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Clustering of particles falling through a turbulent flow

K. Gustavsson, B. Mehlig, S.Vajedi, PRL (2014) arXiv:1401.0513

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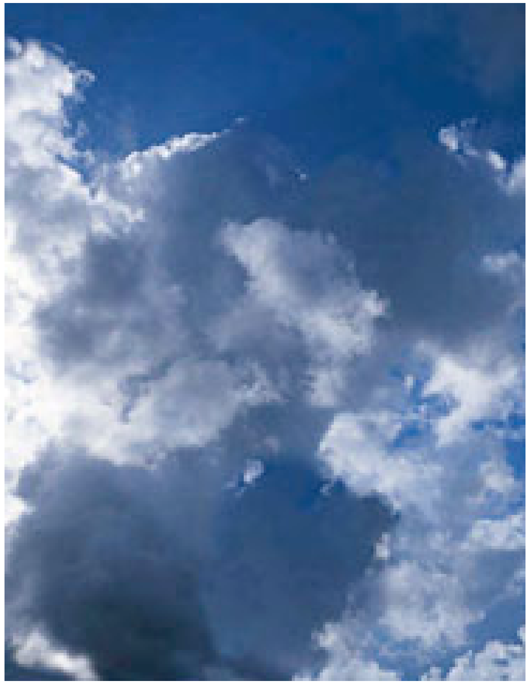
Water droplets in turbulent rain clouds



Forces on small droplet

Gravity (Newton's second law):

$$\boldsymbol{F}_{\mathrm{G}} = m\,\boldsymbol{g} = \frac{4\pi\rho_{\mathrm{p}}}{3} a^3 \boldsymbol{g}$$

ρ_{p} density of water droplet

g gravitational acceleration

a particle size

Friction (Stokes' law):

$$\boldsymbol{F}_{\mathrm{S}} = \mu\left(\boldsymbol{u}(\boldsymbol{r},\mathrm{t}) - \boldsymbol{v}\right)$$

where

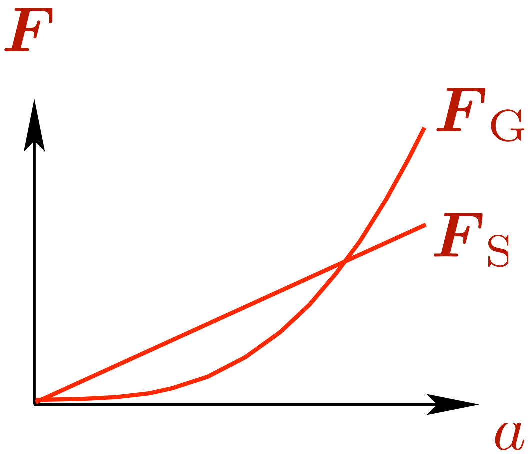
$$\mu = 6\pi\rho_{\mathrm{p}}\nu a \quad (= m\gamma)$$

ν viscosity

$\boldsymbol{u}(\boldsymbol{r},\mathrm{t})$ velocity of turbulent air in cloud

$\boldsymbol{r}$ droplet position

$\boldsymbol{v}$ droplet velocity



Model

Spherical droplets move independently in stationary incompressible, homogeneous, isotropic random velocity field $\boldsymbol{u}(\boldsymbol{x},t)$.
Single-scale flow: typical length scale η , time scale τ and speed u_0 .

Particle equation of motion (dedimensionalized with flow scales)

$$\dot{\boldsymbol{r}} = \mathrm{Ku}\,\boldsymbol{v}$$
$$\dot{\boldsymbol{v}} = (\boldsymbol{u}(\boldsymbol{r},t) - \boldsymbol{v})/\,\mathrm{St} + F\hat{\boldsymbol{g}}$$

$\boldsymbol{r}$	particle position
$\boldsymbol{v}$	particle velocity
$\hat{\boldsymbol{g}}$	direction of gravity

Parameters

Ku	Kubo number (dimensionless flow strength, $\mathrm{Ku} = u_0\tau/\eta$)
St	Stokes number (particle inertia)
F	Inverse Froude number (gravity strength)

Question: How do particles cluster within this model?

‘Unmixing’ of inertial particles ($\mathrm{F} = 0$)

$\mathrm{Ku} = 1 \;\; \mathrm{St} = 0.1$ $\qquad$ $\mathrm{Ku} = 0.1 \;\; \mathrm{St} = 10$

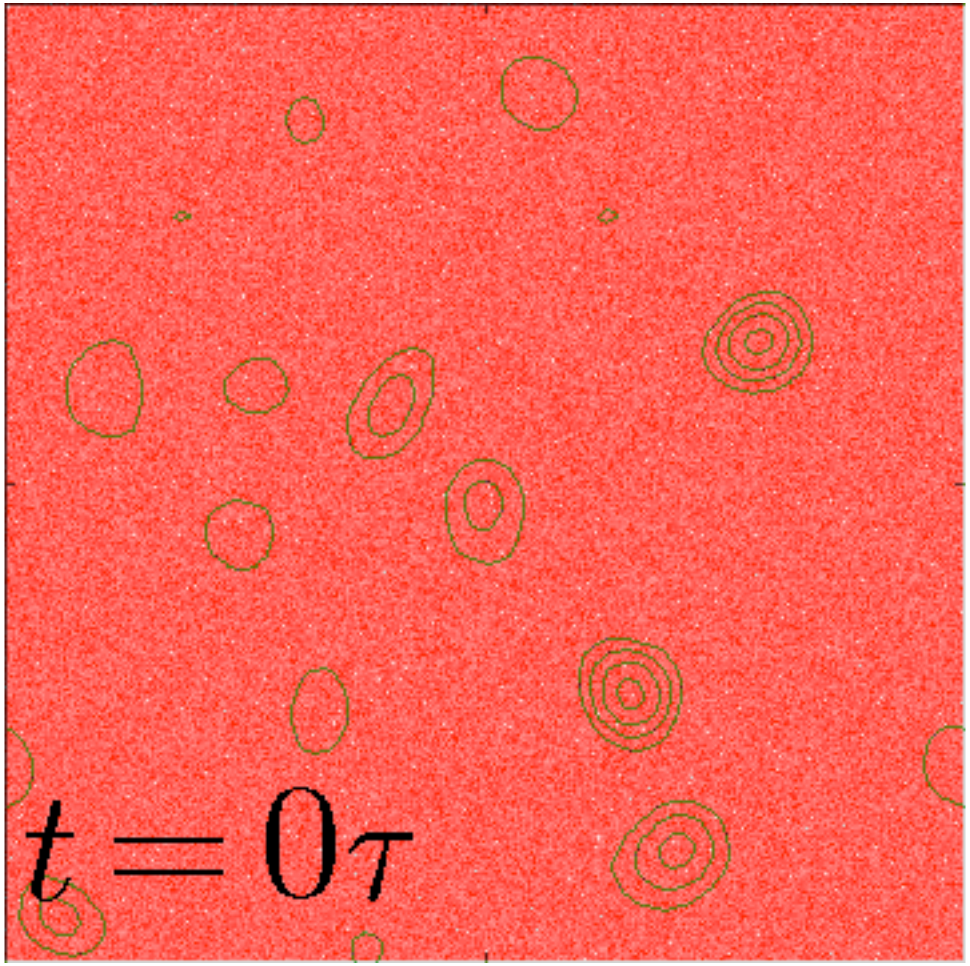
Maxey centrifuge effect

Multiplicative amplification

 Particle density  Region of high vorticity

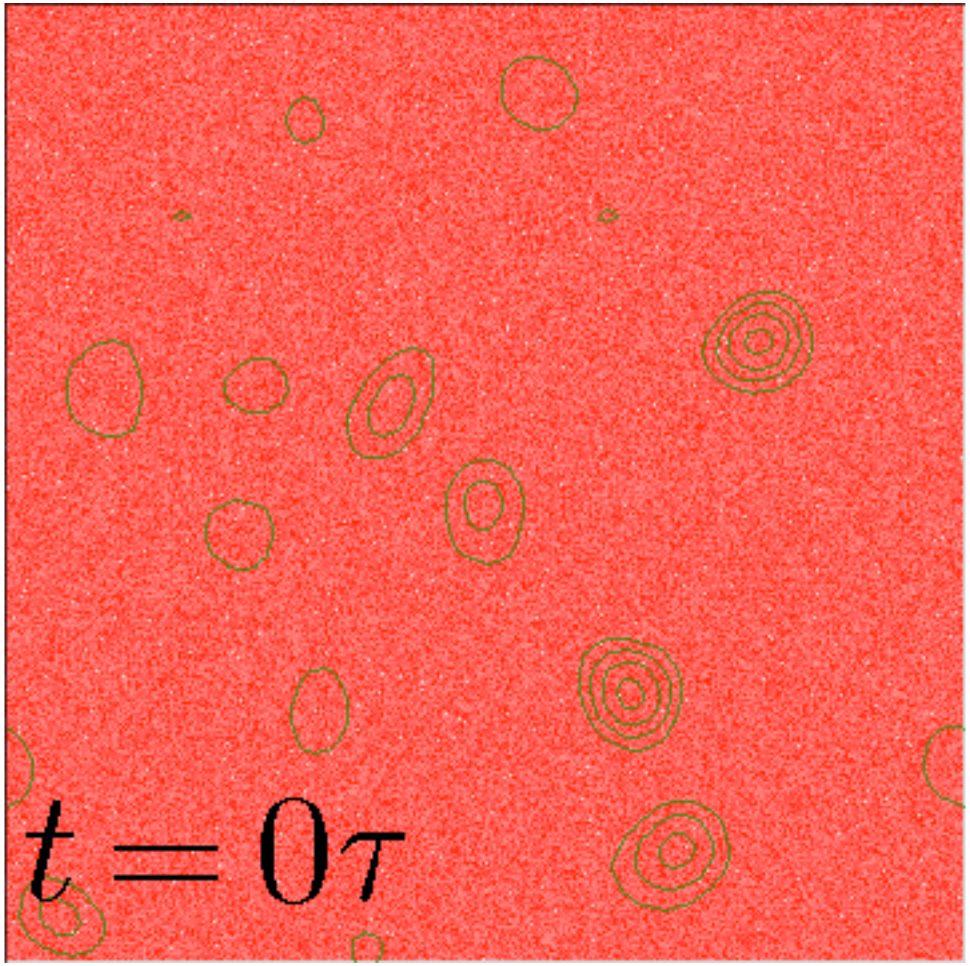
'Unmixing' of inertial particles ($F = 0$)

