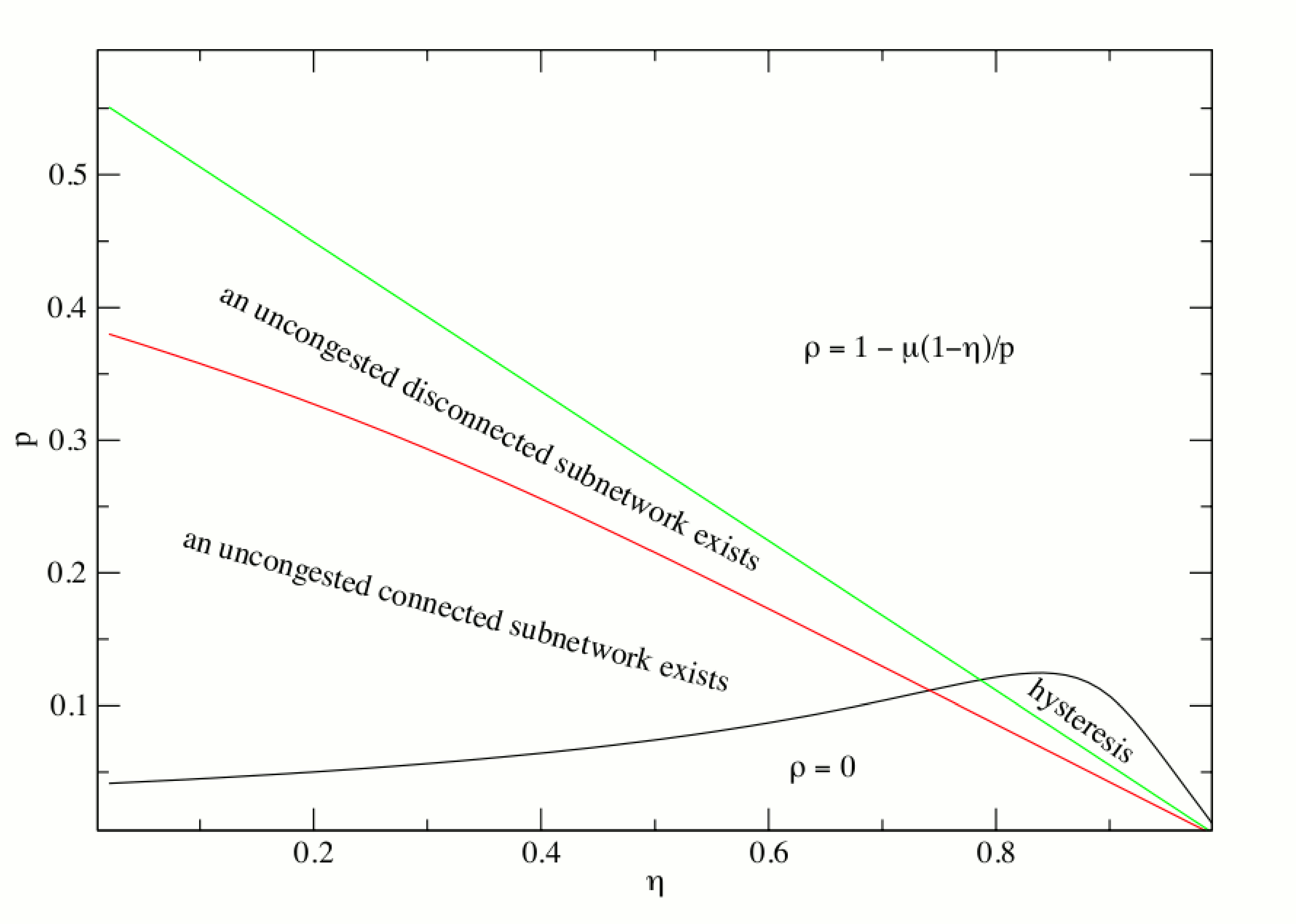
Results for graph ensembles (I)



Results for graph ensembles (II)



Results for graph ensembles (III)



Conclusions

- we have defined a minimal model of traffic flow on networks
- the model is analytically solvable (single graph, ensembles)
- it presents all features of observed congestion phenomena: continuous/discontinuous characters of the phase transition are understood as effects of the routing rule

To do:

- Investigate the relation with jamming transitions in granular media and k-core percolation
- focus on single node activity