$\mathrm{Ku} = 1 \;\; \mathrm{St} = 0.1$



Maxey centrifuge effect

$\mathrm{Ku} = 0.1 \;\; \mathrm{St} = 10$



Multiplicative amplification

Particle density   Region of high vorticity

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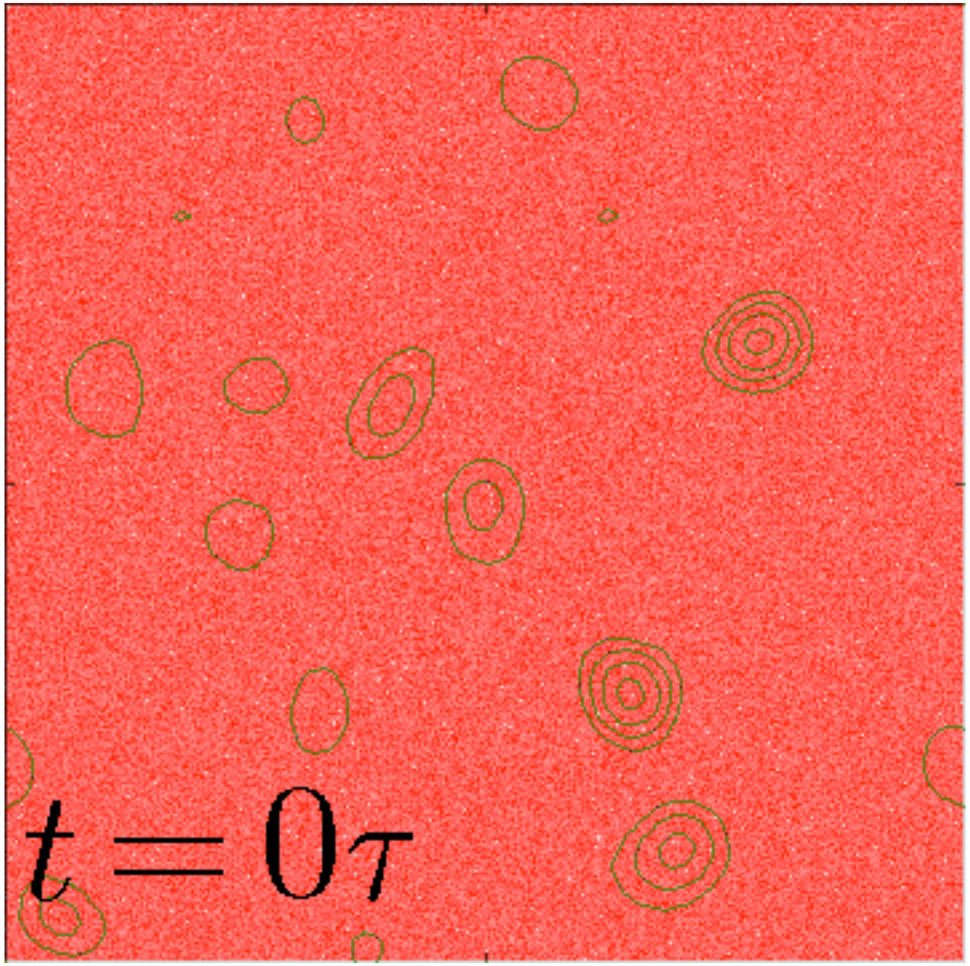

Maxey centrifuge effect

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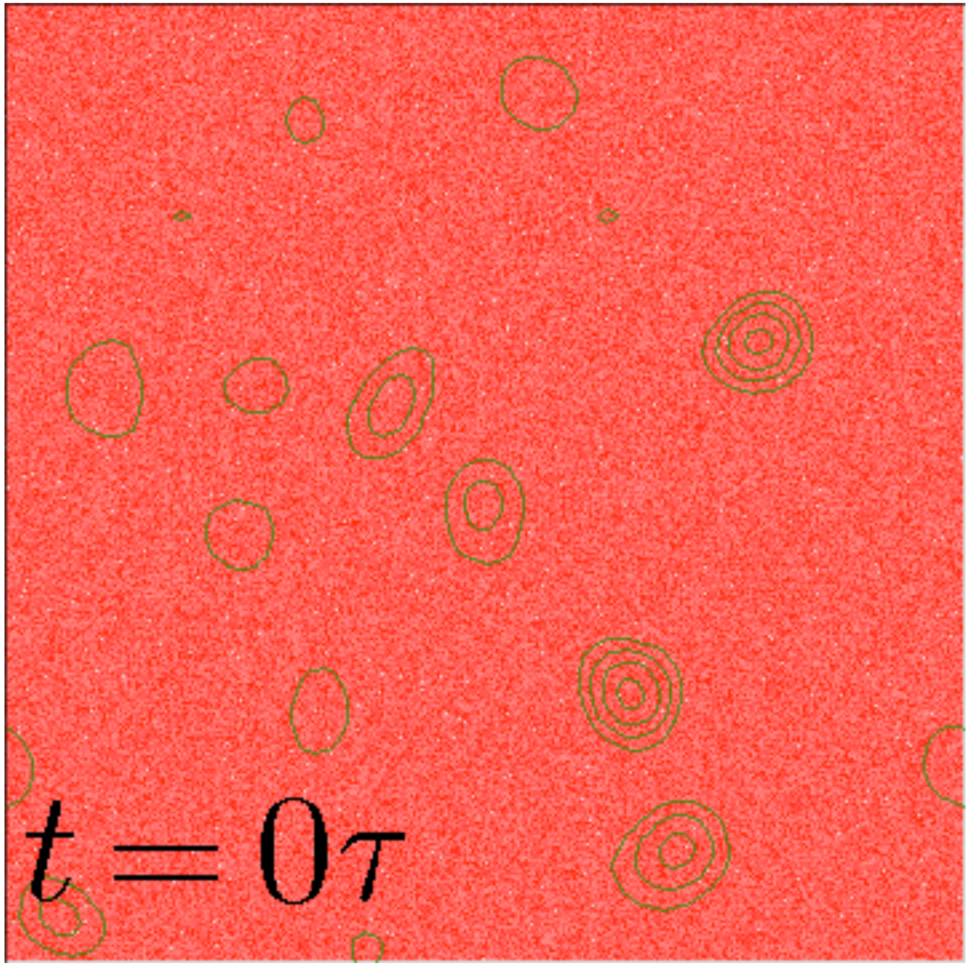


Multiplicative amplification

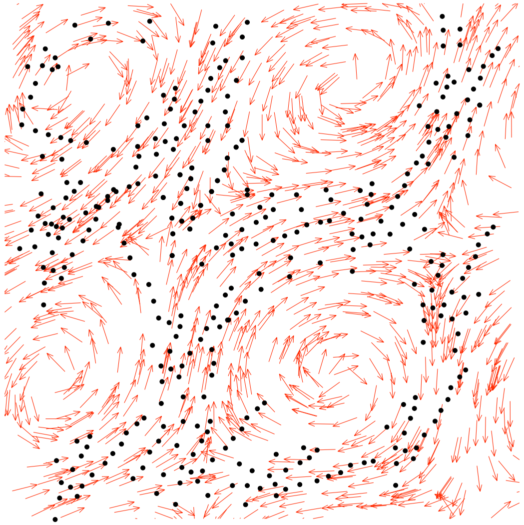
Particle density   Region of high vorticity

Clustering mechanisms ($F = 0$)

Maxey centrifuge effect

Droplets avoid regions of high vorticity and gather in region of high strain.

Strong correlation between instantaneous flow structures and particle positions.



Maxey, J. Fluid Mech. **174**, 441, (1987)

Multiplicative amplification

Clustering as a net effect of many small deformations of volumes spanned by close-by particles.

Particle positions are uncorrelated to instantaneous structures in the flow.

White-noise modelling possible. Dynamics described by single parameter: $\epsilon^2 \sim \mathrm{Ku}^2\mathrm{St}$ ($\mathrm{Ku} \to 0$ and $\mathrm{St} \to \infty$ so that ϵ constant)

Mehlig & Wilkinson, PRL **92** (2004) 250602
Duncan et al., PRL **95** (2005)
Wilkinson et al., Phys. Fluids **19** (2007) 113303

Quantification of clustering ($d = 2$)

Lyapunov exponents $\lambda_1 > \lambda_2$ describe rate of contraction or expansion of small length element δr_t , and area element $\delta \mathcal{A}_t$ of particle flow

$$\lambda_1 = \lim_{t\to\infty} t^{-1} \ln(\delta r_t)$$

$$\lambda_1 + \lambda_2 = \lim_{t\to\infty} t^{-1} \ln(\delta \mathcal{A}_t)$$

J. Sommerer & E. Ott, Science 259 (1993) 351

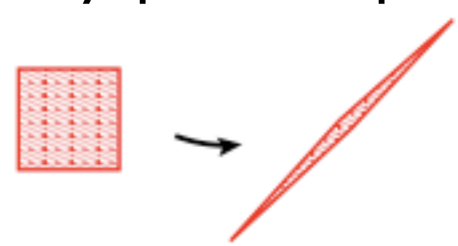
When $\mathrm{St} > 0$ and not too large, the dynamics is:

- chaotic (positive maximal Lyapunov exponent)

$$\lambda_1 > 0$$

- compressible (sum of two maximal Lyapunov exponents negative)

$$\lambda_1 + \lambda_2 < 0$$

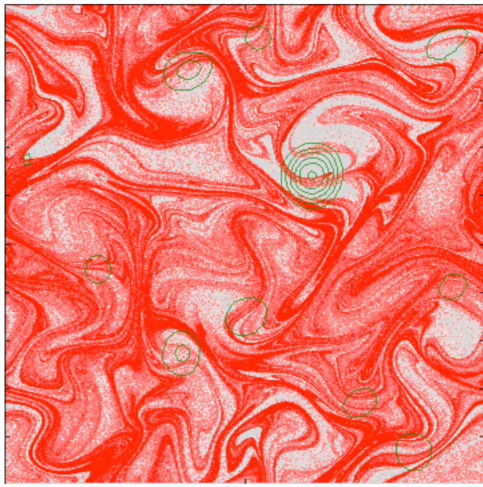


Fractal dimension $d_{\mathrm{L}} \equiv 2 - \dfrac{\lambda_1 + \lambda_2}{\lambda_2}$

Kaplan & Yorke, Springer Lecture Notes in Mathematics **730**, 204, (1979)

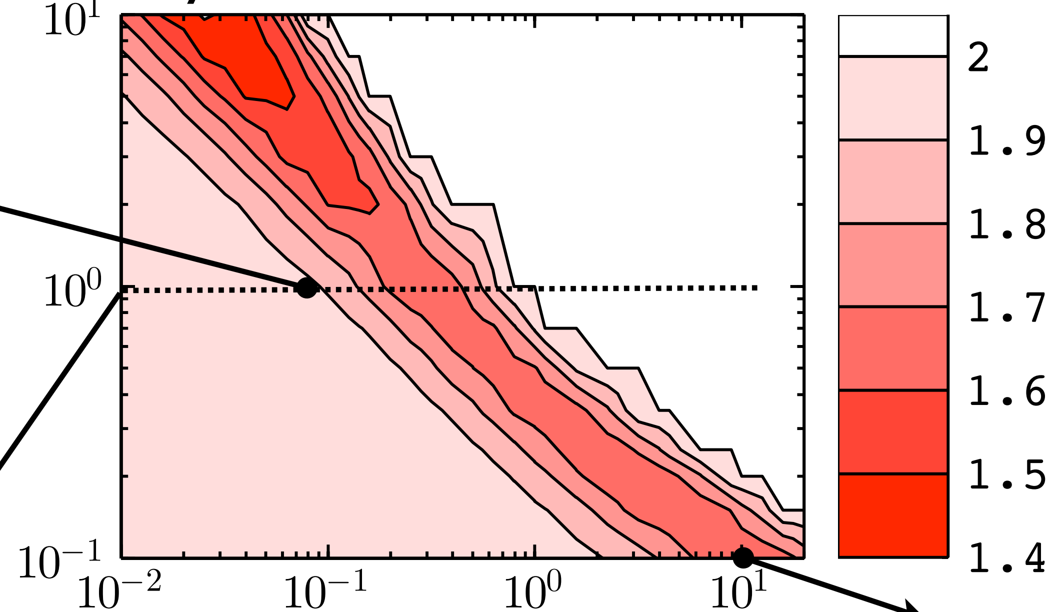
Clustering without gravity

$$d_{\mathrm{L}} \equiv 2 - \frac{\lambda_1 + \lambda_2}{\lambda_2}$$

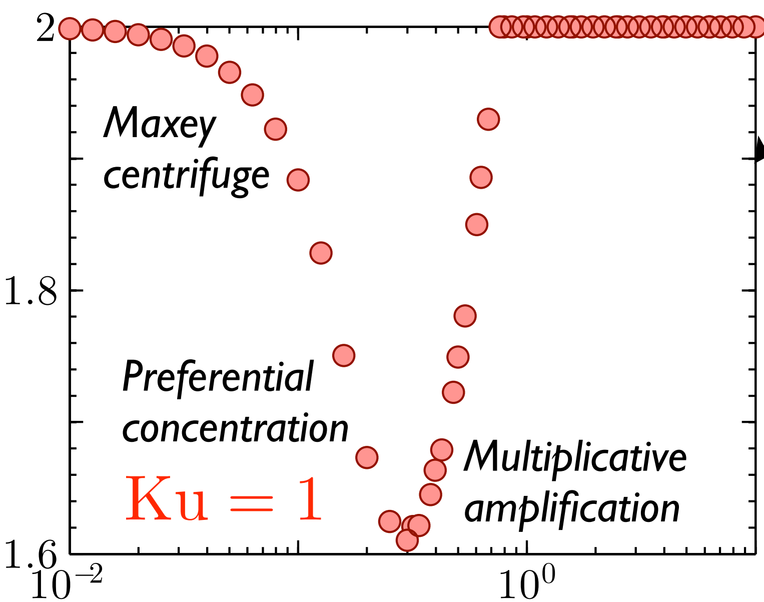


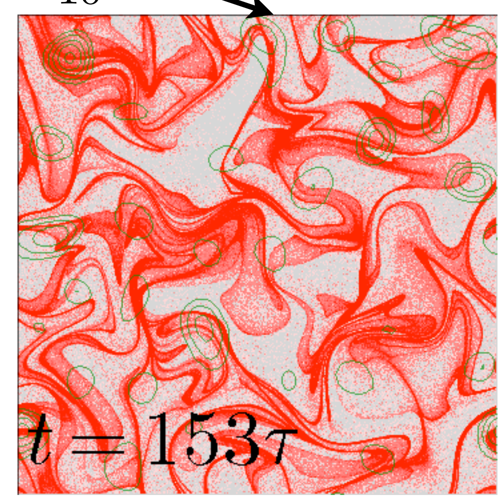
Maxey centrifuge effect

Flow intensity Ku   Fractal dimension d_{L}



Inertia St





Multiplicative amplification

Deterministic dynamics with gravity

Dynamics in the absence of $\boldsymbol{u}$

$$\dot{\boldsymbol{r}} = \mathrm{Ku}\,\boldsymbol{v}$$
$$\dot{\boldsymbol{v}} = (\cancel{\boldsymbol{u}(\boldsymbol{r},t)} - \boldsymbol{v})/\,\mathrm{St} + F\hat{\boldsymbol{g}}$$

Deterministic solution

$$\boldsymbol{r} = \boldsymbol{r}_0 + \mathrm{Ku}\,\boldsymbol{v}_{\mathrm{s}}t + \mathrm{Ku}\,\mathrm{St}(\boldsymbol{v}_0 - \boldsymbol{v}_{\mathrm{s}})(1 - e^{-t/\mathrm{St}})$$
$$\boldsymbol{v} = \boldsymbol{v}_{\mathrm{s}} + (\boldsymbol{v}_0 - \boldsymbol{v}_{\mathrm{s}})e^{-t/\,\mathrm{St}}$$

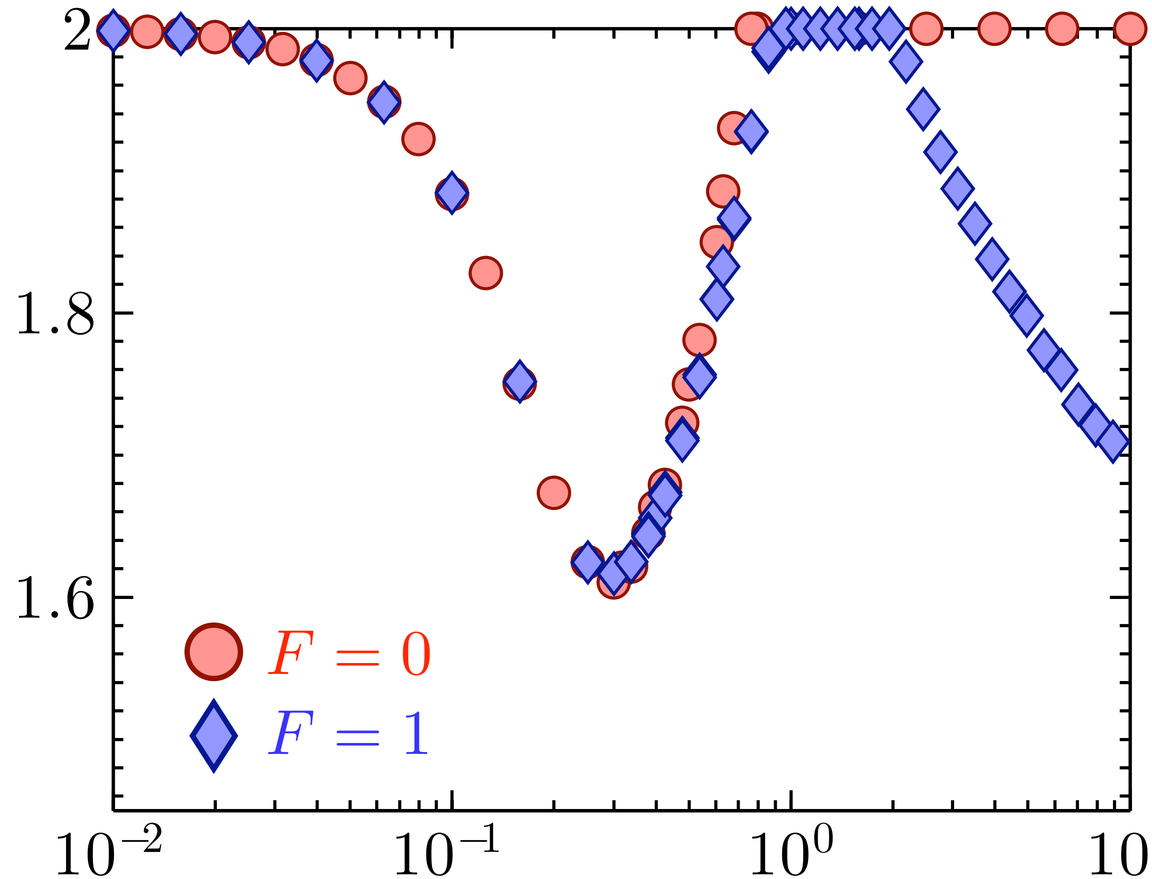
Particles reach a terminal 'settling velocity' $\boldsymbol{v}_{\mathrm{s}} \equiv \mathrm{F}\,\mathrm{St}\,\hat{\boldsymbol{g}}$

The deterministic solution is important if $v_s \gg 1$

Relative motion between two particles is only affected by gravity through the $\boldsymbol{r}$-dependence in $\boldsymbol{u}(\boldsymbol{r},t)$. Gravity is expected to alter correlations between flow and particle trajectories.

Clustering with gravity ($\mathrm{Ku} = 1$)

Fractal dimension d_L



large $F\,\mathrm{St}$ (deterministic solution important)

Inertia St

‘Unmixing’ of falling inertial particles

Non-interacting, non-colliding particles (red) suspended in a random flow

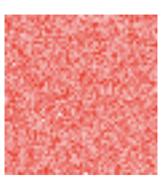
$\mathrm{St} = 10$

$\mathrm{Ku} = 1$

$F = 1$

$\boldsymbol{v}_\mathrm{s}$

Frame moving with velocity $\boldsymbol{v}_\mathrm{s}$



Particle density

‘Unmixing’ of falling inertial particles

Non-interacting, non-colliding particles (red) suspended in a random flow

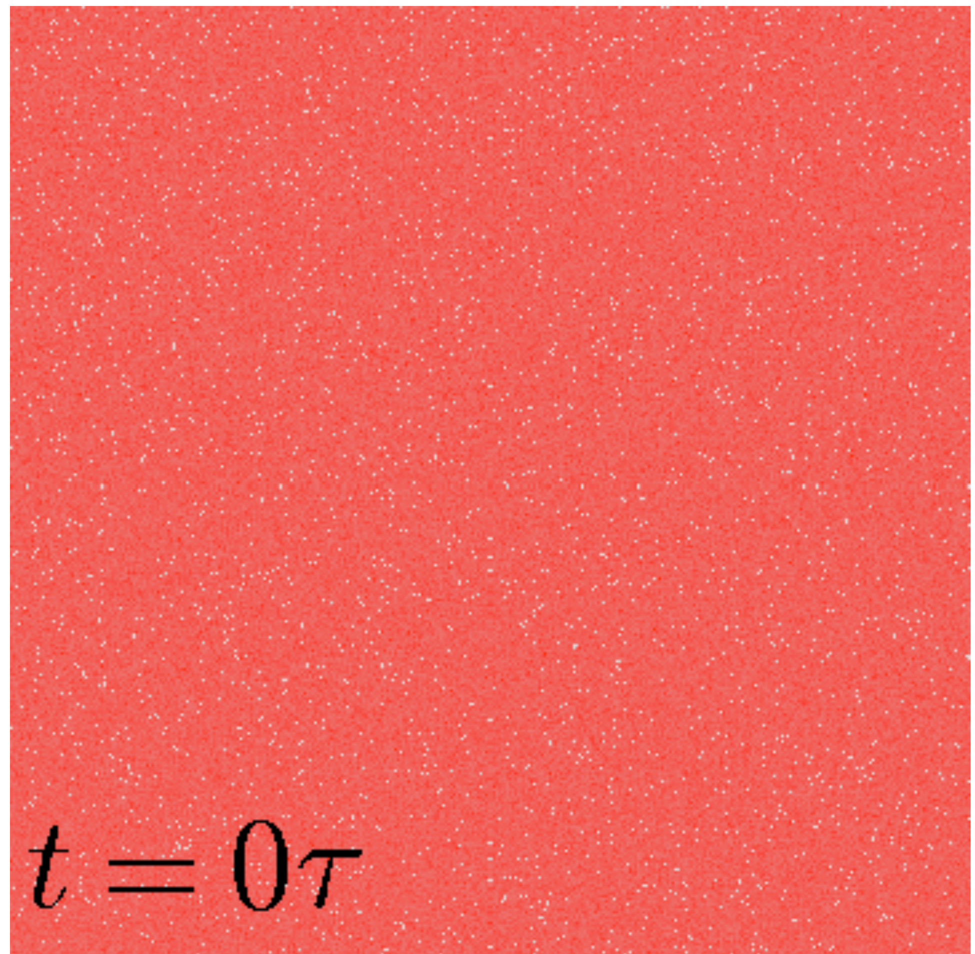
$\mathrm{St} = 10$

$\mathrm{Ku} = 1$

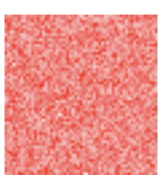
$F = 1$

Frame moving with velocity v_{s}

v_{s}



Large-St gravitational clustering



Particle density

Large-St dynamics

Deterministic solution $\boldsymbol{r} \approx \boldsymbol{r}_0 + \mathrm{Ku}\,\boldsymbol{v}_{\mathrm{s}} t$ with settling velocity $v_{\mathrm{s}} = \mathrm{F\,St}$

Spatial decorrelation becomes faster than time decorrelation.

$t = t_1$ $\boldsymbol{r} = \boldsymbol{r}_1$

$\downarrow \boldsymbol{v}_{\mathrm{s}}$

$t = t_2$ $\boldsymbol{r} \approx \boldsymbol{r}_1 + \mathrm{Ku}\,\boldsymbol{v}_{\mathrm{s}}(t_2 - t_1)$

$\downarrow \boldsymbol{v}_{\mathrm{s}}$

Single-particle correlation function at two different times (our random flow)

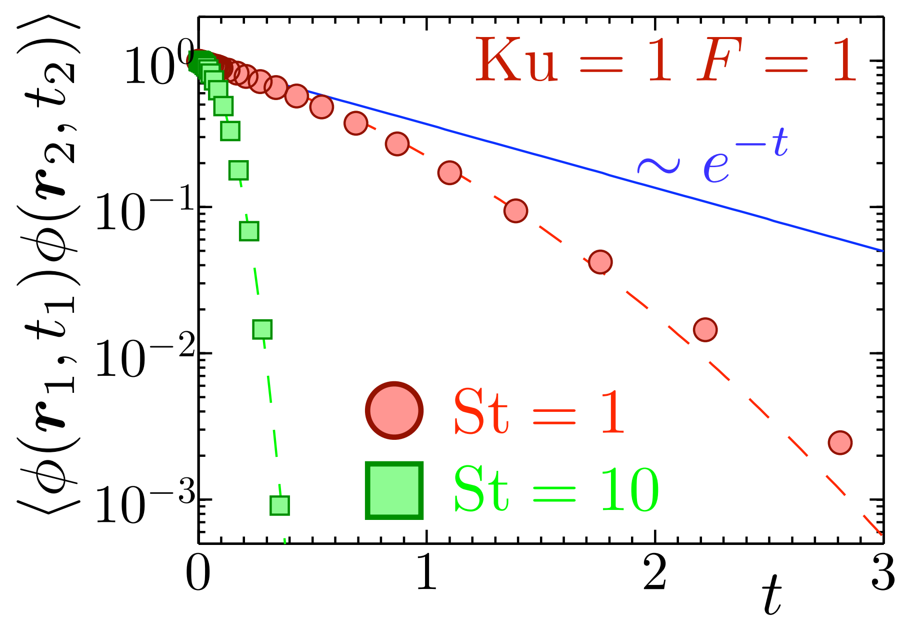
$$\langle u(\boldsymbol{r}_1, t_1) u(\boldsymbol{r}_2, t_2)\rangle$$

$$\sim e^{-|t_1 - t_2| - (\boldsymbol{r}_1 - \boldsymbol{r}_2)^2/2}$$

$$\sim e^{-|t_1 - t_2| - \mathrm{Ku}^2\, v_{\mathrm{s}}^2 (t_1 - t_2)^2/2}$$

Shown as = = = in figure.

When $G \equiv \mathrm{Ku}\, v_{\mathrm{s}} = \mathrm{Ku\,F\,St}$ is large the effective correlation time approaches white noise.



Langevin model

Langevin equation for separations $\boldsymbol{R}' = \boldsymbol{r}_1 - \boldsymbol{r}_2$ and relative velocities $\boldsymbol{V}' = \mathrm{Ku}\,\mathrm{St}(\boldsymbol{v}_1 - \boldsymbol{v}_2)$ ($t' = t/\,\mathrm{St}$)

$$\delta\boldsymbol{R}' = \boldsymbol{V}'\,\delta t'\,, \qquad \delta\boldsymbol{V}' = -\boldsymbol{V}'\,\delta t' + \delta\boldsymbol{F}\,.$$

Increments $\delta\boldsymbol{F}$ are Gaussian white noise with $\langle\delta\boldsymbol{F}\rangle = \boldsymbol{0}$ and $\langle\delta F_i \delta F_j\rangle = 2\delta t' \mathrm{Ku}^2 \mathrm{St}\,\Sigma_{kl} D_{ik,jl} R'_k R'_l$ with $D_{ik,jl}$ obtained by integration of the effective correlation functions

$$D_{ik,jl} \equiv \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t \left\langle \frac{\partial u'_i}{\partial r'^k}(\boldsymbol{r}'(t'), t')\frac{\partial u'_j}{\partial r'^l}(\boldsymbol{0}, 0)\right\rangle\,.$$

We obtain ($\hat{\boldsymbol{g}} = -\boldsymbol{e}_y$)

$$D_{11,11} = D_{22,22} = -D_{11,22} = -D_{22,11} = -D_{12,21} = -D_{21,12} = \frac{1}{2G^2} - \frac{D_{21,21}}{3G^2}$$

$$D_{12,12} = \frac{G^2-1}{2G^4} + \frac{D_{21,21}}{3G^4}\,, \quad D_{21,21} = \frac{3}{\sqrt{8}G}\mathcal{F}\left[\frac{1}{\sqrt{2}G}\right], \mathcal{F}[x] \equiv \sqrt{\pi}e^{x^2}\mathrm{erfc}(x)\,.$$

Gravity introduces anisotropy ($D_{12,12} \neq D_{21,21}$).

Two parameters: $\epsilon^2 \sim \mathrm{Ku}^2\,\mathrm{St}$ and $G = \mathrm{Ku}\,\mathrm{F}\,\mathrm{St}$.

Langevin model, large-G asymptote

Diagonalise and rescale noise

$$A_\pm \equiv \left(\frac{D_{21,21}}{D_{12,12}}\right)^{1/4}\frac{\partial u_1}{\partial r^2} \pm \left(\frac{D_{12,12}}{D_{21,21}}\right)^{1/4}\frac{\partial u_2}{\partial r^1}$$

For large values of $G = \mathrm{Ku\,F\,St}$ the dynamics is governed by a single parameter $D_{++} = D_{--} \sim \mathrm{Ku}^2\mathrm{St}/G^{3/2}$.

Compare this parameter to the parameter of the $F = 0$ white-noise model $\epsilon^2 \sim \mathrm{Ku}^2\mathrm{St}$.

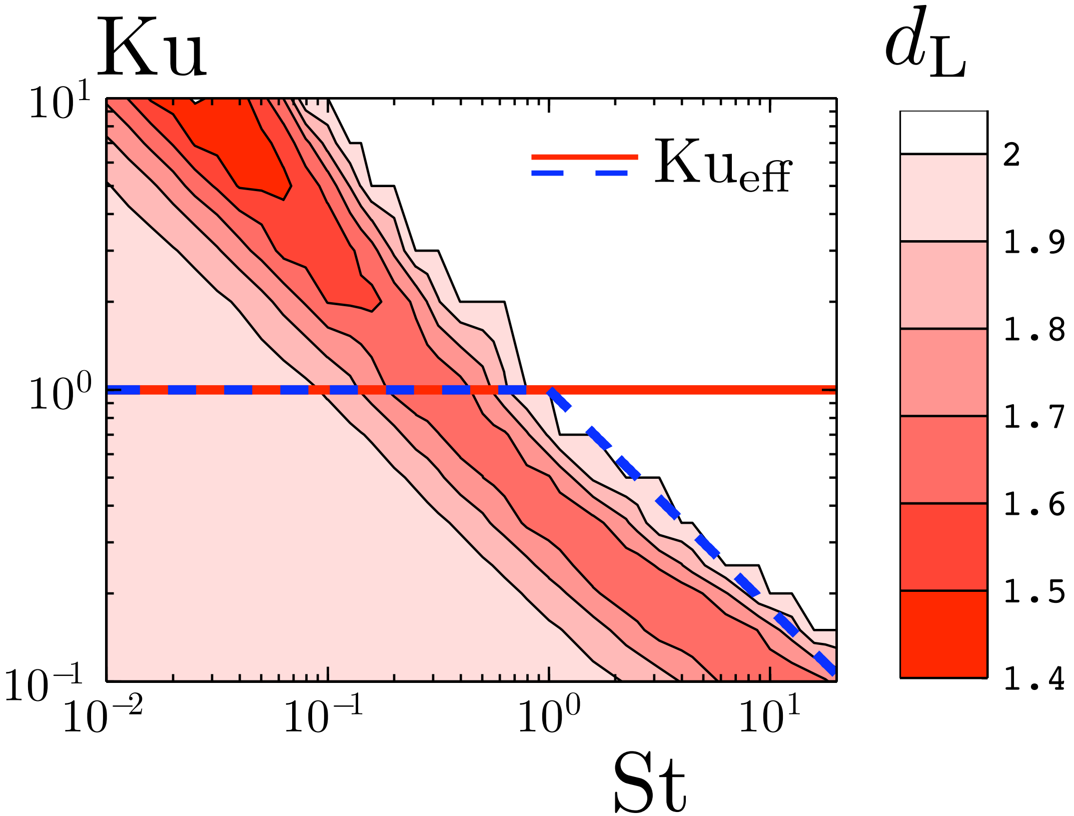
For a given large value of St define an effective Kubo number $\mathrm{Ku}_{\mathrm{eff}}$ in ϵ^2 so that the two parameters are equal

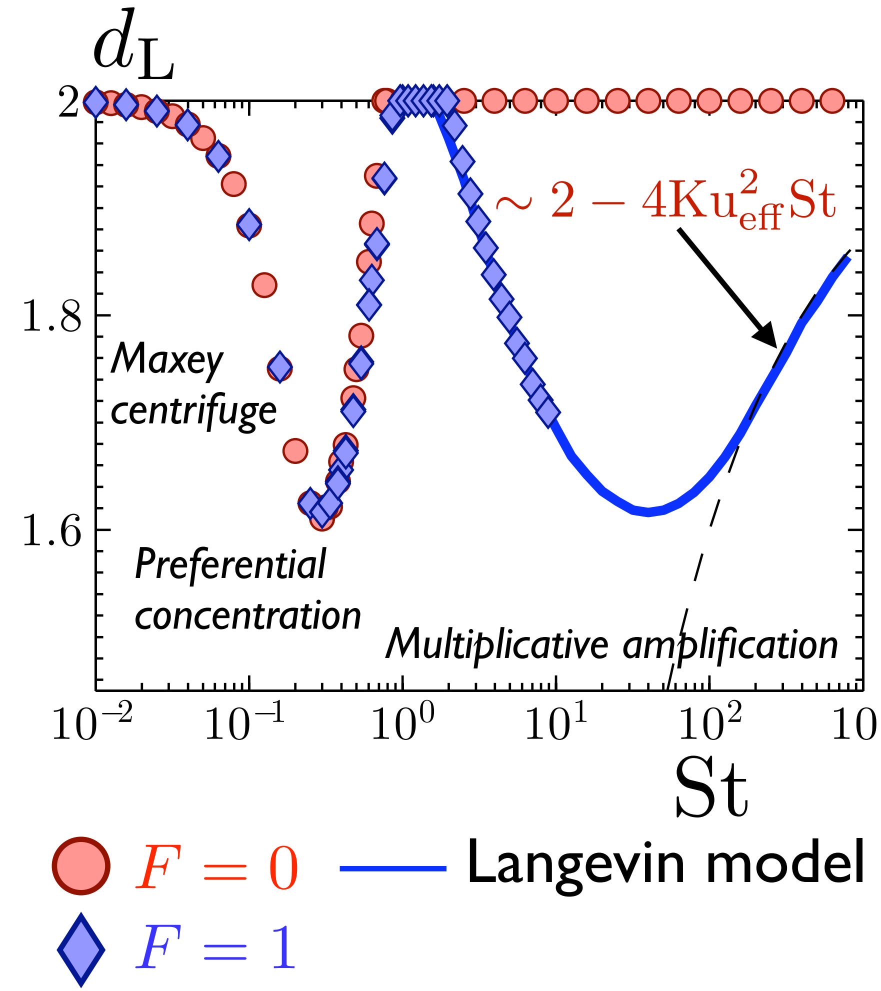
$$\mathrm{Ku}_{\mathrm{eff}} \sim \begin{cases} \mathrm{Ku} & \mathrm{St\ small} \\ \mathrm{Ku}^{1/4}/(F\mathrm{St})^{3/4} & \mathrm{St\ large} \end{cases} .$$

$\mathrm{Ku}_{\mathrm{eff}}$ approximately maps the $F \neq 0$ model with some value of Ku onto the $F = 0$ model with Kubo number $\mathrm{Ku}_{\mathrm{eff}}$.

Large-St gravitational clustering

The effective $\mathrm{Ku_{eff}}$ maps the dynamics with $F>0$, $\mathrm{Ku}=1$ and large St on the $F=0$ -dynamics



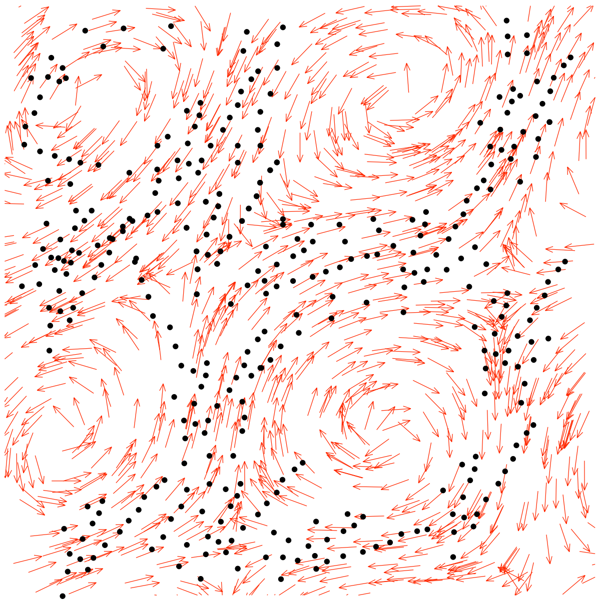


$$\mathrm{Ku_{eff}} \sim \begin{cases} \mathrm{Ku} & \mathrm{St\ small} \\ \mathrm{Ku}^{1/4}/(F\mathrm{St})^{3/4} & \mathrm{St\ large} \end{cases}$$

Clustering due to preferential sampling

As we have seen, gravity tends to enhance clustering due to multiplicative amplification for large values of St .

What is the effect of gravity on preferential sampling (e.g. Maxey centrifuge effect) and anisotropy for general values of F ?



+ gravity = ?

To answer this question we make a series expansion around deterministic trajectories.

Trajectory approximation ($F \neq 0$)

Solve equations of motion (dimensionless units)

$$\dot{\boldsymbol{r}} = \mathrm{Ku}\boldsymbol{v} \quad , \quad \dot{\boldsymbol{v}} = (\boldsymbol{u}(\boldsymbol{r}_t, t) - \boldsymbol{v})/\mathrm{St} + F\hat{\boldsymbol{g}}$$

implicitly

$$\boldsymbol{r}_t = \tilde{\boldsymbol{r}}_t + \frac{\mathrm{Ku}}{\mathrm{St}} \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 e^{-(t_1 - t_2)/\mathrm{St}} \boldsymbol{u}(\boldsymbol{r}_{t_2, t_2})$$

with deterministic part

$$\tilde{\boldsymbol{r}}_t = \boldsymbol{r}_0 + \mathrm{Ku}\boldsymbol{v}_\mathrm{s} t + \mathrm{KuSt}(\boldsymbol{v}_0 - \boldsymbol{v}_\mathrm{s})(1 - e^{-t/\mathrm{St}}) \ .$$

Expand the flow $\boldsymbol{u}(\boldsymbol{r}_t, t)$ around $\tilde{\boldsymbol{r}}_t$ and iterate expansion.

Insert the expanded flow into the equation for the velocity gradient matrix $\mathbb{Z} \equiv \boldsymbol{\nabla}\boldsymbol{v}^\mathrm{T}$: $\dot{\mathbb{Z}} = (\boldsymbol{\nabla}\boldsymbol{u}^\mathrm{T}(\boldsymbol{r}_t, t) - \mathbb{Z})/\mathrm{St} - \mathrm{Ku}\mathbb{Z}^2$.

Expand this equation around the $\mathbb{Z}^2$ -term, solve implicitly and iterate to obtain an expansion of $\mathbb{Z}$.

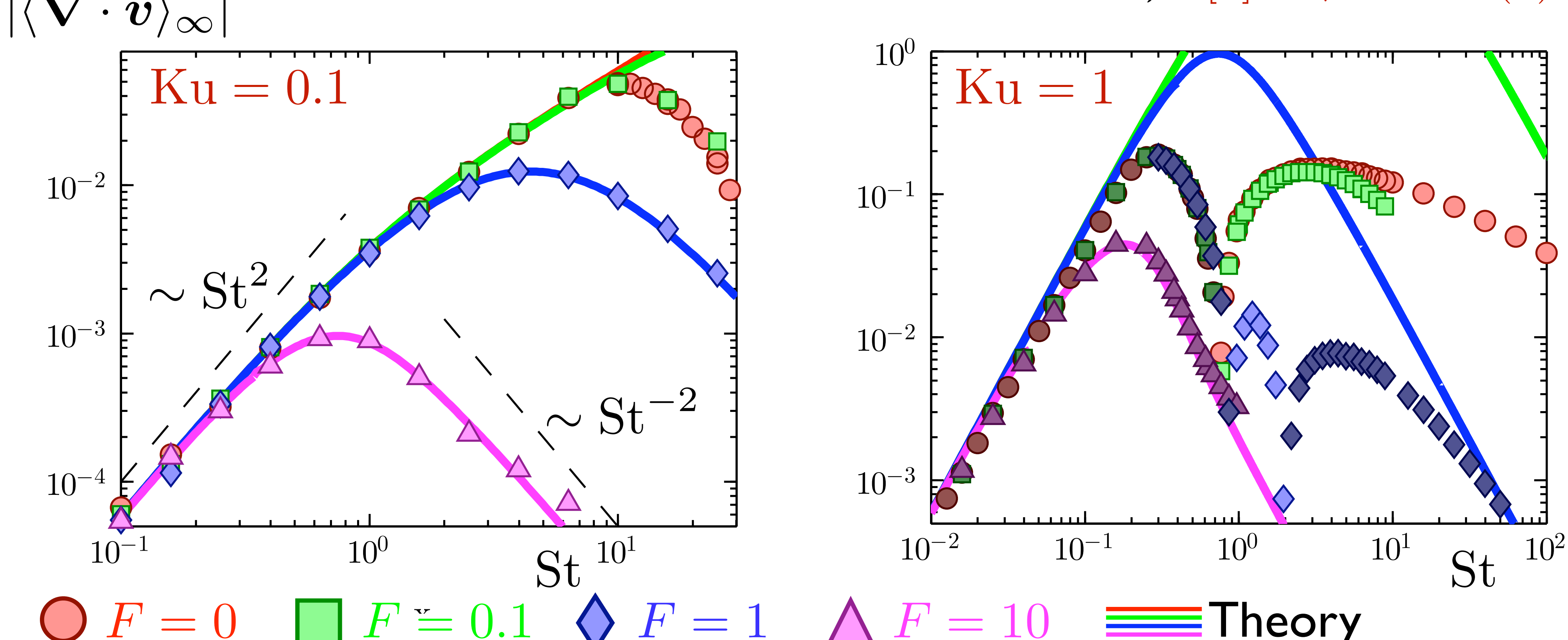
Evaluate average compressibility $\langle \boldsymbol{\nabla} \cdot \boldsymbol{v} \rangle_\infty$ along particle trajectories to determine how areas of close-by particles develop ($\lambda_1 + \lambda_2 = \mathrm{Ku}\langle \boldsymbol{\nabla} \cdot \boldsymbol{v} \rangle_\infty$)

Preferential sampling of $\nabla \cdot \boldsymbol{v}$

We find:

$$\langle \nabla \cdot \boldsymbol{v} \rangle_\infty = \frac{3\mathrm{Ku}^3}{4\mathrm{St}^5 G^8} \Big\{ 2G^2\mathrm{St}^3(5+4\mathrm{St}+3\mathrm{St}^2-G^2\mathrm{St}^2(1+\mathrm{St}))+(1+\mathrm{St})^3(2(1+\mathrm{St})^2-G^2\mathrm{St}^2(\mathrm{St}-3))\mathcal{F}\left[\frac{1+\mathrm{St}}{\sqrt{2}\mathrm{St}G}\right]^2$$
$$-\sqrt{2}G\mathrm{St}^2(13+17\mathrm{St}+15\mathrm{St}^2+3\mathrm{St}^3+G^2\mathrm{St}^2(4-\mathrm{St}-3\mathrm{St}^2)+G^4\mathrm{St}^4)\mathcal{F}\left[\frac{1+\mathrm{St}}{\sqrt{2}\mathrm{St}G}\right]-4G\mathrm{St}(1+\mathrm{St}^2(2+\mathrm{St}^2+G^2))\mathcal{F}\left[\frac{1}{G}\right]$$
$$-2\sqrt{\pi}(1+\mathrm{St}^2)G(-2+\mathrm{St}^2(-2+(-3+\mathrm{St}^2)G^2))\int_0^\infty \mathrm{d}t \exp\left[G^{-2}-t/\mathrm{St}-G^2t^2/4\right]\mathrm{erfc}\left[G^{-1}+Gt/2\right]\Big\}$$

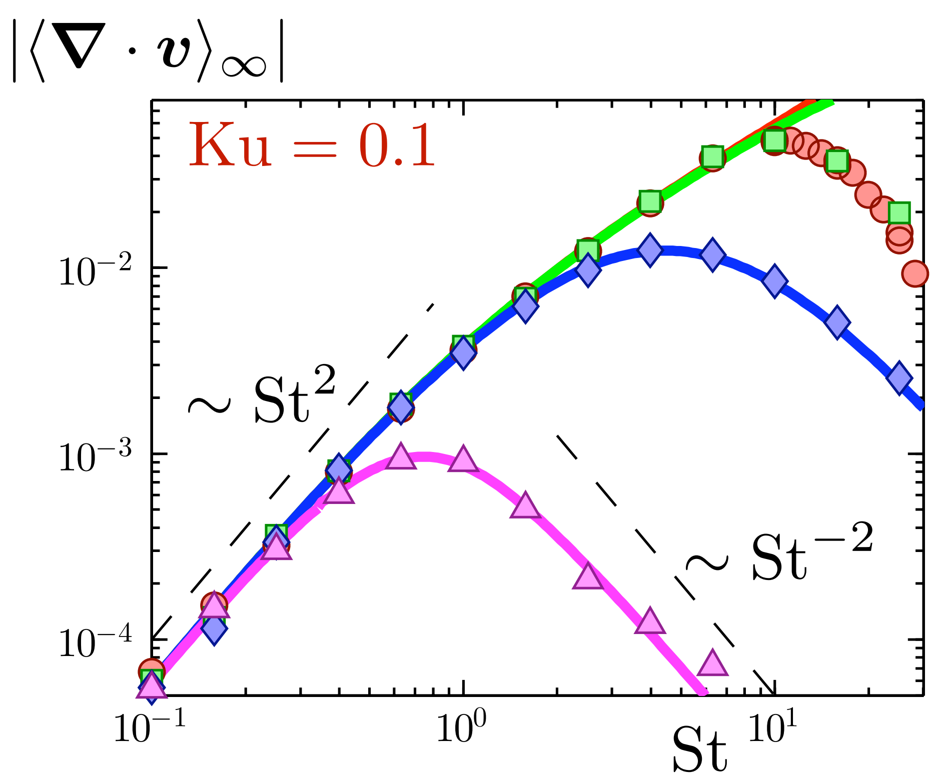
with $G = \mathrm{Ku}\,\mathrm{F}\,\mathrm{St}$, $\mathcal{F}[x] \equiv \sqrt{\pi}e^{x^2}\mathrm{erfc}(x)$

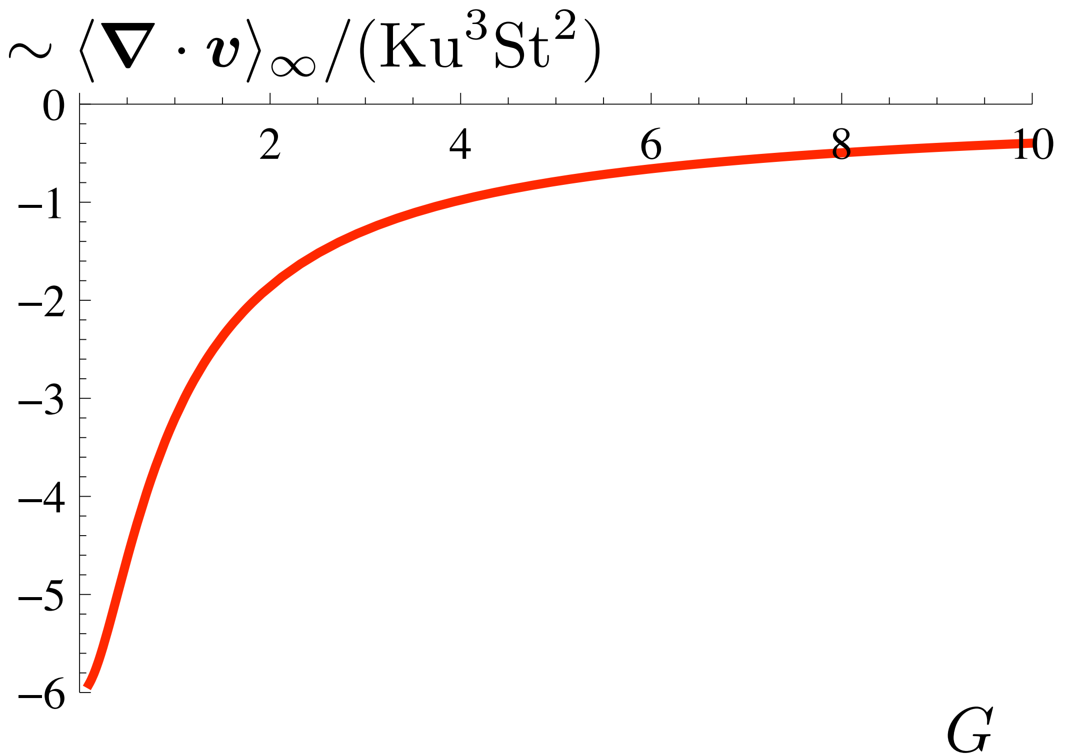


Preferential sampling of $\nabla \cdot \boldsymbol{v}$

Small St: $\langle \nabla \cdot \boldsymbol{v} \rangle_\infty \sim 3\mathrm{Ku}^3\mathrm{St}^2(4G - 6G^3 - (4 - 4G^2 + 3G^4)\mathcal{F}[G^{-1}])/(4G^5)$

As $G \to 0$ known Maxey result is recovered $\langle \nabla \cdot \boldsymbol{v} \rangle_\infty \sim -6\mathrm{Ku}^3\mathrm{St}^2$





$F = 0$ $F = 0.1$ $F = 1$ $F = 10$ Theory

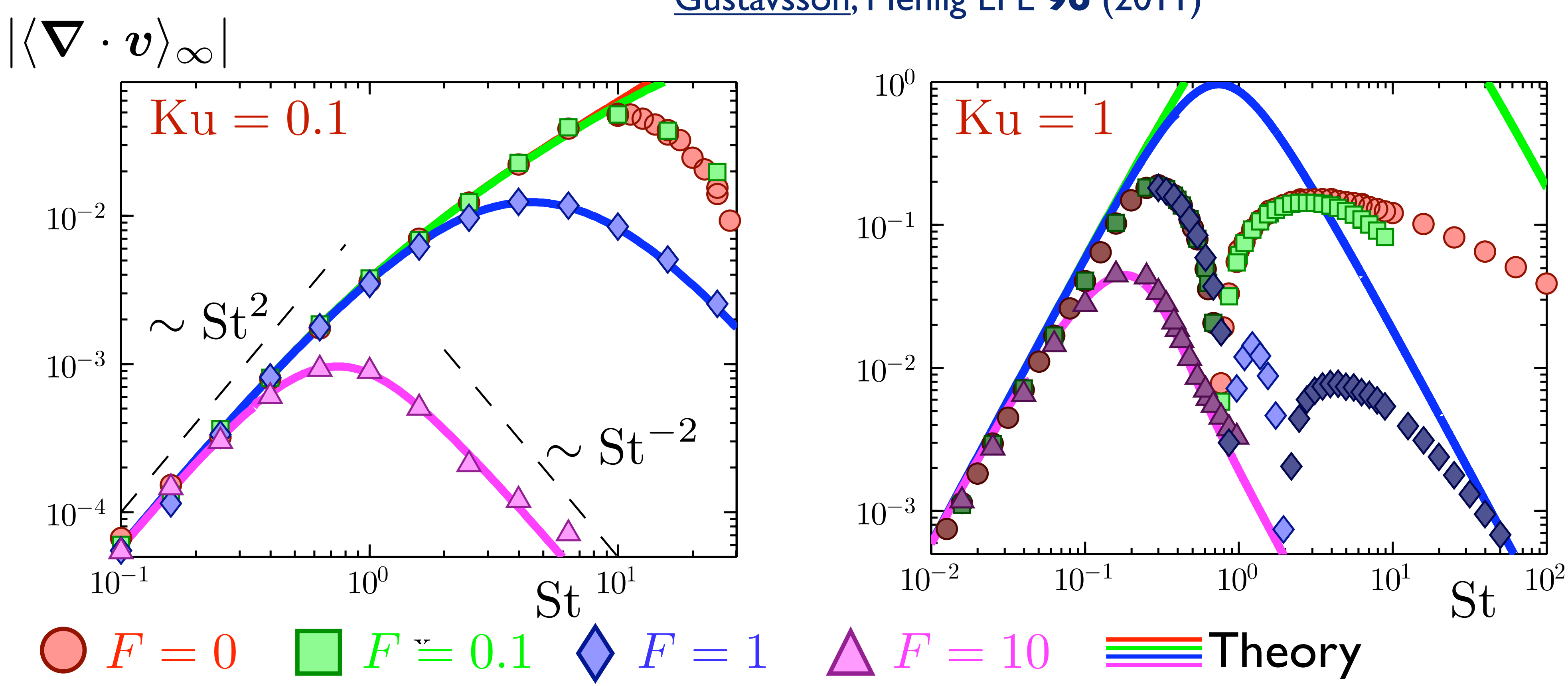
Preferential sampling of $\nabla \cdot v$

Small G :

$$\langle \nabla \cdot v \rangle_\infty \sim -6\mathrm{Ku}^3\mathrm{St}^2 \frac{1+3\mathrm{St}+\mathrm{St}^2}{(1+\mathrm{St})^3} + 9\mathrm{Ku}^3 G^2 \mathrm{St}^2 \frac{1+5\mathrm{St}+12\mathrm{St}^2+20\mathrm{St}^3+4\mathrm{St}^4}{(1+\mathrm{St})^5}$$

As $G \to 0$ earlier results are recovered

Gustavsson, Mehlig EPL **96** (2011)

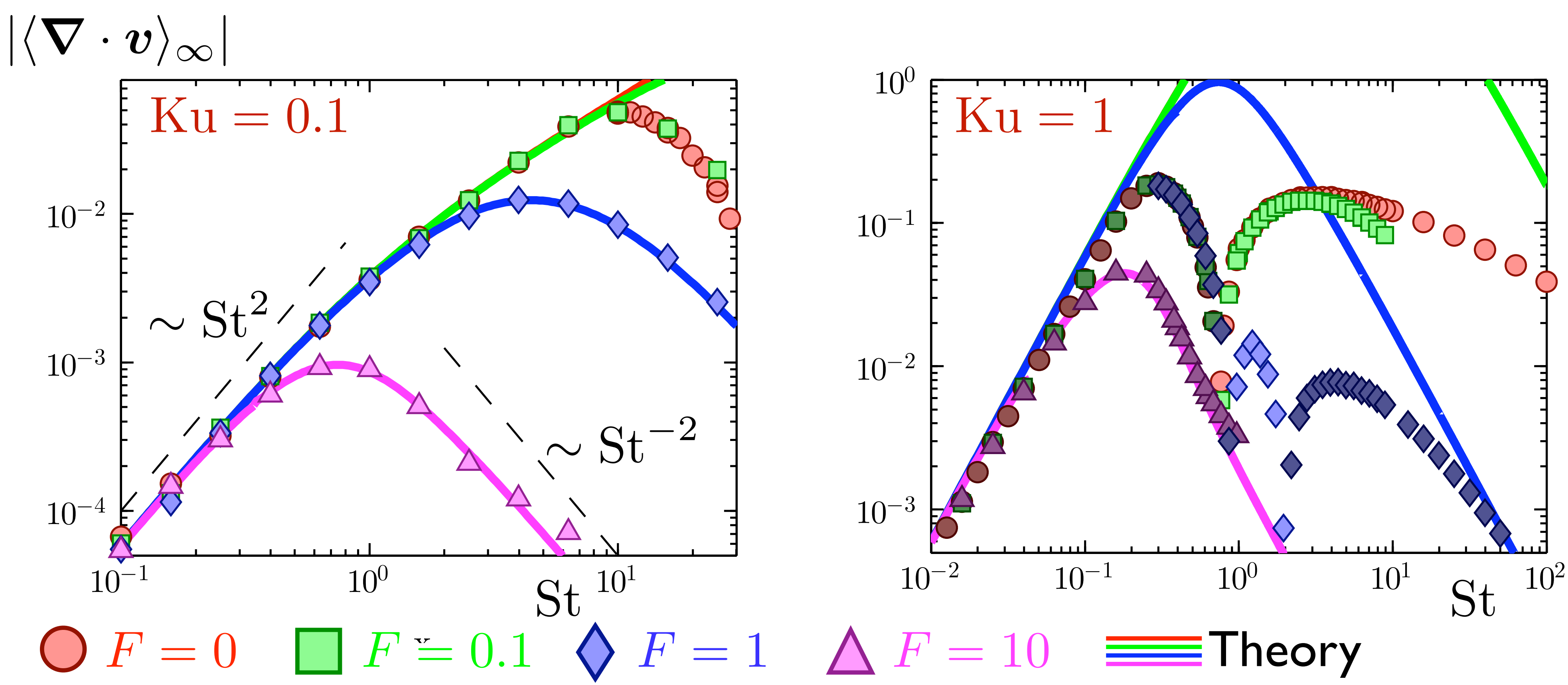


Preferential sampling of $\nabla \cdot v$

Large St , G : $\langle \nabla \cdot v \rangle_\infty \sim -3\mathrm{Ku}^3\mathrm{St}\sqrt{2\pi}/(4G^3)$

Same parameter-dependence as predicted by the Langevin model:

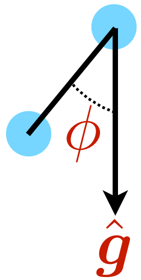
$$\mathrm{KuSt}\langle \nabla' \cdot v' \rangle_\infty \sim -3\sqrt{2\pi}/4[\mathrm{Ku}^2\mathrm{St}/G^{3/2}]^2$$



Maximal Lyapunov exponent λ_1

Similar expansion for λ_1 using

$$\lambda_1 = \lim_{t\to\infty} \frac{1}{t} \ln \frac{|\boldsymbol{R}_t|}{|\boldsymbol{R}_0|} = \lim_{t\to\infty} \frac{\mathrm{Ku}}{t} \int_0^t \mathrm{d}t' \hat{\boldsymbol{R}}_{t'}^{\mathrm{T}} \mathbb{Z}_{t'} \hat{\boldsymbol{R}}_{t'}$$



with $\hat{\boldsymbol{R}}_t \equiv \boldsymbol{R}_t/|\boldsymbol{R}_t|$ gives to lowest order in Ku

$$\lambda_1 = \frac{\mathrm{Ku}^2}{2G^5}\Big[-G^3 + G(1+11G^2)\cos^2\phi_0 - 2G(1+5G^2)\cos^4\phi_0$$
$$+\frac{1}{\sqrt{2}}\Big\{G^2(1+3G^2)-(1+12G^2+9G^4)\cos^2\phi_0+2(1+6G^2+3G^4)\cos^4\phi_0\Big\}\mathcal{F}\Big[\frac{1}{\sqrt{2}G}\Big]\Big]$$

depends on the initial angle ϕ_0 between gravity and the separation between two close-by particles.

Find λ_1 by averaging over ϕ_0 using its steady-state distribution.
We find this distribution from the moments $\langle\cos^{2p}\phi\rangle_\infty$ with $p = 1, 2, \ldots$

Complication: Steady-state averages $\langle\cos^{2p}\phi\rangle_\infty$ in turn depend on ϕ_0 and contain secular terms.

Preferential alignment

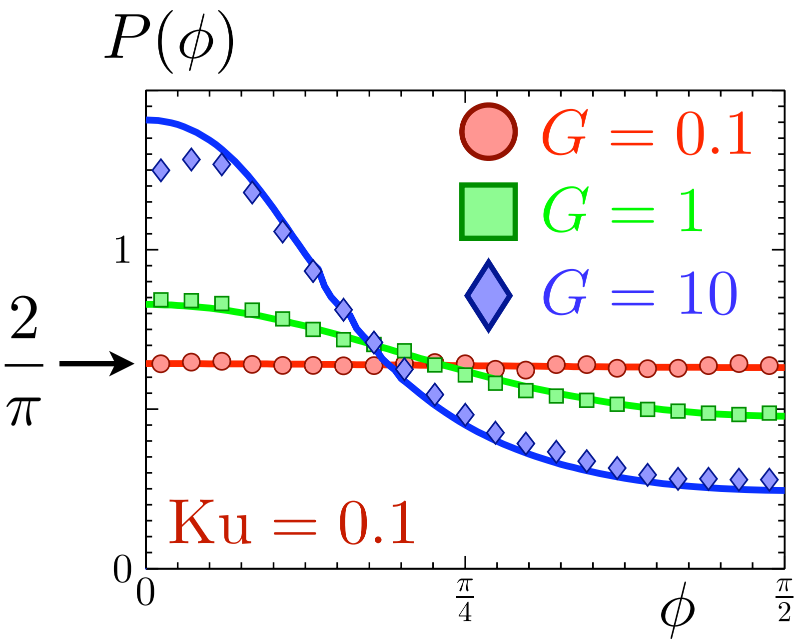
Self-consistency solution to remove the secular terms gives recursion relations for the moments $\langle \cos^{2p}\phi \rangle_\infty$.

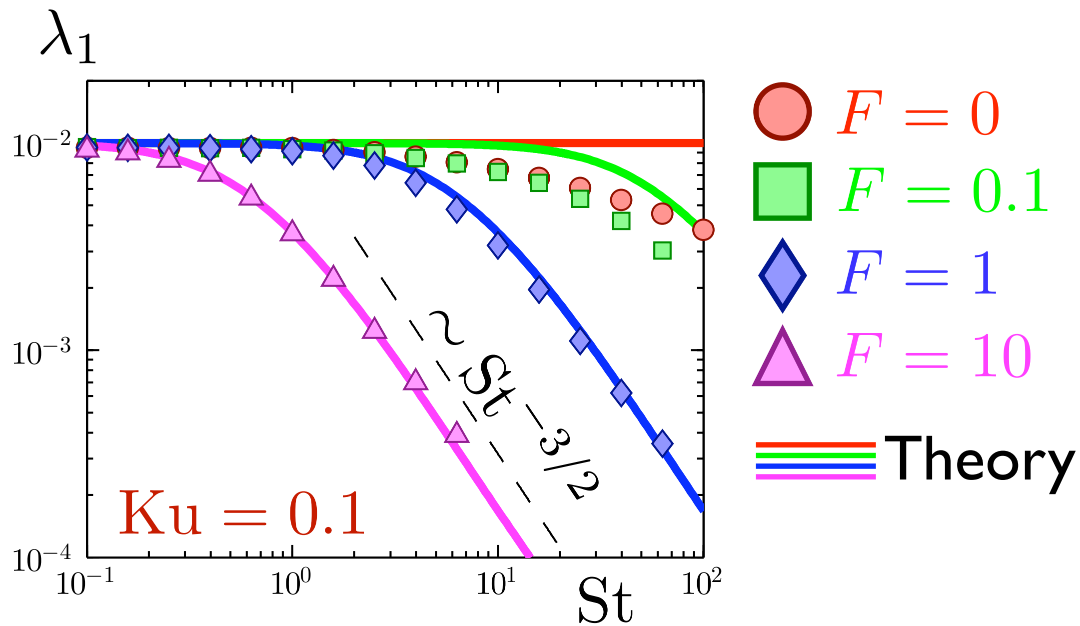
These recursions can be solved if series expanded in small $G = \mathrm{Ku}\,\mathrm{F}\,\mathrm{St}$.

We find the corresponding probability distribution of ϕ

$$P(\phi) = \frac{2}{\pi}\left[1 + \cos(2\phi)G^2 + \left(\frac{1}{4}\cos(4\phi) - 5\cos(2\phi)\right)G^4 + \dots\right]$$

Padé-Borel resum this series to find the theory plotted below





Conclusions

Inertial response to flow fluctuations and the effect of gravity are not additive.

Small St : Gravity reduces clustering because correlations between particles and flow structures are weakened.

Large St : Gravity may increase clustering significantly due to multiplicative amplification.

Gravity introduces an anistropy in the spatial distribution of close-by particles. Particle separations align with $\pm\hat{\boldsymbol{g}}$.